Lect 8. Lippman - Schwinger equation; Born approximation

We will introduce another method of integration equation to treat the scattering amplitude, without ~~tak~~ doing partial wave. In order to solve the Schringer equation

$$(\nabla^2 + k^2)\psi(r) = \frac{2m}{\hbar^2} V(r)\,\psi(r),$$ under the boundary condition

$$\psi(r) \xrightarrow{r\to\infty} e^{i\vec{k}\cdot\vec{r}} + f(\theta,\varphi)\frac{e^{ikr}}{r}.$$

Define Green function statisfying $(\nabla^2 + k^2)\, G(r,r') = \delta(r-r')$

$\Rightarrow$ $\psi(r) = \frac{2m}{\hbar^2}\int dr'\, G(r,r')\, V(r')\, \psi(r')$ satisfies the Schrödinger eq.

check $(\nabla^2+k^2)\psi(r) = \frac{2m}{\hbar^2}\int dr' \,(\nabla^2+k^2)\, G(r,r')\, V(r')\psi(r') = \frac{2m}{\hbar^2}\int dr'\,\delta(r-r')\, V(r')\,\psi(r')$

$$= \frac{2m}{\hbar^2} V(r)\,\psi(r).$$

We can also add the homogeous part $\psi^0(r)$ satisfying $(\nabla^2+k^2)\psi^0(r) = 0$
plane wave

$\Rightarrow$ Scattering problem reduces to the integral equation

$$\boxed{\psi(r) = e^{i\vec{k}\cdot\vec{r}} + \frac{2m}{\hbar^2}\int dr'\, G(r,r')\, V(r')\,\psi(r')}$$

Lippman - Schwinger Eq.

The next question is how to determine $G(r,r')$ to match the outgoing wave boundary condition $\psi_{sc}(r) = \frac{2m}{\hbar^2}\int dr'\, G(r,r')\, V(r')\,\psi(r') \xrightarrow{r\to+\infty} f(\theta,\varphi)\frac{e^{ikr}}{r}$.

due to the translational symmetry $G(r, r') = G(r-r')$

$$G(r-r') = \int d^3\vec{q}\; e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}\; G(\vec{q})$$

$$\Rightarrow \quad (-q^2 + k^2)\, G(\vec{q}) = \frac{1}{(2\pi)^3} \quad \text{or} \quad G(\vec{q}) = \frac{1}{(2\pi)^3}\,\frac{-1}{q^2-k^2}$$

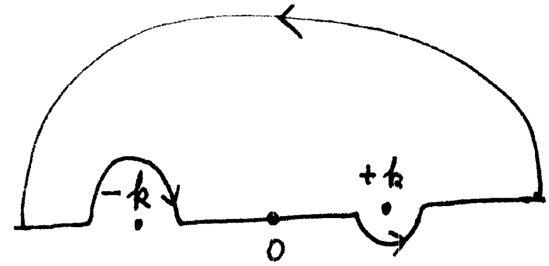
$$G(\vec{r}-\vec{r}') = -\frac{1}{(2\pi)^3}\int d^3\vec{q}\;\frac{1}{q^2-k^2}\; e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}$$

$$= -\frac{1}{(2\pi)^3}\int_0^\infty q^2 dq \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi\; \frac{e^{iq|\vec{r}-\vec{r}'|\cos\theta}}{q^2-k^2} \qquad R = |\vec{r}-\vec{r}'|$$

$$= -\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} \frac{1}{iR}\, dq \cdot \frac{q\, e^{iqR}}{q^2-k^2}$$

, there's two first order poles $q = \pm k$

we need to decide the integral contour



$$= -\frac{2\pi i}{(2\pi)^2\, iR}\; \frac{q\, e^{iqR}}{q+k}\Bigg|_{q=k} = -\frac{1}{4\pi R}\, e^{ikR}$$

Another way is to add a small imaginary part to the Green function as $\frac{-1}{q^2-k^2 \pm i\eta}\,\frac{1}{(2\pi)^3}$
equivalent

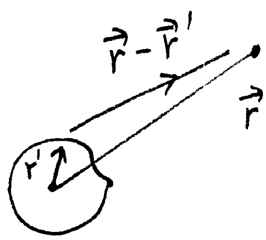
$$\Rightarrow \quad \boxed{\psi(r) = e^{i\vec{k}_i\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}'\; \frac{e^{ik|r-r'|}}{|r-r'|}\, V(r')\, \psi(r')}$$

First order approx: (Born approximation)

$$\psi(r) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\int d^3\vec{r}\,' \frac{e^{ik|r-r'|}}{|r-r'|} V(r')\, e^{i\vec{k}\cdot\vec{r}\,'}$$

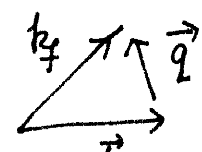
due to the short range of $V(r')$, $|r-r'|$ can be approximated as r in the denominator. But for the phase factor $e^{ik|r-r'|} \approx e^{ikr(1-\vec{r}\cdot\vec{r}\,'/r^2)}$

$= e^{ikr}\, e^{i\vec{k}_f\cdot\vec{r}\,'}$ where $\vec{k}_f = \frac{\vec{k}\cdot\vec{r}}{r}$



$$|\vec{r}-\vec{r}\,'| = (r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,')^{1/2} = r(1 - \vec{r}\cdot\vec{r}\,'/r^2)$$

$$\psi_{scattering}(\vec{r}) \xrightarrow{r\to\infty} -\frac{m e^{ikr}}{2\pi\hbar^2 r}\int d^3r'\, e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}\,'} V(\vec{r}\,')$$



$\vec{q} = \vec{k}_f - \vec{k}$

$$= -\frac{m e^{ikr}}{2\pi\hbar^2 r} V(\vec{k}_f - \vec{k}_i)$$

$$\Rightarrow \psi(r) = e^{i\vec{k}_i\cdot\vec{r}} - \frac{e^{ikr}}{r}\frac{m}{2\pi\hbar^2}V(\vec{k}_f-\vec{k}) \Rightarrow f(\vec{k}_i,\vec{k}_f) = -\frac{m}{2\pi\hbar^2}V(\vec{k}_f-\vec{k}_i)$$

Set $\vec{k}_{in}$ along the z-axis

$$\underset{\vec{q}}{\overset{\parallel}{V(\vec{k}_f-\vec{k}_f)}} = 2\pi\int r'^2 dr' \sin\theta\, d\theta\, V(r')\, e^{-i\vec{q}\cdot\vec{r}\,'\cos\theta} = \frac{2\pi}{2q}\int dr'\, r'\, V(r') \sin qr'$$

$$\Rightarrow f(\theta) = -\frac{2\mu}{\hbar^2 q}\int_0^{+\infty} r' V(r') \sin qr'\, dr' \qquad q = 2k\sin\frac{\theta}{2}$$

$$\Rightarrow \boxed{\sigma(\theta) = |f(\theta)|^2 = \frac{4\mu^2}{\hbar^4 q^2}\left|\int_0^{+\infty} r' V(r') \sin qr'\, dr'\right|^2,} \qquad q = 2k\sin\frac{\theta}{2}$$

§ The condition for Born approximation:

Born approx. is essentially a perturbation theory, we need $|\psi_{scattering}| << e^{i\vec{k}_i \cdot \vec{r}}$ to justify it. At $r=0$, $V_{scattering}$ is ~~strongest~~, so the condition can be justified as $|\psi_{scattering}(0)| << 1$.

$$|\psi_{scattering}(0)| = \frac{m}{2\pi\hbar^2}\left|\int d^3r' \, \frac{e^{ikr'}}{r'} V(r') e^{i\vec{k}\cdot\vec{r}'}\right|$$

$$= \frac{2m}{\hbar^2 k}\left|\int_0^{+\infty} dr' \, e^{ikr'} V(r') \sin kr'\right| << 1$$

For low energy scattering, $\sin kr' \simeq kr'$ $e^{ikr'} \simeq 1 \Rightarrow$

$$\frac{2m}{\hbar^2}\left|\int_0^{+\infty} dr' \, r V(r')\right| \ll 1 .$$

Let us conside $V(r')$ has range r_0 and strength V_0

$$\Rightarrow \frac{m}{\hbar^2}|V_0| r_0^2 << 1 \quad \text{i.e.} \quad \boxed{|V_0| << \frac{\hbar^2}{m r_0^2}}$$

For high energy scattering, $(kr_0 >> 1)$

$$e^{ikr'} \sin kr' = \cos kr' \sin kr' + i \sin^2 kr'$$

the first term vanishes, and the second term has an averged value of $1/2$

$$\Rightarrow \quad \frac{m}{\hbar^2 k} r_0 |V_0| << 1$$

$$\text{or} \quad \boxed{|V_0| \lesssim \frac{\hbar^2 k}{m r_0} = \frac{\hbar^2 (kr_0)}{m r_0^2}}$$

Thus if Born approx is valid in the low energy secter, it's also valid in the high energy secter.

§ phase shift δ_ℓ from Born approximation

$$f(\theta) = -\frac{2m}{\hbar^2|\vec{k}_f-\vec{k}_i|}\int_0^\infty r' V(r') \sin|\vec{k}_f-\vec{k}_i| r' dr'$$

$$\frac{\sin|\vec{k}_f-\vec{k}_i|r}{|\vec{k}_f-\vec{k}_i|r} = \sum_{\ell=0}^{\infty}(2\ell+1)\, j_\ell^2(kr)\, P_\ell(\cos\theta); \quad |\vec{k}_f-\vec{k}_i| = 2k\sin\theta/2$$

$$\Rightarrow f(\theta) = -\frac{2m}{\hbar^2}\int_0^\infty r'^2 V(r') \overbrace{j_\ell^2(kr)}^{\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)} dr' = -\frac{2m}{\hbar^2}\sum_{\ell=0}^{\infty}\sqrt{4\pi(2\ell+1)}\int_0^{+\infty} r'^2 V(r') j_\ell^2(kr') dr'$$

compare to partial wave result $f(\theta) = \sum_\ell \frac{1}{k} e^{i\delta_\ell}\sin\delta_\ell \sqrt{4\pi(2\ell+1)}\, Y_{\ell 0}(\theta)$

as for small phase shift $e^{i\delta_\ell} \sim 1$, $\sin\delta_\ell \sim \delta_\ell$

$$\Rightarrow -\frac{2m}{\hbar^2}\int_0^{+\infty} r'^2 V(r') j_\ell^2(kr') dr' = \frac{\delta_\ell}{k}$$

i.e. $$\delta_\ell = -\frac{2mk}{\hbar^2}\int_0^{+\infty} V(r) j_\ell^2(kr)\, r^2 dr$$

$$j_\ell(kr) \xrightarrow{kr\to 0} \frac{k^\ell r^\ell}{(2\ell+1)!!}$$

if $V(r)$ is short range

$$\Rightarrow \boxed{\delta_\ell \approx -\frac{2mkV_0}{\hbar^2}\int_0^{r_0}\frac{k^{2\ell}r^{2\ell}}{([2\ell+1]!!)^2} r^2 dr \quad \propto k^{2\ell+1}}$$

§ Coulomb scattering:

Rigoursly speaking, due to long range nature of Coulomb scattering, the boundary condition of $e^{ikr} + f(\theta,\varphi)\frac{e^{ikr}}{r}$ does not apply. And also $\int_0^{+\infty} r V(r) dr \to +\infty$, so the condition for Born approximation also fails.

We formally treat Coulomb potential as Yukawa potential

$$V(r) = \frac{\kappa}{r} e^{-\alpha r} \quad \text{as} \quad \alpha \to 0. \quad \text{Born condition applies}$$

$$\text{if} \quad \frac{2m}{\hbar^2}\left|\int_0^\infty r V(r) dr\right| = \frac{2m\kappa}{\hbar^2 \alpha} \ll 1 .$$

For Yukawa potential,

$$f(\theta) = -\frac{m}{2\pi\hbar^2} V(\vec{k}_f - \vec{k}_i) = -\frac{m}{2\pi\hbar^2}\frac{4\pi\kappa}{|k_f - k_i|^2 + \alpha^2} = -\frac{2m\kappa}{\hbar^2}\frac{1}{4k^2\sin^2\frac{\theta}{2} + \alpha^2}$$

$$\sigma(\theta) = |f(\theta)|^2 = \frac{4m^2\kappa^2}{\hbar^4}\frac{1}{(4k^2\sin^2\frac{\theta}{2} + \alpha^2)^2} \xrightarrow{\alpha\to 0} \frac{4m^2\kappa^2}{16\hbar^4 k^4 \sin^4\theta/2} = \frac{\kappa^2}{16E^2\sin^4\theta/2}$$

which is just the Rutherfold formula.