PHYSICS 210A, HOMEWORK ASSIGNMENT #4

May 21, 2009

1. Solve Problem 8.2 of the text; here, you are supposed to show that the ratio T_0 / T_F is about 0.989.

2. Solve Problem 8.13 of the text.
[Hint:- Since the answers to this problem are in terms of the single-particle density of states $g(\varepsilon)$, keep your integrals for N and U over ε --- and do not employ the phase-space approach we normally resort to.
Secondly, to evaluate these integrals, you need to use *only* the first two terms of the formula (E.16); however, in doing so, you'll have to further approximate the first term of this formula (for, in it, the upper limit on the integral is μ / kT, not ε_F / kT.]

3. In the class, I presented a simplified treatment for deriving a formula for the magnetic susceptibility of a gas of spin-half fermions valid at *all* temperatures --- the formula that appears in Problem 8.15 as well. Generalize that treatment to a gas of fermions carrying a magnetic moment μ^*, with spin $J\hbar$ (where $J = 1/2, 3/2, 5/2,$ ----).

4. Re-visit the problem of the "statistical equilibrium of white dwarf stars", for which it is essential that the motion of the electrons be treated as *relativistic*. Using the approximation

$$\epsilon = \sqrt{p^2c^2 + m^2c^4} - mc^2$$
$$\simeq pc - mc^2 + \frac{m^2c^3}{2p},$$

determine the pressure of the electron gas and thereby the mass-radius relationship for such stars --- thus establishing the existence of a limiting mass M_0 that signals the gravitational collapse of the star.

In determining M_0, do not worry about *exact* numerical factors --- just focus on the main characters of the play!

5. In the class, I discussed an Ising-like model in which the spin quantum number of atoms on the various lattice sites is s (instead of ½), with $g = 2s+1$ (instead of 2). Using mean-field approach, I showed that this system undergoes a phase transition at a critical temperature T_c, given by

$$T_c = \{(s+1)/3s\}\, qJ/k.$$

Now, you are supposed to show that, for $T \lesssim T_c$, the order parameter m_0 is given by

$$m_0 \simeq \frac{\sqrt{10}\,(s+1)}{\sqrt{3(2s^2+2s+1)}}\left(1-\frac{T}{T_c}\right)^{1/2}.$$

6. A collection of spin-½ paricles is adsorbed on a surface that has N adsorption sites. For each site, let σ be +1 if the site is occupied by a particle with spin up, −1 if it is occupied by a particle with spin down, 0 if it is unoccupied. The Hamiltonian of the system, in obvious notation, is

$$\mathcal{H} = -W\sum_i \sigma_i^2 - \mu^* H \sum_i \sigma_i - J\sum_{n.n.} \sigma_i\sigma_j \,,$$

where W is the binding energy of an adsorbed atom --- regardless of whether its spin is up or down.

(a) Let p_+ be the probability that a site is occupied by an atom with spin up, p_- the probability that it is occupied by a particle with spin down and p_0 the probability that it is vacant. Let f be the "adsorbate density" and m the "magnetisation density" of the lattice, i.e., $f = p_+ + p_-$ (so f lies between 0 and 1) and $m = p_+ - p_-$ (so m lies between $-f$ and $+f$). Using these probabilities, evaluate the entropy S of the system in terms of the parameters f and m.
Next, using mean-field approximation, write down an expression for the energy U of the system and then for the Helmholtz free energy A --- again as a function of f and m.

(b) Now, with m fixed, minimize A with respect to f to determine the optimal value of f [which you may call $f^*(m)$]. What are the special values of $f^*(m)$ when $m = 0$ and when $m = 1$? And what are its limiting values as T goes to zero and as T goes to infinity?

(c) Finally, with f fixed, minimize A with respect to m to determine the optimal value of m [which you may call $m^*(f)$]. Based on this result, establish a criterion that determines the critical temperature T_c at which a phase transition occurs in this system.