Chapter 1

Dynamical Systems

1.1 Introduction

1.1.1 Phase space and phase curves

Dynamics is the study of motion through phase space. The phase space of a given dynamical system is described as an N-dimensional manifold, $\mathcal{M}$. A (differentiable) manifold $\mathcal{M}$ is a topological space that is locally diffeomorphic to $\mathbb{R}^N$.[1] Typically in this course $\mathcal{M}$ will $\mathbb{R}^N$ itself, but other common examples include the circle $\mathbb{S}^1$, the torus $\mathbb{T}^2$, the sphere $\mathbb{S}^2$, *etc.*

Let $g_t \colon \mathcal{M} \to \mathcal{M}$ be a one-parameter family of transformations from $\mathcal{M}$ to itself, with $g_{t=0} = 1$, the identity. We call g_t the t-advance mapping. It satisfies the composition rule

$$g_t \, g_s = g_{t+s} \ . \tag{1.1}$$

Let us choose a point $\boldsymbol{\varphi}_0 \in \mathcal{M}$. Then we write $\boldsymbol{\varphi}(t) = g_t \, \boldsymbol{\varphi}_0$, which also is in $\mathcal{M}$. The set $\big\{ g_t \, \boldsymbol{\varphi}_0 \, \big| \, t \in \mathbb{R} \, , \, \boldsymbol{\varphi}_0 \in \mathcal{M} \big\}$ is called a *phase curve*. A graph of the motion $\boldsymbol{\varphi}(t)$ in the product space $\mathbb{R} \times \mathcal{M}$ is called an *integral curve*.

1.1.2 Vector fields

The *velocity* vector $\boldsymbol{V}(\boldsymbol{\varphi})$ is given by the derivative

$$\boldsymbol{V}(\boldsymbol{\varphi}) = \frac{d}{dt}\bigg|_{t=0} g_t \, \boldsymbol{\varphi} \ . \tag{1.2}$$

The velocity $\boldsymbol{V}(\boldsymbol{\varphi})$ is an element of the *tangent space* to $\mathcal{M}$ at $\boldsymbol{\varphi}$, abbreviated $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$. If $\mathcal{M}$ is N-dimensional, then so is each $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$ (for all $\boldsymbol{p}$). However, $\mathcal{M}$ and $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$ may

[1] A *diffeomorphism* $F \colon \mathcal{M} \to \mathcal{N}$ is a differentiable map with a differentiable inverse. This is a special type of *homeomorphism*, which is a continuous map with a continuous inverse.

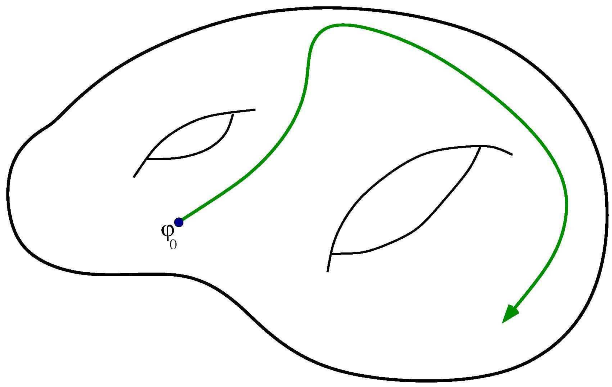

Figure 1.1: An example of a phase curve.

differ topologically. For example, if $\mathcal{M} = \mathbb{S}^1$, the circle, the tangent space at any point is isomorphic to $\mathbb{R}$.

For our purposes, we will take $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ to be an N-tuple, *i.e.* a point in $\mathbb{R}^N$. The equation of motion is then

$$\frac{d}{dt}\,\boldsymbol{\varphi}(t) = \boldsymbol{V}\big(\boldsymbol{\varphi}(t)\big)\ . \tag{1.3}$$

Note that any N^{th} order ODE, of the general form

$$\frac{d^N x}{dt^N} = F\bigg(x, \frac{dx}{dt}, \ldots, \frac{d^{N-1}x}{dt^{N-1}}\bigg)\ , \tag{1.4}$$

may be represented by the first order system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. To see this, define $\varphi_k = d^{k-1}x/dt^{k-1}$, with $k = 1, \ldots, N$. Thus, for $j < N$ we have $\dot{\varphi}_j = \varphi_{j+1}$, and $\dot{\varphi}_N = f$. In other words,

$$\overbrace{\frac{d}{dt}\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{N-1} \\ \varphi_N \end{pmatrix}}^{\dot{\boldsymbol{\varphi}}} = \overbrace{\begin{pmatrix} \varphi_2 \\ \vdots \\ \varphi_N \\ F\big(\varphi_1, \ldots, \varphi_N\big) \end{pmatrix}}^{\boldsymbol{V}(\boldsymbol{\varphi})}\ . \tag{1.5}$$

1.1.3 Existence / uniqueness / extension theorems

Theorem : Given $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}(0)$, if each $\boldsymbol{V}(\boldsymbol{\varphi})$ is a smooth vector field over some open set $\mathcal{D} \in \mathcal{M}$, then for $\boldsymbol{\varphi}(0) \in \mathcal{D}$ the initial value problem has a solution on some finite time interval $(-\tau, +\tau)$ and the solution is unique. Furthermore, the solution has a unique extension forward or backward in time, either indefinitely or until $\boldsymbol{\varphi}(t)$ reaches the boundary of $\mathcal{D}$.

Corollary : *Different trajectories never intersect!*

1.1.4 Linear differential equations

A homogeneous linear $N^{\rm th}$ order ODE,

$$\frac{d^N x}{dt^N} + c_{N-1}\,\frac{d^{N-1}x}{dt^{N-1}} + \ldots + c_1\,\frac{dx}{dt} + c_0\,x = 0 \tag{1.6}$$

may be written in matrix form, as

$$\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & & \vdots\\ -c_0 & -c_1 & -c_2 & \cdots & -c_{N-1}\end{pmatrix}}^{M}\begin{pmatrix}\varphi_1\\ \varphi_2\\ \vdots\\ \varphi_N\end{pmatrix} . \tag{1.7}$$

Thus,

$$\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}\ , \tag{1.8}$$

and if the coefficients c_k are time-independent, *i.e.* the ODE is *autonomous*, the solution is obtained by exponentiating the constant matrix Q:

$$\boldsymbol{\varphi}(t) = \exp(Mt)\,\boldsymbol{\varphi}(0)\ ; \tag{1.9}$$

the exponential of a matrix may be given meaning by its Taylor series expansion. If the ODE is not autonomous, then $M = M(t)$ is time-dependent, and the solution is given by the path-ordered exponential,

$$\boldsymbol{\varphi}(t) = \mathcal{P}\exp\left\{\int\limits_0^t \! dt'\, M(t')\right\}\boldsymbol{\varphi}(0)\ , \tag{1.10}$$

As defined, the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is autonomous, since g_t depends only on t and on no other time variable. However, by extending the phase space from $\mathcal{M}$ to $\mathbb{R}\times\mathcal{M}$, which is of dimension $(N+1)$, one can describe arbitrary time-dependent ODEs.

1.1.5 Lyapunov functions

For a general dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, a *Lyapunov function* $L(\boldsymbol{\varphi})$ is a function which satisfies

$$\boldsymbol{\nabla}L(\boldsymbol{\varphi})\cdot\boldsymbol{V}(\boldsymbol{\varphi}) \le 0\ . \tag{1.11}$$

There is no simple way to determine whether a Lyapunov function exists for a given dynamical system, or, if it does exist, what the Lyapunov function is. However, if a Lyapunov function can be found, then this severely limits the possible behavior of the system. This is because $L\big(\boldsymbol{\varphi}(t)\big)$ must be a monotonic function of time:

$$\frac{d}{dt}\,L\big(\boldsymbol{\varphi}(t)\big) = \boldsymbol{\nabla}L\cdot\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\nabla}L(\boldsymbol{\varphi})\cdot\boldsymbol{V}(\boldsymbol{\varphi}) \le 0\ . \tag{1.12}$$

Thus, the system evolves toward a local minimum of the Lyapunov function. In general this means that oscillations are impossible in systems for which a Lyapunov function exists. For example, the relaxational dynamics of the magnetization M of a system are sometimes modeled by the equation

$$\frac{dM}{dt} = -\Gamma \, \frac{\partial F}{\partial M} \ , \tag{1.13}$$

where $F(M,T)$ is the *free energy* of the system. In this model, assuming constant temperature T, $\dot{F} = F'(M)\,\dot{M} = -\Gamma \left[F'(M)\right]^2 \le 0$. So the free energy $F(M)$ itself is a Lyapunov function, and it monotonically decreases during the evolution of the system. We shall meet up with this example again in the next chapter when we discuss imperfect bifurcations.

1.2 $N = 1$ Systems

We now study phase flows in a one-dimensional phase space, governed by the equation

$$\frac{du}{dt} = f(u) \ . \tag{1.14}$$

Again, the equation $\dot{u} = h(u,t)$ is first order, but not autonomous, and it corresponds to the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} h(u,t) \\ 1 \end{pmatrix} \ . \tag{1.15}$$

The equation 1.14 is easily integrated:

$$\frac{du}{f(u)} = dt \qquad \Longrightarrow \qquad \boxed{\quad t - t_0 = \int\limits_{u_0}^{u} \frac{du'}{f(u')} \quad} \ . \tag{1.16}$$

This gives $t(u)$; we must then invert this relationship to obtain $u(t)$.

Example : Suppose $f(u) = a - bu$, with a and b constant. Then

$$dt = \frac{du}{a - bu} = -b^{-1}\, d\ln(a - bu) \tag{1.17}$$

whence

$$t = \frac{1}{b}\ln\left(\frac{a - b\,u(0)}{a - b\,u(t)}\right) \quad \Longrightarrow \quad u(t) = \frac{a}{b} + \Big(u(0) - \frac{a}{b}\Big)\exp(-bt) \ . \tag{1.18}$$

Even if one cannot analytically obtain $u(t)$, the behavior is very simple, and easily obtained by graphical analysis. Sketch the function $f(u)$. Then note that

$$\dot{u} = f(u) \quad \Longrightarrow \quad \begin{cases} f(u) > 0 & \dot{u} > 0 \quad \Rightarrow \quad \text{move to right} \\ f(u) < 0 & \dot{u} < 0 \quad \Rightarrow \quad \text{move to left} \\ f(u) = 0 & \dot{u} = 0 \quad \Rightarrow \quad \text{fixed point} \end{cases} \tag{1.19}$$

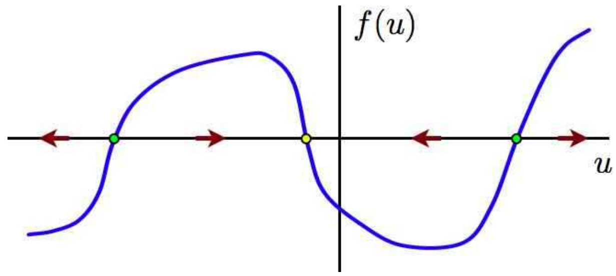

Figure 1.2: Phase flow for an $N = 1$ system.

The behavior of $N = 1$ systems is particularly simple: $u(t)$ flows to the first stable fixed point encountered, where it then (after a logarithmically infinite time) stops. The motion is monotonic – the velocity $\dot{u}$ never changes sign. Thus, *oscillations never occur for $N = 1$ phase flows.*[2]

1.2.1 Classification of fixed points ($N = 1$)

A *fixed point* u^* satisfies $f(u^*) = 0$. Generically, $f'(u^*) \neq 0$ at a fixed point.[3] Suppose $f'(u^*) < 0$. Then to the left of the fixed point, the function $f(u < u^*)$ is positive, and the flow is to the right, *i.e.* toward u^*. To the right of the fixed point, the function $f(u > u^*)$ is negative, and the flow is to the left, *i.e.* again toward u^*. Thus, when $f'(u^*) < 0$ the fixed point is said to be *stable*, since the flow in the vicinity of u^* is to u^*. Conversely, when $f'(u^*) > 0$, the flow is always away from u^*, and the fixed point is then said to be *unstable*. Indeed, if we linearize about the fixed point, and let $\epsilon \equiv u - u^*$, then

$$\dot{\epsilon} = f'(u^*)\,\epsilon + \tfrac{1}{2}\,f''(u^*)\,\epsilon^2 + \mathcal{O}(\epsilon^3) \ , \tag{1.20}$$

and dropping all terms past the first on the RHS gives

$$\epsilon(t) = \exp\Big[f'(u^*)\,t\Big]\,\epsilon(0) \ . \tag{1.21}$$

The deviation decreases exponentially for $f'(u^*) < 0$ and increases exponentially for $f(u^*) > 0$. Note that

$$t(\epsilon) = \frac{1}{f'(u^*)}\,\ln\left(\frac{\epsilon}{\epsilon(0)}\right) \ , \tag{1.22}$$

so the approach to a stable fixed point takes a logarithmically infinite time. For the unstable case, the deviation grows exponentially, until eventually the linearization itself fails.

[2]When I say ‘never’ I mean ‘sometimes’ – see the section 1.3.

[3]The system $f(u^*) = 0$ and $f'(u^*) = 0$ is overdetermined, with two equations for the single variable u^*.

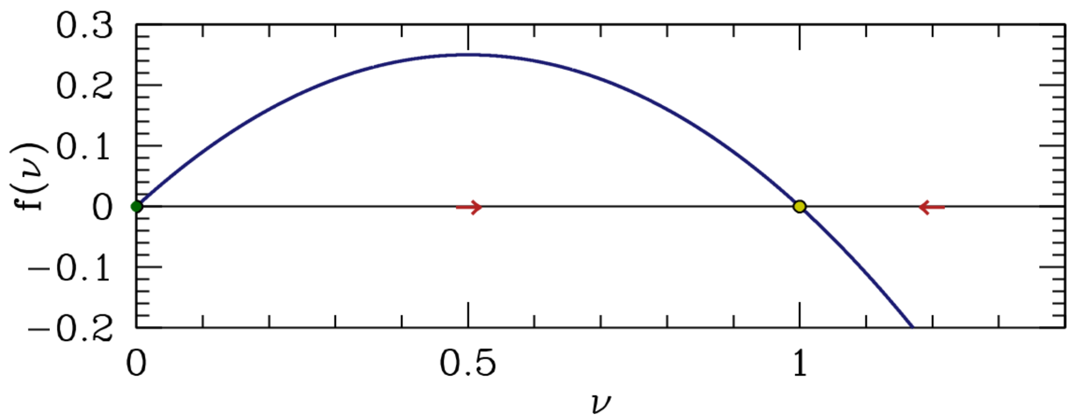

Figure 1.3: Flow diagram for the logistic equation.

1.2.2 Logistic equation

This model for population growth was first proposed by Verhulst in 1838. Let N denote the population in question. The dynamics are modeled by the first order ODE,

$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big) \ , \tag{1.23}$$

where N, r, and K are all positive. For $N \ll K$ the growth rate is r, but as N increases a quadratic nonlinearity kicks in and the rate vanishes for $N = K$ and is negative for $N > K$. The nonlinearity models the effects of competition between the organisms for food, shelter, or other resources. Or maybe they crap all over each other and get sick. Whatever.

There are two fixed points, one at $N^* = 0$, which is unstable ($f'(0) = r > 0$). The other, at $N^* = K$, is stable ($f'(K) = -r$). The equation is adimensionalized by defining $\nu = N/K$ and $s = rt$, whence

$$\dot{\nu} = \nu(1 - \nu) \ . \tag{1.24}$$

Integrating,

$$\frac{d\nu}{\nu(1-\nu)} = d\ln\Big(\frac{\nu}{1-\nu}\Big) = ds \qquad \Longrightarrow \qquad \boxed{\nu(s) = \frac{\nu_0}{\nu_0 + \big(1 - \nu_0\big)\exp(-s)}} \ . \tag{1.25}$$

As $s \to \infty$, $\nu(s) = 1 - \big(\nu_0^{-1} - 1\big)\, e^{-s} + \mathcal{O}(e^{-2s})$, and the relaxation to equilibrium ($\nu^* = 1$) is exponential, as usual.

Another application of this model is to a simple autocatalytic reaction, such as

$$A + X \rightleftharpoons 2X \ , \tag{1.26}$$

i.e. X catalyses the reaction $A \longrightarrow X$. Assuming a fixed concentration of A, we have

$$\dot{x} = \kappa_+ \, a\, x - \kappa_- \, x^2 \ , \tag{1.27}$$

where x is the concentration of X, and $\kappa_\pm$ are the forward and backward reaction rates.

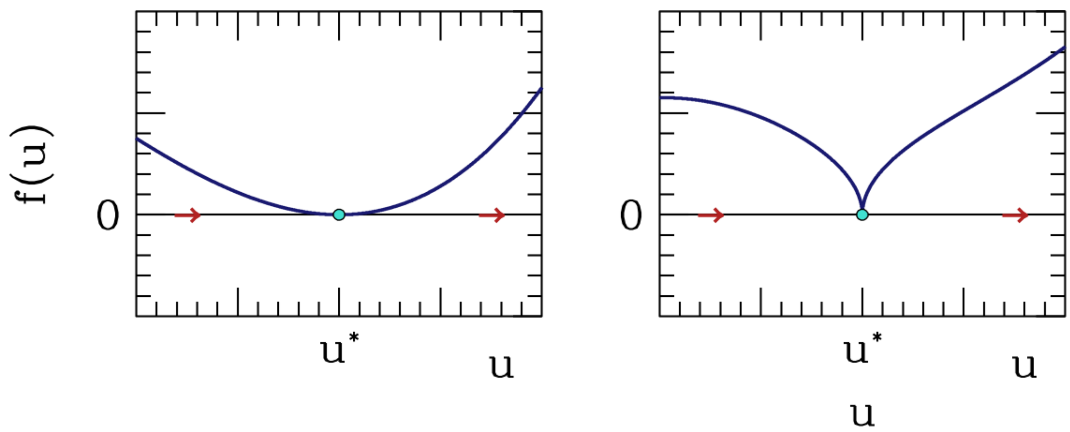

Figure 1.4: $f(u) = A\,\big|u - u^*\big|^{\alpha}$, for $\alpha > 1$ and $\alpha < 1$.

1.2.3 Singular $f(u)$

Suppose that in the vicinity of a fixed point we have $f(u) = A\,\big|u - u^*\big|^{\alpha}$, with $A > 0$. We now analyze both sides of the fixed point.

$u < u^*$: Let $\epsilon = u^* - u$. Then

$$\dot{\epsilon} = -A\,\epsilon^{\alpha} \quad \Longrightarrow \quad \frac{\epsilon^{1-\alpha}}{1-\alpha} = \frac{\epsilon_0^{1-\alpha}}{1-\alpha} - At \ , \tag{1.28}$$

hence

$$\epsilon(t) = \Big[\epsilon_0^{1-\alpha} + (\alpha - 1)A\,t\Big]^{\frac{1}{1-\alpha}} \ . \tag{1.29}$$

This, for $\alpha < 1$ the fixed point $\epsilon = 0$ is reached in a finite time: $\epsilon(t_c) = 0$, with

$$t_c = \frac{\epsilon_0^{1-\alpha}}{(1-\alpha)\,A} \ . \tag{1.30}$$

For $\alpha > 1$, we have $\lim_{t\to\infty} \epsilon(t) = 0$, but $\epsilon(t) > 0 \ \forall \ t < \infty$.

The fixed point $u = u^*$ is now *half-stable* – the flow from the left is toward u^* but from the right is away from u^*. Let's analyze the flow on either side of u^*.

$u > u^*$: Let $\epsilon = u - u^*$. Then $\dot{\epsilon} = A\,\epsilon^{\alpha}$, and

$$\epsilon(t) = \Big[\epsilon_0^{1-\alpha} + (1-\alpha)A\,t\Big]^{\frac{1}{1-\alpha}} \ . \tag{1.31}$$

For $\alpha < 1$, $\epsilon(t)$ escapes to $\epsilon = \infty$ only after an infinite time. For $\alpha > 1$, the escape to infinity takes a finite time: $\epsilon(t_c) = \infty$, with

$$t_c = \frac{\epsilon_0^{1-\alpha}}{(\alpha-1)\,A} \ . \tag{1.32}$$

In both cases, higher order terms in the (nonanalytic) expansion of $f(u)$ about $u = u^*$ will eventually come into play.

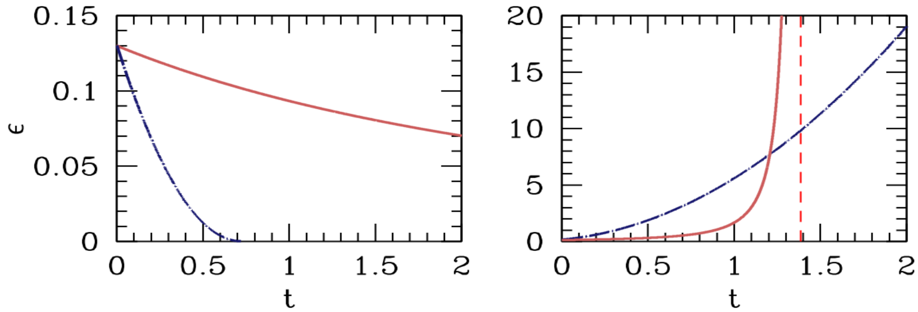

Figure 1.5: Solutions to $\dot{\epsilon} = \mp A\,\epsilon^{\alpha}$. Left panel: $\epsilon = u^* - u$, with $\alpha = 1.5$ (solid red) and $\alpha = 0.5$ (dot-dashed blue); $A = 1$ in both cases. Right panel: $\epsilon = u - u^*$, $\alpha = 1.5$ (solid red) and $\alpha = 0.5$ (dot-dashed blue); $A = 4$ in both cases

1.2.4 Recommended exercises

It is constructive to sketch the phase flows for the following examples:

$$\begin{aligned} \dot{v} &= -g & \qquad \dot{u} &= A\sin(u) \\ m\dot{v} &= -mg - \gamma v & \qquad \dot{u} &= A\,(u-a)(u-b)(u-c) \\ m\dot{v} &= -mg - cv^2\,\mathrm{sgn}(v) & \qquad \dot{u} &= au^2 - bu^3\ . \end{aligned}$$

In each case, identify all the fixed points and assess their stability. Assume all constants A, a, b, c, γ, *etc.* are positive.

1.2.5 Non-autonomous ODEs

Non-autonomous ODEs of the form $\dot{u} = h(u,t)$ are in general impossible to solve by quadratures. One can always go to the computer, but it is worth noting that in the *separable* case, $h(u,t) = f(u)\,g(t)$, one can obtain the solution

$$\frac{du}{f(u)} = g(t)\,dt \qquad \Longrightarrow \qquad \boxed{\int\limits_{u_0}^{u} \frac{du'}{f(u')} = \int\limits_0^t dt'\,g(t')}\ , \tag{1.33}$$

which implicitly gives $u(t)$. Note that $\dot{u}$ may now change sign, and $u(t)$ may even oscillate. For an explicit example, consider the equation

$$\dot{u} = A\,(u+1)\,\sin(\beta t)\ , \tag{1.34}$$

the solution of which is

$$u(t) = -1 + \big(u_0 + 1\big)\,\exp\left\{\frac{A}{\beta}\,\big[1-\cos(\beta t)\big]\right\}\ . \tag{1.35}$$

In general, the non-autonomous case defies analytic solution. Many have been studied, such as the Riccati equation,

$$\frac{du}{dt} = P(t)u^2 + Q(t)u + R(t) \ . \tag{1.36}$$

Riccati equations have the special and remarkable property that one can generate *all* solutions (*i.e.* with arbitrary boundary condition $u(0) = u_0$) from *any* given solution (*i.e.* with any boundary condition).

1.3 Flows on the Circle

We had remarked that oscillations are impossible for the equation $\dot{u} = f(u)$ because the flow is to the first stable fixed point encountered. If there are no stable fixed points, the flow is unbounded. However, suppose phase space itself is bounded, *e.g.* a circle $\mathbb{S}^1$ rather than the real line $\mathbb{R}$. Thus,

$$\dot{\theta} = f(\theta) \ , \tag{1.37}$$

with $f(\theta + 2\pi) = f(\theta)$. Now if there are no fixed points, $\theta(t)$ endlessly winds around the circle, and in this sense we can have oscillations.

1.3.1 Nonuniform oscillator

A particularly common example is that of the nonuniform oscillator,

$$\dot{\theta} = \omega - \sin\theta \ , \tag{1.38}$$

which has applications to electronics, biology, classical mechanics, and condensed matter physics. Note that the general equation $\dot{\theta} = \omega - A\sin\theta$ may be rescaled to the above form. A simple application is to the dynamics of a driven, overdamped pendulum. The equation of motion is

$$I\ddot{\theta} + b\,\dot{\theta} + I\omega_0^2 \sin\theta = N \ , \tag{1.39}$$

where I is the moment of inertia, b is the damping parameter, N is the external torque (presumed constant), and ω_0 is the frequency of small oscillations when $b = N = 0$. When b is large, the inertial term $I\ddot{\theta}$ may be neglected, and after rescaling we arrive at eqn. 1.38.

The book by Strogatz provides a biological example of the nonuniform oscillator: fireflies. An individual firefly will on its own flash at some frequency f. This can be modeled by the equation $\dot{\phi} = \beta$, where $\beta = 2\pi f$ is the angular frequency. A flash occurs when $\phi = 2\pi n$ for $n \in \mathbb{Z}$. When subjected to a periodic stimulus, fireflies will attempt to synchronize their flash to the flash of the stimulus. Suppose the stimulus is periodic with angular frequency Ω. The firefly synchronization is then modeled by the equation

$$\dot{\phi} = \beta - A\sin(\phi - \Omega t) \ . \tag{1.40}$$

Here, A is a measure of the firefly's ability to modify its natural frequency in response to the stimulus. Note that when $0 < \phi - \Omega t < \pi$, *i.e.* when the firefly is leading the stimulus,

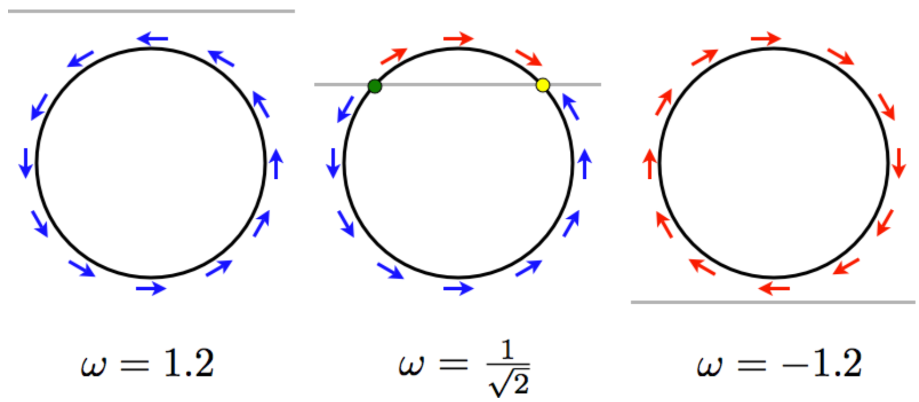

Figure 1.6: Flow for the nonuniform oscillator $\dot{\theta} = \omega - \sin\theta$ for three characteristic values of ω.

the dynamics tell the firefly to slow down. Conversely, when $-\pi < \phi - \Omega t < 0$, the firefly is lagging the stimulus, the the dynamics tell it to speed up. Now focus on the difference $\theta \equiv \phi - \Omega t$. We have

$$\dot{\theta} = \beta - \Omega - A \sin\theta \ , \tag{1.41}$$

which is the nonuniform oscillator. We can adimensionalize by defining

$$s \equiv At \quad , \quad \omega \equiv \frac{\beta - \Omega}{A} \ , \tag{1.42}$$

yielding $\frac{d\theta}{ds} = \omega - \sin\theta$.

Fixed points occur only for $\omega < 1$, for $\sin\theta = \omega$. To integrate, set $z = \exp(i\theta)$, in which case

$$\begin{aligned} \dot{z} &= -\tfrac{1}{2}(z^2 - 2i\omega z - 1) \\ &= -\tfrac{1}{2}(z - z_-)(z - z_+) \ , \end{aligned} \tag{1.43}$$

where $\nu = \sqrt{1-\omega^2}$ and $z_\pm = i\omega \pm \nu$. Note that for $\omega^2 < 1$, ν is real, and $z(t \to \mp\infty) = z_\pm$. This equation can easily be integrated, yielding

$$z(t) = \frac{(z(0) - z_-)\, z_+ - (z(0) - z_+)\, z_- \ \exp(\nu t)}{(z(0) - z_-) - (z(0) - z_+) \ \exp(\nu t)} \ , \tag{1.44}$$

For $\omega^2 > 1$, the motion is periodic, with period

$$T = \int\limits_0^{2\pi} \frac{d\theta}{|\omega| - \sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - 1}} \ . \tag{1.45}$$

The situation is depicted in Fig. 1.6.

1.4 Appendix I : Evolution of Phase Space Volumes

Recall the general form of a dynamical system, $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. Usually we are interested in finding integral curves $\boldsymbol{\varphi}(t)$. However, consider for the moment a collection of points in phase space comprising a region $\mathcal{R}$. As the dynamical system evolves, this region will also evolve, so that $\mathcal{R} = \mathcal{R}(t)$. We now ask: how does the volume of $\mathcal{R}(t)$,

$$\text{vol}\big[\mathcal{R}(t)\big] = \int\limits_{\mathcal{R}(t)} d\mu \ , \tag{1.46}$$

where $d\mu = d\varphi_1 d\varphi_2 \cdots d\varphi_N$ is the phase space measure, change with time. We have, explicitly,

$$\begin{aligned}\text{vol}\big[\mathcal{R}(t+dt)\big] &= \int\limits_{\mathcal{R}(t+dt)} d\mu \\ &= \int\limits_{\mathcal{R}(t)} d\mu \left\| \frac{\partial\varphi_i(t+dt)}{\partial\varphi_j(t)} \right\| \\ &= \int\limits_{\mathcal{R}(t)} d\mu \left\{ 1 + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{V}\, dt + \mathcal{O}\big((dt)^2\big) \right\} \ ,\end{aligned} \tag{1.47}$$

since

$$\frac{\partial\varphi_i(t+dt)}{\partial\varphi_j(t)} = \delta_{ij} + \frac{\partial V_i}{\partial\varphi_j}\bigg|_{\boldsymbol{\varphi}(t)} dt + \mathcal{O}\big((dt)^2\big) \ , \tag{1.48}$$

and, using $\ln\det M = \text{Tr}\ln M$,

$$\det(1+\epsilon A) = 1 + \epsilon\,\text{Tr}A + \mathcal{O}(\epsilon^2) \ . \tag{1.49}$$

Thus,

$$\frac{d}{dt}\text{vol}\big[\mathcal{R}(t)\big] = \int\limits_{\mathcal{R}(t)} d\mu\, \boldsymbol{\nabla}\!\cdot\!\boldsymbol{V} \tag{1.50}$$

$$= \int\limits_{\partial\mathcal{R}(t)} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{V} \ , \tag{1.51}$$

where in the last line we have used Stokes' theorem to convert the volume integral over $\mathcal{R}$ to a surface integral over its boundary $\partial\mathcal{R}$.

1.5 Appendix II : Lyapunov Characteristic Exponents

Suppose $\boldsymbol{\varphi}(t)$ is an integral curve – *i.e.* a solution of $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. We now ask: how do nearby trajectories behave? Do they always remain close to $\boldsymbol{\varphi}(t)$ for all t? To answer this, we write $\widetilde{\boldsymbol{\varphi}}(t) \equiv \boldsymbol{\varphi}(t) + \boldsymbol{\eta}(t)$, in which case

$$\frac{d}{dt}\,\eta_i(t) = M_{ij}(t)\,\eta_j(t) + \mathcal{O}\big(\eta^2\big) \ , \tag{1.52}$$

where

$$M_{ij}(t) = \frac{\partial V_i}{\partial \varphi_j}\bigg|_{\boldsymbol{\varphi}(t)} \ . \tag{1.53}$$

The solution, valid to first order in $\delta\varphi$, is

$$\eta_i(t) = Q_{ij}(t,t_0)\,\eta_j(t_0) \ , \tag{1.54}$$

where the matrix $Q(t,t_0)$ is given by the *path ordered exponential,*

$$Q(t,t_0) = \mathcal{P}\exp\left\{\int\limits_{t_0}^{t} dt'\,M(t')\right\} \tag{1.55}$$

$$\equiv \lim_{N\to\infty}\Big(1+\frac{\Delta t}{N}\,M(t_{N-1})\Big)\cdots\Big(1+\frac{\Delta t}{N}\,M(t_1)\Big)\Big(1+\frac{\Delta t}{N}\,M(t_0)\Big) \ , \tag{1.56}$$

with $\Delta t = t - t_0$ and $t_j = t_0 + (j/N)\Delta t$. $\mathcal{P}$ is the *path ordering operator*, which places earlier times to the right:

$$\mathcal{P}\,A(t)\,B(t') = \begin{cases} A(t)\,B(t') & \text{if } t > t' \\ \\ B(t')\,A(t) & \text{if } t < t' \ . \end{cases} \tag{1.57}$$

The distinction is important if $\big[A(t),B(t')\big] \neq 0$. Note that Q satisfies the composition property,

$$Q(t,t_0) = Q(t,t_1)\,Q(t_1,t_0) \tag{1.58}$$

for any $t_1 \in [t_0,t]$. When M is time-independent, as in the case of a *fixed point* where $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$, the path ordered exponential reduces to the ordinary exponential, and $Q(t,t_0) = \exp\big(M(t-t_0)\big)$.

Generally it is impossible to analytically compute path-ordered exponentials. However, the following example may be instructive. Suppose

$$M(t) = \begin{cases} M_1 & \text{if } t/T \in \big[2j, 2j+1\big] \\ \\ M_2 & \text{if } t/T \in \big[2j+1, 2j+2\big] \ , \end{cases} \tag{1.59}$$

for all integer j. $M(t)$ is a 'matrix-valued square wave', with period $2T$. Then, integrating over one period, from $t=0$ to $t=2T$, we have

$$A \equiv \ \exp\left\{\int\limits_0^{2T} dt\,M(t)\right\} = e^{(M_1+M_2)\,T} \tag{1.60}$$

$$A_{\mathcal{P}} \equiv \mathcal{P}\exp\left\{\int\limits_0^{2T} dt\,M(t)\right\} = e^{M_2\,T}\,e^{M_1\,T} \ . \tag{1.61}$$

In general, $A \neq A_{\mathcal{P}}$, so the path ordering has a nontrivial effect[4].

The Lyapunov exponents are defined in the following manner. Let $\hat{\boldsymbol{e}}$ be an N-dimensional unit vector. Define

$$\Lambda(\boldsymbol{\varphi}_0, \hat{\boldsymbol{e}}) \equiv \lim_{t\to\infty} \lim_{b\to 0} \frac{1}{t-t_0} \ln\left(\frac{\|\boldsymbol{\eta}(t)\|}{\|\boldsymbol{\eta}(t_0)\|}\right)_{\boldsymbol{\eta}(t_0)=b\,\hat{\mathbf{e}}} , \tag{1.62}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector, and where $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}\big(t_0\big)$. A theorem due to Oseledec guarantees that there are N such values $\Lambda_i(\boldsymbol{\varphi}_0)$, depending on the choice of $\hat{\boldsymbol{e}}$, for a given $\boldsymbol{\varphi}_0$. Specifically, the theorem guarantees that the matrix

$$\hat{\mathcal{Q}} \equiv \big(Q^{\mathrm{t}} Q\big)^{1/(t-t_0)} \tag{1.63}$$

converges in the limit $t \to \infty$ for almost all $\boldsymbol{\varphi}_0$. The eigenvalues Λ_i correspond to the different eigenspaces of R. Oseledec's theorem (also called the 'multiplicative ergodic theorem') guarantees that the eigenspaces of Q either grow ($\Lambda_i > 1$) or shrink ($\Lambda_i < 1$) *exponentially* fast. That is, the norm any vector lying in the i^{th} eigenspace of Q will behave as $\exp\big(\Lambda_i\,(t-t_0)\big)$, for $t \to \infty$.

Note that while $\hat{\mathcal{Q}} = \hat{\mathcal{Q}}^{\mathrm{t}}$ is symmetric by construction, Q is simply a general real-valued $N \times N$ matrix. The left and right eigenvectors of a matrix $M \in \mathrm{GL}(N, \mathbb{R})$ will in general be different. The set of eigenvalues λ_α is, however, common to both sets of eigenvectors. Let $\{\psi_\alpha\}$ be the right eigenvectors and $\{\chi^*_\alpha\}$ the left eigenvectors, such that

$$M_{ij}\,\psi_{\alpha,j} = \lambda_\alpha\,\psi_{\alpha,i} \tag{1.64}$$
$$\chi^*_{\alpha,i}\,M_{ij} = \lambda_\alpha\,\chi^*_{\alpha,j} \ . \tag{1.65}$$

We can always choose the left and right eigenvectors to be orthonormal, *viz.*

$$\big\langle\, \chi_\alpha \,\big|\, \psi_\beta \,\big\rangle = \chi^*_{\alpha,i}\,\psi_{\beta,j} = \delta_{\alpha\beta} \ . \tag{1.66}$$

Indeed, we can define the matrix $S_{i\alpha} = \psi_{\alpha,i}$, in which case $S^{-1}_{\alpha j} = \chi^*_{\alpha,j}$, and

$$S^{-1}\,M\,S = \mathrm{diag}\big(\lambda_1, \ldots, \lambda_N\big) \ . \tag{1.67}$$

The matrix M can always be decomposed into its eigenvectors, as

$$M_{ij} = \sum_\alpha \lambda_\alpha\,\psi_{\alpha,i}\,\chi^*_{\alpha,j} \ . \tag{1.68}$$

If we expand $\boldsymbol{u}$ in terms of the right eigenvectors,

$$\boldsymbol{\eta}(t) = \sum_\beta C_\beta(t)\,\boldsymbol{\psi}_\beta(t) \ , \tag{1.69}$$

[4] If $\big[M_1, M_2\big] = 0$ then $A = A_{\mathcal{P}}$.

then upon taking the inner product with $\boldsymbol{\chi}_\alpha$, we find that C_α obeys

$$\dot{C}_\alpha + \langle\, \chi_\alpha \,|\, \dot{\psi}_\beta \,\rangle\, C_\beta = \lambda_\alpha\, C_\alpha \ . \tag{1.70}$$

If $\dot{\boldsymbol{\psi}}_\beta = 0$, *e.g.* if M is time-independent, then $C_\alpha(t) = C_\alpha(0)\, e^{\lambda_\alpha t}$, and

$$\eta_i(t) = \sum_\alpha \overbrace{\sum_j \eta_j(0)\, \chi^*_{\alpha,j}}^{C_\alpha(0)}\ e^{\lambda_\alpha t}\, \psi_{\alpha,i} \ . \tag{1.71}$$

Thus, the component of $\boldsymbol{\eta}(t)$ along $\boldsymbol{\psi}_\alpha$ increases exponentially with time if $\mathrm{Re}(\lambda_\alpha) > 0$, and decreases exponentially if $\mathrm{Re}(\lambda_\alpha) < 0$.