Nonlinear gyrokinetic equations for low-frequency electromagnetic waves in general plasma equilibria

E. A. Frieman[a] and Liu Chen

Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08544
(Received 6 October 1981; accepted 6 January 1982)

A nonlinear gyrokinetic formalism for low-frequency (less than the cyclotron frequency) microscopic electromagnetic perturbations in general magnetic field configurations is developed. The nonlinear equations thus derived are valid in the strong-turbulence regime and contain effects due to finite Larmor radius, plasma inhomogeneities, and magnetic field geometries. The specific case of axisymmetric tokamaks is then considered and a model nonlinear equation is derived for electrostatic drift waves. Also, applying the formalism to the shear Alfvén wave heating scheme, it is found that nonlinear ion Landau damping of kinetic shear-Alfvén waves is modified, both qualitatively and quantitatively, by the diamagnetic drift effects. In particular, wave energy is found to cascade in wavenumber instead of frequency.

I. INTRODUCTION

Electromagnetic instabilities with frequencies lower than the ion-cyclotron frequency and perpendicular (to the magnetic field) wavelengths comparable to the ion Larmor radius are believed to be important for the transport processes in magnetically confined plasmas. One complicating factor in analyzing these low-frequency microscopic instabilities, which are driven by plasma inhomogeneities and shall be loosely termed kinetic drift-Alfvén waves, is that the destabilizing mechanisms are sensitive to effects associated with magnetic field configurations such as magnetic shear, magnetic gradient and curvature drifts, and trapped particles. To overcome this difficulty, Rutherford and Frieman[1] as well as Taylor and Hastie[2] have, independently, developed a formalism, now known as the gyrokinetic formalism, to treat the linear aspects of kinetic drift waves in general magnetic configurations. Recently, the linear gyrokinetic formalism has been extended to include electromagnetic perturbations[3,4] associated with shear and compressional Alfvén waves. Since the transport induced by the instabilities is ultimately determined by the nonlinear processes, it is, therefore, desirable to further extend the gyrokinetic formalism into the nonlinear regime while retaining crucial features such as finite Larmor radius and arbitrary magnetic field configurations. This constitutes the principal motivation of the present research.

In this work, we develop the nonlinear gyrokinetic formalism based on a multiple-scale expansion;[5] that is, microscopic fluctuations vary on the fast (linear) time scale (i.e., typically, the diamagnetic drift frequency time scale), while macroscopic quantities are assumed to vary on the slow transport time scale. Here, the cyclotron frequency time scale is the fastest time scale. Furthermore, consistent with experimental observations,[6,7] our formulation allows nonlinear time scales to be comparable to the linear ones and, hence, the results are valid in the strong-turbulence regime.

[a]Present address: Science Application Inc., La Jolla, CA 92038 and University of California, San Diego, California 92092.

Section II contains the theoretical formulation and derivations of the nonlinear gyrokinetic equations. The specific case of axisymmetric tokamaks is further considered in Sec. III using the ballooning-mode representation[8-10] and a nonlinear drift-wave equation is derived in a limiting case. Noting that our results are also applicable to nonlinear heating via externally launched low-frequency waves, in Sec. IV, we also consider nonlinear ion Landau damping (ion induced scattering) of the mode-converted kinetic (shear) Alfvén waves[11] and find that the inclusion of diamagnetic drift effects modifies, both qualitatively and quantitatively, the decay processes. Final conclusions and discussion are given in Sec. V.

II. THEORETICAL ANALYSES

A. Guiding-center transformation and the two spatial scales

As in the linear formalism,[1,2,12] it is more convenient to carry out the analysis in the guiding-center phase space, $(\mathbf{X}, \mathbf{V})$, which is related to the particle phase space, $(\mathbf{x}, \mathbf{v})$, via the following guiding-center transformation

$$\mathbf{X} = \mathbf{x} + \mathbf{v}\times\mathbf{e}_{\parallel}/\Omega, \tag{1}$$

$$\mathbf{V} = \mathbf{V}(\epsilon, \mu, \alpha), \tag{2}$$

where $\epsilon = v^2/2 + q\Phi_0/m$, $\mu = v_\perp^2/2B$, Φ_0 and $\mathbf{B}$ are, respectively, the macroscopic (equilibrium) electric potential and magnetic field, $\Omega = qB/mc$, $\mathbf{e}_{\parallel} = \mathbf{B}/B$, α is the gyrophase, and

$$\mathbf{v}_\perp = v_\perp(\mathbf{e}_1 \cos\alpha + \mathbf{e}_2 \sin\alpha), \tag{3}$$

with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_{\parallel}$ being the local orthogonal unit vectors. Furthermore, noting that perturbations of interest here have perpendicular (to B) wavelengths of the order of the Larmor radius, ρ, which is much smaller than the macroscopic scale length, L_0 (i.e., $\lambda \equiv \rho/L_0$ is a small parameter), we may separate physical quantities into microscopic and macroscopic parts by averaging over the microscopic spatial variations. Thus,

$$\mathbf{p} = \mathbf{P} + \delta\mathbf{P}, \tag{4}$$

where

 0031-9171/82/030502-07$01.90

$$\mathbf{P}=\frac{\int d^2\mathbf{X}_{\perp 0}\,\mathbf{p}}{\int d^2\mathbf{X}_{\perp 0}}\equiv\langle \mathbf{p}\rangle_{\mathbf{X}_{\perp 0}}\,, \tag{5}$$

$\langle\delta\mathbf{P}\rangle_{\mathbf{X}_{\perp 0}}=0$ and $\mathbf{X}_{\perp 0}$ corresponds to fast perpendicular spatial variations.

In the $(\mathbf{X},\mathbf{V})$ phase space, the Vlasov equation then becomes

$$L_g f=-(q/m)(\delta R f)\,, \tag{6}$$

where

$$L_g=\frac{\partial}{\partial t}+\frac{\partial\mathbf{V}}{\partial t}\cdot\nabla_V+\frac{\partial\mathbf{X}}{\partial t}\cdot\nabla_X+v_\parallel\mathbf{e}_\parallel\cdot\nabla_X$$
$$+\mathbf{v}\cdot(\boldsymbol{\lambda}_{B1}+\boldsymbol{\lambda}_{B2})+\frac{q}{m}\,(\mathbf{E}-\mathbf{E}_0)\cdot\mathbf{v}\,\frac{\partial}{\partial\epsilon}$$
$$+\frac{q}{m}\,\mathbf{E}\cdot\left(\frac{\mathbf{v}_\perp}{B}\,\frac{\partial}{\partial\mu}+\frac{\mathbf{e}_\alpha}{v_\perp}\,\frac{\partial}{\partial\alpha}\right)+\mathbf{v}_E\cdot\nabla_X-\Omega\,\frac{\partial}{\partial\alpha}\,, \tag{7}$$

$$\boldsymbol{\lambda}_{B1}=\mathbf{v}\times\nabla_x(\mathbf{e}_\parallel/\Omega)\cdot\nabla_X\,, \tag{8}$$

$$\boldsymbol{\lambda}_{B2}=(\nabla_x\mu)\,\frac{\partial}{\partial\mu}+(\nabla_x\alpha)\,\frac{\partial}{\partial\alpha}\,, \tag{9}$$

$$\nabla_x\mu=-[\mu\nabla_x B+v_\parallel\nabla_x\mathbf{e}_\parallel\cdot\mathbf{v}_\perp]/B\,, \tag{10}$$

$$\nabla_x\alpha=(\nabla_x\mathbf{e}_2)\cdot\mathbf{e}_1+(v_\parallel/v_\perp^2)\nabla_x\mathbf{e}_\parallel\cdot(\mathbf{v}_\perp\times\mathbf{e}_\parallel)\,, \tag{11}$$

$$\delta R=\delta\mathbf{a}\cdot\nabla_v=\delta\mathbf{a}\cdot\nabla_V+\delta\mathbf{a}\times\mathbf{e}_\parallel/\Omega\cdot\nabla_X\,, \tag{12}$$

$$\delta\mathbf{a}=\delta\mathbf{E}+\mathbf{v}\times\delta\mathbf{B}/c\,, \tag{13}$$

$\mathbf{E}_0=-\nabla_x\Phi_0$, $\mathbf{e}_\alpha=-\mathbf{e}_1\sin\alpha+\mathbf{e}_2\cos\alpha$, and $\mathbf{v}_E=c\mathbf{E}\times\mathbf{e}_\parallel/B$. Here, L_g contains $\partial\mathbf{V}/\partial t$ and $\partial\mathbf{X}/\partial t$ because the macroscopic quantities are in general time-dependent. Performing spatial averaging on Eq. (6), we obtain

$$L_g F=-(q/m)\langle\delta R\,\delta F\rangle_{\mathbf{X}_{\perp 0}}\,, \tag{14}$$

and

$$L_g\delta F=-(q/m)(\delta RF+\delta R\delta F-\langle\delta R\delta F\rangle_{\mathbf{X}_{\perp 0}})\,. \tag{15}$$

B. Ordering

To proceed further with Eqs. (14) and (15), we adopt the following ordering for the microscopic fluctuations:

$$\frac{|\partial/\partial t|}{|\Omega|}\sim|\rho\mathbf{e}_\parallel\cdot\nabla_X|\sim\left|\frac{\delta F}{F}\right|\sim\left|\frac{q\delta\Phi}{T}\right|\sim\left|\frac{\delta B}{B}\right|\sim\left|\frac{v_E}{v_t}\right|\sim O(\lambda)\,, \tag{16}$$

where v_t is the characteristic (thermal) velocity, and

$$|\rho\nabla_{\mathbf{X}_{\perp 0}}|\sim O(1)\,. \tag{17}$$

For macroscopic quantities, however, since they evolve on the transport (including that induced by turbulence) time scale, we take

$$\frac{|\partial/\partial t|}{|\Omega|}\sim O(\lambda^3)\,, \tag{18}$$

in addition to

$$|\rho\nabla_X|\sim O(\lambda)\,. \tag{19}$$

We remark that the ordering adopted here is consistent with experimental observations as well as most of the proposed phenomenological anomalous transport mechanisms.[6,7] Furthermore, since the nonlinear term, $\delta R\delta F$, is of $O(\lambda^2)$, i.e., comparable to the linear term, $L_g\delta F$, our ordering, in principle, contains strong-turbulence effects. Also, as will be shown later, in Eq. (47), the ordering of the transport time scale, Eq. (18), is consistent with that of the fluctuations, Eq. (16).

C. Solution of F

Using the small parameter λ, we have, with $F=F_0+F_1+\ldots$,[1,2]

$$F_0=F_0(\epsilon,\mu,\mathbf{X}_\perp)\,; \tag{20}$$

i.e., $\mathbf{e}_\parallel\cdot\nabla_X F_0=0$,

$$\tilde{F}_1=\frac{\tilde{\beta}}{B}\,\frac{\partial F_0}{\partial\mu}\,, \tag{21}$$

$$\tilde{\beta}=-\left[\mathbf{v}_\perp\cdot v_D+\int^\alpha\left(\frac{d\alpha'}{\Omega}\right)\left(v_\parallel(\mathbf{v}_\perp\cdot\nabla_x\mathbf{e}_\parallel\cdot\mathbf{v}_\perp)-\frac{v_\perp^2\nabla_x\cdot\mathbf{e}_\parallel}{2}\right)\right]\,, \tag{22}$$

$$\mathbf{v}_D=\mathbf{v}_d+\mathbf{v}_{E0}\,, \tag{23}$$

$$\mathbf{v}_d=\mathbf{e}_\parallel\times[(v_\perp^2/2)\nabla_x\ln B+v_\parallel^2\mathbf{e}_\parallel\cdot\nabla\mathbf{e}_\parallel]/\Omega\,, \tag{24}$$

and $\tilde{F}_1$ is the α-dependent part of F_1. To determine the α-independent part of F_1, one needs to go to $O(\lambda^2)$, where turbulence effects, $\langle\delta R\delta F\rangle_{\mathbf{X}_{\perp 0}}$, would also enter. In fact, following the procedures of neoclassical theory,[13] a formal transport theory including turbulence could be developed; this is, however, beyond the scope of the present work and will be left for future investigations. For the present purpose of obtaining a nonlinear gyrokinetic equation from Eq. (15), knowing F_0 and $\tilde{F}_1$ is sufficient.

D. Nonlinear gyrokinetic equations

We now concentrate on Eq. (15) for the fluctuating δF. Since only terms up to $O(\lambda^2)$ are of interest here, the macroscopic background can be treated as frozen and L_g becomes accurate to $O(\lambda)$,

$$L_g\simeq L_{gf}=\frac{\partial}{\partial t}+v_\parallel\mathbf{e}_\parallel\cdot\nabla_X+\mathbf{v}\cdot(\boldsymbol{\lambda}_{B1}+\boldsymbol{\lambda}_{B2})$$
$$+\frac{q}{m}\,\mathbf{E}_0\cdot\left(\frac{\mathbf{v}_\perp}{B}\,\frac{\partial}{\partial\mu}+\frac{\mathbf{e}_\alpha}{v_\perp}\,\frac{\partial}{\partial\alpha}\right)$$
$$+\mathbf{v}_{E0}\cdot\nabla_X-\Omega\,\frac{\partial}{\partial\alpha}\,. \tag{25}$$

Following the linear formalism,[3,4] we let

$$\delta F=(q/m)\delta F_a+\delta G\,, \tag{26}$$

where

$$\delta F_a=\left[\frac{\delta\Phi\,\partial}{\partial\epsilon}+\left(\delta\Phi-\frac{v_\parallel\delta A_\parallel}{c}\right)\frac{\partial}{B\,\partial\mu}\right]F_0\,, \tag{27}$$

in order to remove the $O(\lambda)$ term, $\delta R\hat{F}_0$, in Eq. (15). We note that we have adopted $\delta\Phi$ and $\delta\mathbf{A}$ as field variables. Thus, $\delta\mathbf{B}=\nabla_x\times\delta\mathbf{A}$ and $\delta\mathbf{E}=-(\nabla_x\delta\Phi+\partial\delta\mathbf{A}/c\partial t)$. Substituting Eqs. (25) and (26) into Eq. (15), we obtain

$$L_{gf}\delta G=-(q/m)(R_l+R_{nl})\,, \tag{28}$$

where R_l is the linear term[3,4] given by

$$R_l=R_{l1}+R_{l2}+R_{l3}+R_{l4}\,, \tag{29}$$

$$R_{l1}=\frac{\partial \delta L}{\partial t}\frac{\partial F_0}{\partial \epsilon}-\frac{\nabla_0 \delta L \times \mathbf{e}_\parallel}{\Omega}\cdot \nabla_X F_0\,, \tag{30}$$

$$\delta L \equiv \delta\Phi - \mathbf{v}\cdot\delta\mathbf{A}/c\,, \tag{31}$$

$$\nabla_0 \equiv \frac{\partial}{\partial \mathbf{X}_{\perp 0}}\,, \tag{32}$$

$$R_{l2}=\frac{\partial F_0}{B\partial\mu}\left(\frac{\partial}{\partial t}+v_\parallel \mathbf{e}_\parallel\cdot\nabla_x\right)\delta L\,, \tag{33}$$

$$R_{l3}=\Big[\delta\Phi(v_\parallel\mathbf{e}_\parallel+\mathbf{v}_D)\cdot\nabla_X-\delta\Phi\,\mathbf{v}_\perp\cdot\nabla_x\ln B - (\nabla_V\tilde{\beta})\cdot\nabla_X\delta\Phi+\Omega(\delta\Phi\tilde{\beta})'_\alpha\left(\frac{\partial}{\partial\epsilon}+\frac{\partial}{B\partial\mu}\right)\Big]\frac{\partial F_0}{B\partial\mu}\,, \tag{34}$$

$$R_{l4}=-\frac{1}{c}\Big(v_\parallel(\mathbf{v}\cdot\nabla_x\mathbf{e}_\parallel)\cdot\delta\mathbf{A}+\delta A_\parallel(\mathbf{v}\cdot\nabla\mathbf{e}_\parallel)\cdot\mathbf{v} + (q/m)\mathbf{e}_\parallel\cdot\mathbf{E}_0\delta A_\parallel - v_\parallel\delta A_\parallel\mathbf{v}\cdot\nabla_x\ln B -(\nabla_V\tilde{\beta})\cdot[\nabla_0(\mathbf{v}\cdot\delta\mathbf{A})-(\mathbf{v}\cdot\nabla_0)\delta\mathbf{A}] + \Omega(\tilde{\beta}v_\parallel\delta A_\parallel)'_\alpha\frac{\partial}{B\partial\mu}\Big)\frac{\partial F_0}{B\partial\mu}\,, \tag{35}$$

$$R_{nl}=\delta R_0\delta F-\langle\delta R_0\delta F\rangle_{\mathbf{X}_{\perp 0}}\,, \tag{36}$$

$$\delta\mathbf{R}_0=\delta\mathbf{a}_0\cdot\nabla_V+\delta\mathbf{a}_0\times\mathbf{e}_\parallel/\Omega\cdot\nabla_0\,, \tag{37}$$

$$\delta\mathbf{a}_0=-\nabla_0\delta\Phi+[\nabla_0(\mathbf{v}\cdot\delta\mathbf{A})-(\mathbf{v}_\perp\cdot\nabla_0)\delta\mathbf{A}]/c\,, \tag{38}$$

and $(a)'_\alpha \equiv \partial a/\partial\alpha$. Expanding $\delta G=\delta G_0+\delta G_1+\ldots$ and noting that the right-hand side of Eq. (28) is of $O(\lambda^2)$, we have, for $O(\lambda)$,

$$\partial\delta G_0/\partial\alpha=0\,. \tag{39}$$

Gyrophase averaging the $O(\lambda^2)$ equation then yields

$$\langle L_{gf}\rangle_\alpha\delta G_0=-(q/m)\langle R_l+R_{nl}\rangle_\alpha\,, \tag{40}$$

where

$$\langle L_{gf}\rangle_\alpha=\partial/\partial t+v_\parallel\mathbf{e}_\parallel\cdot\nabla_X+\mathbf{v}_D\cdot\nabla_0\,, \tag{41}$$

and $\langle\cdots\rangle_\alpha\equiv(1/2\pi)\int_0^{2\pi}d\alpha(\cdots)$. If we further let

$$\delta G_0=-(q/m)\langle\delta L\rangle_\alpha\frac{\partial F_0}{B\partial\mu}+\delta H_0\,, \tag{42}$$

Eq. (40) becomes

$$\langle L_{gf}\rangle_\alpha\delta H_0=(-q/m)(S_{l1}+S_{l2}+\langle R_{nl}\rangle_\alpha)\,, \tag{43}$$

where

$$S_{l1}=\frac{\partial\langle\delta L\rangle_\alpha}{\partial t}\frac{\partial F_0}{\partial\epsilon}-\nabla_0\langle\delta L\rangle_\alpha\times\frac{\mathbf{e}_\parallel}{\Omega}\cdot\nabla_X F_0\,, \tag{44}$$

$$S_{l2}=\frac{\partial F_0}{B\partial\mu}\{v_\parallel\langle\mathbf{e}_\parallel\cdot\nabla_x\delta L\rangle_\alpha-v_\parallel\mathbf{e}_\parallel\cdot\nabla_X\langle\delta L\rangle_\alpha - \mathbf{v}_D\cdot\nabla_X\langle\delta L\rangle_\alpha-\langle\mathbf{v}_\perp\delta\Phi\rangle_\alpha\cdot\nabla_x\ln B -\langle\nabla_V\tilde{\beta}\cdot\nabla_0\delta\Phi\rangle_\alpha-\langle\mathbf{v}\cdot\delta\mathbf{A}/c\rangle_\alpha v_\parallel\mathbf{e}_\parallel\cdot\nabla_x\ln B + v_\parallel\langle\mathbf{v}\delta A_\parallel/c\rangle_\alpha\cdot\nabla_x\ln B+\langle(\nabla_V\tilde{\beta})\cdot[\nabla_0(\mathbf{v}\cdot\delta\mathbf{A}) -(\mathbf{v}\cdot\nabla_0)\delta\mathbf{A}]/c\rangle_\alpha-v_\parallel\langle(\mathbf{v}\cdot\nabla_x\mathbf{e}_\parallel)\cdot\delta\mathbf{A}/c\rangle_\alpha -\langle\delta A_\parallel(\mathbf{v}\cdot\nabla_x)\mathbf{e}_\parallel\cdot\mathbf{v}/c\rangle_\alpha-(q/m)\mathbf{e}_\parallel\cdot\mathbf{E}_0\langle\delta A_\parallel/c\rangle_\alpha\}\,, \tag{45}$$

and, after some algebraic manipulations,

$$\langle R_{nl}\rangle_\alpha=\langle\delta R_0\delta F_0\rangle_\alpha=\langle\delta R_0[(q/m)\delta F_a+\delta G_0]\rangle_\alpha =-\nabla_0\langle\delta L\rangle_\alpha\times\mathbf{e}_\parallel/\Omega\cdot\nabla_0\delta H_0\,. \tag{46}$$

Here, we have noted that

$$\langle\langle\delta R_0\delta F_0\rangle_\alpha\rangle_{\mathbf{X}_{\perp 0}}=0\,. \tag{47}$$

Equation (47) indicates that the effect of turbulence on the α-independent part of F is of $O(\lambda^3)$ and, therefore, consistent with the ordering of the transport time scale, Eq. (18). Combining Eqs. (43) and (46) gives

$$\overline{L}_g\delta H_0=\left[\frac{\partial}{\partial t}+v_\parallel\mathbf{e}_\parallel\cdot\nabla_X+\left(\mathbf{v}_D+\frac{c}{B}\mathbf{e}_\parallel\times\nabla_0\langle\delta L\rangle_\alpha\right)\cdot\nabla_0\right]\delta H_0 =-(q/m)(S_{l1}+S_{l2})\,. \tag{48}$$

Equation (48) is the desired nonlinear gyrokinetic equation. Combining Maxwell's equations and δF, given by Eqs. (26), (42), and (48), the microscopic dynamics is then, in principle, completely determined.

Since the theoretical analyses employ two spatial scales, it is instructive to proceed further with the following WKB ansatz

$$\delta F(\mathbf{x},\mathbf{v})=\sum_{\mathbf{k}_\perp}\delta\overline{F}(\mathbf{x},\mathbf{v};\mathbf{k}_\perp)\exp\left(i\int^{\mathbf{x}_{\perp 0}}\mathbf{k}_\perp\cdot d\mathbf{x}_\perp\right) =\sum_{\mathbf{k}_\perp}\delta\overline{F}(\mathbf{X},\mathbf{V};\mathbf{k}_\perp)\exp\left(i\int^{\mathbf{X}_{\perp 0}}\mathbf{k}_\perp\cdot d\mathbf{X}_\perp-iL(\mathbf{k}_\perp)\right)\,, \tag{49}$$

where $L(\mathbf{k}_\perp)=\mathbf{k}_\perp\cdot\mathbf{v}\times\mathbf{e}_\parallel/\Omega$, and $\delta\overline{F}$ as well as $\mathbf{k}_\perp$ contain slow spatial variations. Equation (48) then reduces to

$$\left(\frac{\partial}{\partial t}+v_\parallel\mathbf{e}_\parallel\cdot\nabla_X+i\mathbf{k}_\perp\cdot\mathbf{v}_D\right)\delta\overline{H}_0(\mathbf{k}_\perp)+\frac{c}{B}\sum_{\mathbf{k}'_\perp,\mathbf{k}''_\perp}[\mathbf{e}_\parallel\cdot(\mathbf{k}''_\perp\times\mathbf{k}'_\perp)] \times\langle\delta\overline{L}\rangle_\alpha(\mathbf{k}'_\perp)\delta\overline{H}_0(\mathbf{k}''_\perp)\exp\left(i\int^{\mathbf{X}_{\perp 0}}(\mathbf{k}'_\perp+\mathbf{k}''_\perp-\mathbf{k}_\perp)\cdot d\mathbf{X}_\perp\right) =-(q/m)S_{l1}(\mathbf{k}_\perp)\,, \tag{50}$$

where

$$\langle\delta\overline{L}\rangle_\alpha(\mathbf{k}_\perp)=(\delta\overline{\Phi}-v_\parallel\delta A_\parallel/c)J_0(\gamma)+v_\perp J_1(\gamma)\delta\overline{b}_\parallel/k_\perp c\,, \tag{51}$$

$$S_{l1}(\mathbf{k}_\perp)=\left(\frac{\partial F_0}{\partial\epsilon}\frac{\partial}{\partial t}+i\mathbf{e}_\parallel\times\frac{\mathbf{k}_\perp}{\Omega}\cdot\nabla_X F_0\right)\langle\delta\overline{L}\rangle_\alpha(\mathbf{k}_\perp)\,, \tag{52}$$

$\gamma=k_\perp v_\perp/\Omega$, and $S_{l2}(\mathbf{k}_\perp)$ is of higher order and ignorable here. Equation (50) and, hence, Eq. (48) show that the nonlinearity arises from a gyrophase-averaged effective $\delta\mathbf{E}_{eff}\times\mathbf{B}\cdot\nabla_X$ coupling; here,

$$\delta\mathbf{E}_{\mathrm{eff}}=-\nabla_0(\delta\Phi-\mathbf{v}\cdot\delta\mathbf{A}/c)\,. \tag{53}$$

It is interesting to note that the nonlinear polarization drift is contained, not very obviously, within the finite Larmor radius corrections in J_0 and J_1. In fact, the electrostatic nonlinear drift-wave equation first obtained by Hasegawa and Mima[14] can be readily derived from Eq. (50) in the appropriate limits; i.e., adiabatic electrons and cold fluid ions (Sec. III). Equation (48) [or Eq. (50)], of course, is much more general and also is not easy to solve. In the next two sections, we consider more specific applications of the general results obtained so far.

III. AXISYMMETRIC TOKAMAKS

A. General formulation

Here, we consider the specific case of axisymmetric tokamaks and, using the ballooning-mode representation,[8-10,15] further explore the properties of the nonlinear gyrokinetic equation, Eq. (48). Thus, employing the ψ (poloidal flux), ξ (toroidal angle), and χ (poloidal angle-like) coordinates, we have

$$\mathbf{B}=\nabla_x\xi\times\nabla_x\psi+I(\psi,\chi)\nabla_x\xi\,, \tag{54}$$

$$F_0=F_0(\psi,\mu,\epsilon)\,, \tag{55}$$

$$\delta H_0=\sum_n\sum_m \delta h_{n,m}(\psi,\mathrm{v})\exp\left[i\left(n\xi-m\chi+n\int^{\psi}k\,d\psi\right)\right]\,, \tag{56}$$

$$\delta h_{n,m}(\psi,\mathrm{v})=\int_{-\infty}^{\infty}d\hat\theta_n\,\delta\bar h_n(\psi,\hat\theta_n,\mathrm{v})\,g_{n,m}(\hat\theta_n)\,, \tag{57}$$

$$g_{n,m}(\hat\theta_n)=\exp\left[i\left(m\hat\theta_n-n\int^{\theta_n}\nu\,d\hat\theta\right)\right]\,, \tag{58}$$

$\nu=IJ/R^2$, and $J=(\nabla_x\psi\times\nabla_x\xi\cdot\nabla_x\chi)^{-1}$ is the Jacobian. Equation (48) or, equivalently, Eq. (43) then becomes

$$\int_{-\infty}^{\infty}d\hat\theta_n\,g_{n,m}(\hat\theta_n)\left[\left(\frac{\partial}{\partial t}+\frac{v_\parallel}{JB}\frac{\partial}{\partial\hat\theta_n}+i\mathbf{k}_\perp\cdot\mathbf{v}_D\right)\delta\bar h_n+\frac{q}{m}\bar S_{ln}\right]=-\frac{q}{m}\langle R_{nl}\rangle_\alpha(m,n)\,, \tag{59}$$

where

$$\mathbf{k}_\perp=\frac{nB}{RB_\chi}(\mathbf{e}_\parallel\times\mathbf{e}_\psi)-nRB_\chi\left(\int_0^{\theta_n}\frac{\partial\nu}{\partial\psi}\,d\hat\theta-k\right)\mathbf{e}_\psi\,,$$

$$\bar S_{ln}=\left(\frac{\partial F_0}{\partial\epsilon}\frac{\partial}{\partial t}-i\frac{nB}{\Omega}\frac{\partial F_0}{\partial\psi}\right)\delta\bar L_n\,, \tag{60}$$

$$\delta\bar L_n=J_0(\gamma)(\delta\bar\Phi-v_\parallel\delta\bar A_\parallel/c)_n+J_1(\gamma)v_\perp\delta\bar b_{\parallel n}/k_\perp c\,, \tag{61}$$

$$\langle R_{nl}\rangle_\alpha(m,n)=-\frac{1}{2}\sum_{\substack{n'+n''\\=n}}\sum_{\substack{m'+m''\\=m}}\int_{-\infty}^{\infty}d\hat\theta_{n'}\int_{-\infty}^{\infty}d\hat\theta_{n''}\,g_{n',m'}\times g_{n'',m''}C_{n',n''}(\delta\bar L_{n'}\,\delta\bar h_{n''}-\delta\bar L_{n''}\,\delta\bar h_{n'})W_{n',n''}\,, \tag{62}$$

$$C_{n',n''}=\frac{\mathbf{e}_\parallel}{\Omega}\cdot(\mathbf{k}'_\perp\times\mathbf{k}''_\perp)=\frac{Bn'n''}{\Omega}\left(\int_{\theta_{n'}}^{\theta_{n''}}\frac{\partial\nu}{\partial\psi}\,d\hat\theta+k'-k''\right)\,, \tag{63}$$

$$W_{n',n''}=\exp\left(i\int^{\psi}(n'k'+n''k''-nk)d\psi\right)\,, \tag{64}$$

and, again, $\gamma=k_\perp v_\perp/\Omega$. After some manipulations, Eq. (62) can be shown to be

$$\langle R_{nl}\rangle_\alpha(m,n)=\int_{-\infty}^{\infty}d\hat\theta_n\,g_{n,m}(\hat\theta_n)(\bar R_{nl})_n\,, \tag{65}$$

where

$$(\bar R_{nl})_n=-\pi\sum_{\substack{n'+n''\\=n}}W_{n',n''}\sum_p\exp(-in''2\pi pQ)\times\tilde C_{n',n''}\int_{-\infty}^{\infty}d\hat\theta_{n'}\,\delta(\hat\theta_n-\hat\theta_{n'})\int_{-\infty}^{\infty}d\hat\theta_{n''}\,\delta(\hat\theta_{n''}-\hat\theta_n-2\pi p)\times(\delta\bar L_{n'}\delta\bar h_{n''}-\delta\bar L_{n''}\delta\bar h_{n'})\,, \tag{66}$$

$$Q\equiv(1/2\pi)\int_0^{2\pi}\nu\,d\hat\theta\,, \tag{67}$$

and

$$\tilde C_{n',n''}=\frac{Bn'n''}{\Omega}\left(2\pi p\frac{\partial Q}{\partial\psi}+k'-k''\right)\,. \tag{68}$$

Combining Eqs. (59) and (65), we finally derive the nonlinear gyrokinetic equation for axisymmetric tokamaks

$$\left(\frac{\partial}{\partial t}+\frac{v_\parallel}{JB}\frac{\partial}{\partial\hat\theta_n}+i\mathbf{k}_\perp\cdot\mathbf{v}_D\right)\delta\bar h_n=-\frac{q}{m}[\bar S_{ln}+(\bar R_{nl})_n]\,. \tag{69}$$

We now make some qualitative remarks. For simplicity, we shall ignore global amplitude modulations by letting $k=k'=k''=0$ and, hence, $\tilde C_{n',n''}\propto p$. For flute-like modes, which are highly localized about the mode-rational surfaces, $\exp(-in''2\pi pQ)\simeq 1$ and $\delta\bar h$ as well as $\delta\bar L$ are weakly dependent on $\hat\theta$. Therefore, we have, roughly, $(\bar R_{nl})_n\propto\sum_p p=0$; i.e., nonlinear coupling is much reduced, which may be expected because there is little overlap between the modes in the ψ (radial) coordinate. On the other hand, for modes which are strongly ballooning, and hence, $(\delta\bar\Phi,\delta\bar L)(\hat\theta+p2\pi)\simeq 0$ for $p\neq 0$; the nonlinear coupling, again, is small. This is because, in this limit, radial structures are rather broad and, hence, the $\delta\mathbf{E}\times\mathbf{B}\cdot\nabla_X$ coupling is rendered ineffective. Thus, qualitatively speaking, Eq. (66) appears to indicate that the most effective nonlinear coupling occurs among modes which are moderately ballooning.

B. Electrostatic drift waves with adiabatic electrons and cold fluid ions

For simplicity, we assume F_0 to be a local Maxwellian; i.e.,

$$F_0=F_M(\psi,\epsilon)\,, \tag{70}$$

and neglect any equilibrium electric potential Φ_0. Since electrons are adiabatic, their nonlinear contributions are negligible and the quasineutrality condition becomes

$$(1+\tau)\delta\psi_n=2\pi\int B\,d\mu\int dv_\parallel\,\delta\bar h_{in}J_0(\gamma_i)\equiv\delta\bar n_{in}\,, \tag{71}$$

where, $\delta\psi_n\equiv e\delta\bar\Phi_n/T_e$, $\tau=T_e/T_i\gg 1$, and $J_0(\gamma_i)\simeq 1-\gamma_i^2/4$. Meanwhile, neglecting the ion-sound term, we multiply the ion equation (69) by $J_0(\gamma_i)$ and carry out the velocity integration to obtain

$$\left(\frac{\partial}{\partial t}+i\mathbf{k}_\perp\cdot\bar{\mathbf{v}}_{di}\right)\delta\bar n_{in}=\tau(1-b_i)\left(\frac{\partial}{\partial t}-i\omega_{*in}\right)\delta\psi_n-\frac{e}{m_i}2\pi\int B\,d\mu\int dv_\parallel J_0(\gamma_i)(\bar R_{nl})_{in}\,, \tag{72}$$

where, for $j=e,i$,

$$\bar{\mathbf{v}}_{dj}=\mathbf{e}_\parallel\times(v_j^2/2)(\nabla_x\ln B+\mathbf{e}_\parallel\cdot\nabla\mathbf{e}_\parallel)/\Omega_j\,, \tag{73}$$

$$\omega_{*j}=\frac{v_j^2nB}{2\Omega_j}\left|\frac{\partial\ln N_{0j}}{\partial\psi}\right|\,, \tag{74}$$

and $b_i=k_\perp^2\rho_i^2/2$. In deriving Eq. (72), we have noted that in the zeroth-order approximation, $\delta\bar h_{in}\simeq\delta\bar n_{in}F_{Mi}$. Expanding the J_0's in $(\bar R_{nl})_{in}$ and using the quasineutrality condition, Eq. (71), it is then straightforward to derive

$$\left((1+k_\perp^2\rho_s^2)\frac{\partial}{\partial t}-i\mathbf{k}_\perp\cdot\bar{\mathbf{v}}_{de}+i\omega_{*e}\right)\delta\psi_n=\pi C_s^2\sum_{\substack{n'+n''\\=n}}W_{n',n''}\sum_p\exp(-in''2\pi pQ)\tilde C_{n',n''}$$

$$\times \int_{-\infty}^{\infty} d\hat{\theta}_{n'}\, \delta(\hat{\theta}_n - \hat{\theta}_{n'}) \int_{-\infty}^{\infty} d\hat{\theta}_{n''}\, \delta(\hat{\theta}_{n''} - \hat{\theta}_n - 2\pi p)$$

$$\times \rho_s^2 [(k''_\perp)^2 - (k'_\perp)^2] \delta\psi_{n'}\, \delta\psi_{n''}\,, \tag{75}$$

where $C_s^2 = \tau v_i^2/2$ and $\rho_s = C_s/\Omega_i$. Equation (75) may be regarded as the tokamak version of the nonlinear drift-wave equation of Hasegawa and Mima.[14] It contains the poloidal mode-coupling effects due to the $\nabla_x B$ and curvature drifts and employs the ballooning-mode representation. The assumption of adiabatic electrons and cold fluid ions thus results in a single nonlinear equation instead of coupled ones, as in more general cases. To make Eq. (75) more tractable, we further assume concentric, circular magnetic surfaces and ignore the global wavenumber (k). Equation (75) then reduces to

$$\left((1 + \tilde{k}_\perp^2 \rho_s^2) \frac{\partial}{\partial t} - i\bar{\omega}_{de} T_c(\hat{\theta}) + i\omega_{*e}\right) \delta\psi(n, \hat{\theta})$$

$$= \pi \Omega_i \hat{S} \sum_{\substack{n'+n'' \\ =n}} \sum_p \exp(-in'' 2\pi p Q) \rho_s^4 (k_{n'} k_{n''} 2\pi p)$$

$$\times [(\tilde{k}''_\perp)^2 - (\tilde{k}'_\perp)^2] \delta\psi(n', \hat{\theta}) \delta\psi(n'', \hat{\theta} + 2\pi p)\,, \tag{76}$$

where $\omega_{*e} = k_n \rho_s C_s / r_n$, $r_n^{-1} = |d \ln N/dr|$, $k_n = nQ/r$, $\bar{\omega}_{de} = 2 r_n \omega_{*e}/R$, $\hat{S} = r(\partial Q/\partial r)/Q$,

$$T_c = \cos\hat{\theta} + \hat{S}\hat{\theta}\sin\hat{\theta}\,, \tag{77}$$

$$\tilde{k}_\perp^2 = k_n^2 (1 + \hat{S}^2 \hat{\theta}^2)\,, \tag{78}$$

$$(\tilde{k}'_\perp)^2 = k_{n'}^2 (1 + \hat{S}^2 \hat{\theta}^2)\,, \tag{79}$$

and

$$(\tilde{k}''_\perp)^2 = k_{n''}^2 [1 + \hat{S}^2 (\hat{\theta} + 2\pi p)^2]\,. \tag{80}$$

We note that Eq. (76) is a two-dimensional $(n, \hat{\theta})$ nonlinear equation and may be viewed as the simplest model equation for electrostatic drift waves in axisymmetric tokamaks. Detailed studies of Eq. (76) will be reported in a future publication.

IV. NONLINEAR ION LANDAU DAMPING OF KINETIC ALFVÉN WAVES

As noted in Sec. I, the results obtained in Sec. II are rather general and, therefore, are also useful for describing nonlinear processes associated with low-frequency wave heating. To illustrate this potential application, we consider nonlinear ion Landau damping of the mode-converted kinetic (shear) Alfvén waves.[11] The major difference between our work and Ref. 16 is that we include the diamagnetic drift effects. Another minor difference is that no small ion-Larmor-radius expansion is taken here.

For the sake of simplicity, we adopt the WKB description, Eqs. (49) and (50), and ignore the wavenumber mismatch and geometrical effects. Furthermore, F_0 is taken to be Maxwellian and, for the present purpose, we can assume a weak-turbulence subsidiary ordering as well as ignore the compressional Alfvén perturbations $\delta A_\perp = 0$ [i.e., $\beta \equiv$ (plasma pressure/magnetic pressure) $\ll 1$]. The relevant equations, thus, are

$$\delta\bar{F}(\mathbf{k}) = -(q/T)\delta\bar{\Phi}(\mathbf{k}) F_0 + \delta\bar{H}_0(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{v}\times\mathbf{e}_\parallel/\Omega)\,, \tag{81}$$

and

$$(\omega_{\mathbf{k}} - k_\parallel v_\parallel)\delta H_0(\mathbf{k}) = (q/T)(\omega + \omega_*)_{\mathbf{k}} F_0 \delta\bar{L}(\mathbf{k}) J_0(\gamma)$$

$$- i(c/B) \sum_{\substack{\mathbf{k}'+\mathbf{k}'' \\ =\mathbf{k}}} [\delta\bar{L}(\mathbf{k}') J_0(\gamma') \delta\bar{H}_0(\mathbf{k}'')$$

$$\times (\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_\parallel]\,, \tag{82}$$

where $\delta\bar{L}(\mathbf{k}) = \delta\bar{\Phi}(\mathbf{k}) - v_\parallel \delta\bar{A}_\parallel(\mathbf{k})/c$, $\omega_{*\mathbf{k}} = k_y cT/eBL_n$, $L_n^{-1} = |d\ln N_0/dx|$, x is the nonuniformity direction, and temperature gradients are ignored. Expanding $\delta\bar{H}_0 = \delta H_0^{(1)} + \delta H_0^{(2)} + \delta H_0^{(3)} + \cdots$, we have, in the linear order,

$$\delta H_0^{(1)}(\mathbf{k}) = \frac{q}{T}\, \frac{(\omega + \omega_*)_{\mathbf{k}}}{\omega_{\mathbf{k}} - k_\parallel v_\parallel}\, F_0 \delta\bar{L}(\mathbf{k}) J_0(\gamma)\,. \tag{83}$$

To calculate $\delta H_0^{(2),(3)}$, we let $(\omega_{\mathbf{k}}, \mathbf{k})$ and $(\omega_{\mathbf{k}'}, \mathbf{k}')$ be the normal modes; while $(\omega_{\mathbf{q}} = \omega_{\mathbf{k}} - \omega_{\mathbf{k}'}, \mathbf{q} = \mathbf{k} - \mathbf{k}')$ is the virtual or quasimode. Also, we have in mind

$$|\omega_{\mathbf{q}}| \sim |q_\parallel v_i| \sim |k_\parallel, k'_\parallel| v_i \ll |\omega_{*q}| \sim |\omega_{\mathbf{k},\mathbf{k}'}| \ll |k_\parallel, k'_\parallel| v_e\,. \tag{84}$$

Let us now calculate the q-mode response. For the ions, we find, noting Eq. (84), that

$$\delta H_{0i}^{(2)}(\mathbf{q}) = [\omega_{*\mathbf{q}e}/(\omega_{\mathbf{q}} - q_\parallel v_\parallel)] \delta\tilde{P}_{\mathbf{q}} \delta\tilde{\Phi}(\mathbf{k}) \delta\tilde{\Phi}(\mathbf{k}') J_0(\gamma) J_0(\gamma') F_{0i}\,, \tag{85}$$

where

$$\delta\tilde{P}_{\mathbf{q}} = i(\Omega_i/\omega_{\mathbf{k}'}) \rho_s^2 [(\mathbf{k}\times\mathbf{k}')\cdot\mathbf{e}_\parallel]\,, \tag{86}$$

and $\delta\tilde{\Phi} \equiv e\delta\bar{\Phi}/T_e$. In deriving Eq. (85), we have observed that $|\delta\bar{\Phi}| \sim |\omega\delta\bar{A}_\parallel/ck_\parallel|$ for the $\mathbf{k}$ and $\mathbf{k}'$ modes. As to electron nonlinearity, it is $O(|\omega/\omega_{*e}|_{\mathbf{q}})$ smaller and, hence, negligible. Furthermore, for $\beta \ll 1$, we have $|\omega_{\mathbf{q}}| \ll |q_\parallel V_A|$ and the q-mode response is predominately electrostatic. Here, V_A is the Alfvén speed. The quasineutrality condition then yields

$$\delta\tilde{\Phi}_{\mathbf{q}}^{(1)} = -(F_1/\Gamma_{0\mathbf{q}}) \delta\tilde{P}_{\mathbf{q}} \delta\tilde{\Phi}(\mathbf{k}) \delta\tilde{\Phi}(\mathbf{k}')\,, \tag{87}$$

where

$$F_1(\mathbf{k}, \mathbf{k}') = 2\pi \int_0^\infty v_\perp dv_\perp J_0(\gamma) J_0(\gamma') J_0(\gamma_{\mathbf{q}}) F_{Mi}(v_\perp)$$

$$\equiv \langle J_0(\gamma) J_0(\gamma') J_0(\gamma_{\mathbf{q}}) \rangle_\perp\,, \tag{88}$$

$$\Gamma_{0\mathbf{q}} = I_0(b_{\mathbf{q}}) \exp(-b_{\mathbf{q}})\,, \tag{89}$$

$\gamma_{\mathbf{q}} = q_\perp v_\perp/\Omega_i$ and $b_{\mathbf{q}} = q_\perp^2 \rho_i^2/2$. Equations (83) and (87) then give $\delta H_{0i}^{(1)}(\mathbf{q})$.

For the k mode, the ion nonlinear response can be shown to be

$$\delta H_{0i}^{(3)}(\mathbf{k}) = |\delta\tilde{P}_{\mathbf{q}}|^2 |\delta\tilde{\Phi}(\mathbf{k}')|^2 \delta\tilde{\Phi}(\mathbf{k}) F_{0i} \left(\frac{\omega_{*e}}{\omega - q_\parallel v_\parallel}\right)_{\mathbf{q}}$$

$$\times [J_0^2(\gamma') J_0(\gamma) - F_1 J_0(\gamma_{\mathbf{q}}) J_0(\gamma')/\Gamma_{0\mathbf{q}}]\,. \tag{90}$$

From Eq. (90), we can readily calculate $\delta n_i^{(3)}(\mathbf{k})$. Meanwhile, the electron nonlinearity, again, is found to be $O(|\omega/\omega_{*e}|_{\mathbf{q}})$ smaller. Substituting $\delta n_i^{(3)}(\mathbf{k})$ and the linear response into the quasineutrality condition and parallel Ampere's law, it is straightforward to derive the following nonlinear dispersion relation:

$$D_i(\mathbf{k}) = -\frac{\omega_{*eq}}{|q_\parallel| v_i} Z_{iq}\left[1-\left(1-\frac{\omega_{*e}}{\omega}\right)_{\mathbf{k}}\left(\frac{\omega}{k_\parallel V_A k_\perp \rho_s}\right)^2_{\mathbf{k}}\right] \times |\delta\bar{P}_q \delta\bar{\Phi}(\mathbf{k}')|^2 G(\mathbf{k},\mathbf{k}'), \quad (91)$$

where $D_l(\mathbf{k})$ is the linear dispersion relation given by

$$D_l(\mathbf{k}) = (\tau + \omega_{*e}/\omega)_{\mathbf{k}}(1-\Gamma_{0\mathbf{k}}) + (1-\omega_{*e}/\omega)_{\mathbf{k}} \cdot [1-\omega^2_{\mathbf{k}}(\tau+\omega_{*e}/\omega)_{\mathbf{k}}(1-\Gamma_{0\mathbf{k}})/(k_\parallel V_A k_\perp \rho_s)^2], \quad (92)$$

$$G(\mathbf{k},\mathbf{k}') = \langle J_0^2(\gamma) J_0^2(\gamma')\rangle_\perp - F_1^2/\Gamma_{0q}, \quad (93)$$

$Z_{iq} \equiv Z(\omega_q/|q_\parallel| v_i)$ with Z being the standard plasma dispersion function. In the low-β limit of interest here, $|\omega_{*e}/k_\parallel V_A|_{\mathbf{k}} < 1$, and assuming $\omega_{\mathbf{k}} = \omega_{r\mathbf{k}} + i\omega_{i\mathbf{k}}$ with $|\omega_i/\omega_r| \ll 1$, Eq. (92) gives the known result that there exist two branches of waves; the drift waves with

$$\omega_{r\mathbf{k},d} \simeq \omega_{*e\mathbf{k}}\Gamma_{0\mathbf{k}}/[1+\tau(1-\Gamma_{0\mathbf{k}})], \quad (94)$$

and the kinetic shear Alfvén waves with

$$\omega^2_{r\mathbf{k},A} \simeq (k_\parallel V_A)^2[1+\tau(1-\Gamma_{0\mathbf{k}})]k_\perp^2\rho_s^2/\tau(1-\Gamma_{0\mathbf{k}}). \quad (95)$$

We note that nonlinear ion Landau damping of the drift waves, as described by Eq. (91), has previously been investigated.[17] Concentrating on the kinetic shear-Alfvén waves, we find

$$\left(\frac{\omega_i}{\omega_r}\right)_{\mathbf{k},A} = \frac{(\omega_{*\mathbf{k}'} - \omega_{*\mathbf{k}})_e}{2\tau(1-\Gamma_{0\mathbf{k}})[1+\tau(1-\Gamma_{0\mathbf{k}})]} \times G(\mathbf{k},\mathbf{k}')(\mathrm{Im}\, Z_{iq}/|q_\parallel v_i|)|\delta\bar{\Phi}(\mathbf{k}')\delta\bar{P}_q|^2. \quad (96)$$

Comparing the ω_i given by Eq. (96) with that of Ref. 16 (which, we recall, does not include the ω_* effects), we find the present parametric-decay growth rate is larger by a factor of $O(|\omega_{*iq}|/|q_\parallel v_i|) > 1$. Furthermore, noting that $\mathrm{Im}\, Z_{iq} \simeq |q_\parallel| v_i \pi\delta(\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})$, $G(\mathbf{k},\mathbf{k}') \geq 0$, and $\omega_i \propto (\omega_{*\mathbf{k}'} - \omega_{*\mathbf{k}}) \propto (k_y' - k_y)$; hence, the daughter wave, $(\omega_{\mathbf{k}}, \mathbf{k})$, has a smaller k_y but the same ω as the pump wave $(\omega_{\mathbf{k}'}, \mathbf{k}')$. This is qualitatively different from the results of Ref. 16, where $\omega_i \propto (\omega_{\mathbf{k}'} - \omega_{\mathbf{k}})$. In fact, as might be expected, one can show that the results of Ref. 16 are valid for $|\omega_{*iq}| < |q_\parallel v_i|$; i.e., in the opposite limit.

Finally, we briefly discuss the implication of the present results on the shear Alfvén wave heating scheme. Since the pump wave (i.e., the mode-converted kinetic shear Alfvén wave) has $k_x'\rho_i \sim O(1)$ and $k_y'\rho_i \sim O(\rho_i/a) \ll 1$ with a being the tokamak minor radius, we have $\omega_i \propto |\mathbf{k}\times\mathbf{k}' \cdot \mathbf{e}_\parallel|^2 \simeq (k_x' k_y)^2$ and, hence, nonlinear coupling becomes appreciable for $k_y\rho_i \sim O(1) \gg k_y'\rho_i$; that is, $\omega_i < 0$. Thus, our results suggest that parametric decay through nonlinear ion Landau damping will in general be unlikely to occur.

V. CONCLUSIONS AND DISCUSSIONS

In this work, a systematic formalism for the nonlinear interactions of microscopic low-frequency electromagnetic waves has been developed. This formalism extends the linear gyrokinetic formalism of Refs. 1 and 2 into the nonlinear regime. The corresponding nonlinear gyrokinetic equations, valid for general magnetic field configurations as well as the strong turbulence regime, are derived. Effects due to fully electromagnetic perturbations, finite Larmor radii, plasma inhomogeneities, magnetic drifts, and magnetic trapping are retained. The results are, thus, rather general and should have wide application. Note that it is straightforward to extend the treatment to include collisional effects by retaining the Fokker-Planck collision operator in Eqs. (7) and (15). As a specific example of possible applications, we consider axisymmetric tokamaks and explore the properties of the nonlinear gyrokinetic equations in more detail via the ballooning-mode representation. Furthermore, a single nonlinear equation is derived for electrostatic drift waves in the limit of adiabatic electrons and cold fluid ions which retains the crucial features of toroidal geometry and nonlinear coupling and, therefore, may be exploited as a useful model equation. On the other hand, we have also applied the results to the shear Alfvén wave heating scheme and considered the parametric decay of the mode-converted kinetic Alfvén waves via nonlinear ion Landau damping (ion-induced scattering). Here, it is found that the diamagnetic drift effects not only enhance the parametric growth rate but also modify the decay process qualitatively. That is, the daughter waves tend to have smaller perpendicular (to B and the density gradient) wavenumbers (i.e., smaller poloidal mode numbers for tokamak plasmas) instead of frequencies as suggested by uniform plasma calculations. This property, therefore, suggests that the mode-converted kinetic Alfvén waves with small poloidal mode numbers will not, in general, parametrically decay via nonlinear ion Landau damping. Other possible channels such as resonant decay to drift waves, of course, are not ruled out and need to be investigated.

Let us comment on some other possible applications of the general results obtained here. One possible application is the following: by taking the limit of adiabatic electrons and fluid ions, a sufficiently simple nonlinear equation may also be derived for the kinetic shear-Alfvén waves, and, hence, could serve as a model equation for studying the nonlinear evolution and associated transport of kinetic ballooning-mode instabilities.[18] Another interesting application is to simulate plasmas employing the nonlinear gyrokinetic equation, Eq. (48) or, more transparently, Eq. (50). This application will further extend the gyrokinetic simulation scheme initially proposed by Lee[19] for electrostatic waves in simple (slab) geometries to fully electromagnetic perturbations in general plasma equilibria. The details of these applications, however, remain to be worked out.

ACKNOWLEDGMENT

This work was supported by United States Department of Energy Contract No. DE-AC02-76-CH03073.

[1]P. H. Rutherford and E. A. Frieman, Phys. Fluids 11, 569 (1968).
[2]J. B. Taylor and R. J. Hastie, Plasma Phys. 10, 479 (1968).
[3]T. M. Antonsen and B. Lane, Phys. Fluids 23, 1205 (1980).
[4]P. J. Catto, W. M. Tang, and D. E. Baldwin, Plasma Phys. 23, 639 (1981).

[5]E. A. Frieman and L. Chen, Bull. Am. Phys. Soc. **24**, 1009 (1979). A similar approach has been adopted by P. H. Diamond and M. N. Rosenbluth, Phys. Fluids (to be published).
[6]E. Mazzucato, Phys. Rev. Lett. **36**, 792 (1976).
[7]C. M. Surko and R. E. Slusher, Phys. Rev. Lett. **37**, 1747 (1976).
[8]J. W. Connor, R. J. Hastie, and J. B. Taylor, Proc. R. Soc. Lond. Ser. **A365**, 1 (1979).
[9]Y. C. Lee and J. W. van Dam, in *Proceedings of the Finite-Beta Theory Workshop*, Varenna, Italy, edited by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, Virginia, 1979), p. 93.
[10]A. H. Glasser, in *Proceedings of the Finite-Beta Theory Workshop*, Varenna, Italy, edited by B. Coppi and W. Sadowski (National Technical Information Service, Springfield, Virginia, 1979), p. 55; also R. L. Dewar, M. S. Chance, A. H. Glasser, J. M. Greene, and E. A. Frieman, Princeton Plasma Physics Laboratory Report PPPL-1587 (1979).
[11]A. Hasegawa and L. Chen, Phys. Rev. Lett. **35**, 370 (1975).
[12]P. J. Catto, Plasma Phys. **20**, 719 (1978).
[13]F. L. Hinton and R. D. Hazeltine, Rev. Mod. Phys. **48**, 239 (1976).
[14]A. Hasegawa and K. Mima, Phys. Rev. Lett. **39**, 205 (1977); also, A. Hasegawa, C. G. Maclennan, and Y. Kodama, Phys. Fluids **22**, 2122 (1979).
[15]J. W. Connor, R. J. Hastie, and J. B. Taylor, Plasma Phys. **22**, 757 (1980); E. A. Frieman, G. W. Rewoldt, W. M. Tang; and A. H. Glasser, Phys. Fluids **23**, 1750 (1980); F. Pegoraro and J. J. Schep, *ibid*., **24**, 478 (1981).
[16]A. Hasegawa and L. Chen, Phys. Rev. Lett. **36**, 1362 (1976); also Phys. Fluids **19**, 1924 (1976).
[17]R. Z. Sagdeev and A. A. Galeev, *Nonlinear Plasma Theory* (Benjamin, New York, 1969), p. 103.
[18]W. M. Tang, J. W. Connor, and R. J. Hastie, Nucl. Fusion **20**, 1439 (1980).
[19]W. W. Lee, Bull. Am. Phys. Soc. **25**, 966 (1980).