Formulas:
Time dilation; Length contraction: $\Delta t = \gamma \Delta t' = \gamma \Delta t_p$; $L = L_p / \gamma$; $c = 3 \times 10^8 \text{m/s}$

Lorentz transformation: $x' = \gamma(x - vt)$; $y' = y$; $z' = z$; $t' = \gamma(t - vx/c^2)$; inverse: $v \rightarrow -v$

Spacetime interval: $(ds)^2 = (c dt)^2 - (dx^2 + dy^2 + dz^2)$

Velocity transformation: $u_x = \frac{u_x - v}{1 - u_x v/c^2}$; $u_y = \frac{u_y}{\gamma(1 - u_x v/c^2)}$; inverse: $v \rightarrow -v$

Relativistic Doppler shift: $f_{\text{obs}} = f_{\text{source}} \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}}$ (approaching)

Momentum: $p = \gamma m \bar{u}$; Energy: $E = \gamma mc^2$; Kinetic energy: $K = (\gamma - 1)mc^2$

Rest energy: $E_0 = mc^2$; $E = \sqrt{p^2c^2 + m^2c^4}$

Electron: $m_e = 0.511 \text{MeV}/c^2$; Proton: $m_p = 938.26 \text{MeV}/c^2$; Neutron: $m_n = 939.55 \text{MeV}/c^2$

Atomic mass unit: $1\ u = 931.5 \text{MeV}/c^2$; electron volt: $1\ eV = 1.6 \times 10^{-19} \text{J}$

Stefan’s law: $e_{\text{tot}} = \sigma T^4$, $e_{\text{tot}} = \text{power/unit area}$; $\sigma = 5.67 \times 10^{-8} \text{W/m}^2\text{K}^4$

$e_{\text{tot}} = c U/4$, $U = \text{energy density} = \int_0^\infty u(\lambda, T) d\lambda$; Wien’s law: $\lambda_{\text{max}} = \frac{hc}{kT} = \frac{3.68 \times 10^{-13} \text{m}}{T}$

Boltzmann distribution: $P(E) = C e^{-E/kT}$

Planck’s law: $u_\alpha(\lambda, T) = N_\alpha(\lambda) \times \overline{E}_\alpha(\lambda, T) = \frac{8\pi \lambda^4}{hc^3} \times e^{hc/\lambda kT} - 1$; $N(f) = \frac{8\pi f^2}{c^3}$

Photons: $E = hf = pc$; $f = c/\lambda$; $hc = 12,400 \text{eV} \cdot \text{A}$; $k_B = (1/11,600) \text{eV} / K$

Photoelectric effect: $eV_\alpha = K_{\text{max}} = hf - \phi$, $\phi = m \text{work function}$; Bragg equation: $n\lambda = 2d\sin \theta$

Compton scattering: $\lambda' - \lambda = \frac{h}{m \gamma v} (1 - \cos \theta)$; $m = 0.0243 \text{A}$; Coulomb constant: $ke^2 = 14.4 \text{eV} \cdot \text{A}$

Force in electric and magnetic fields (Lorentz force): $F = qE + qv \times B$; Drag force: $D = 6\pi \eta v$

Rutherford scattering: $\Delta n = \frac{C}{\sin^4(\phi/2)}$; $hc = 1973 \text{eV} \cdot \text{A}$

Hydrogen spectrum: $\frac{1}{\lambda_{\text{min}}} = R \left( \frac{1}{m^2 - \frac{1}{n^2}} \right)$; $R = 1.097 \times 10^7 \text{m}^{-1} = \frac{1}{911.3 \text{A}}$

Electrostatic force, energy: $F = \frac{kq_1q_2}{r^2}$; $U = \frac{kq_1q_2}{r}$. Centripetal force: $F_c = \frac{mv^2}{r}$

Bohr atom: $E_n = -\frac{ke^2Z}{2r_n} = -\frac{Z^2E_0}{n^2}$; $E_0 = \frac{ke^2}{2a_0} = 13.6eV$; $K = \frac{mv^2}{2}$; $U = -\frac{ke^2Z}{r}$

$hf = E_i - E_f$; $r_n = r_n h^2$; $r_0 = \frac{a_0}{Z}$; $a_0 = \frac{h^2}{m \cdot ke^2} = 0.529 \text{A}$; $L = m \cdot vr = nh$ angular momentum

de Broglie: $\lambda = \frac{h}{p}$; $f = \frac{E}{h}$; $\omega = 2\pi f$; $k = \frac{2\pi}{\lambda}$; $E = h\omega$; $p = hK$; $E = \frac{p^2}{2m}$

Wave packets: $y(x,t) = \sum a_j \cos(k_j x - \omega_j t)$, or $y(x,t) = \int dk a(k) e^{ikx - \omega(t)}$, $\Delta k \Delta x \sim 1$; $\Delta \omega \Delta t \sim 1$

group and phase velocity: $v_x = \frac{d\omega}{dk}$; $v_p = \frac{\omega}{k}$; Heisenberg: $\Delta x \Delta p \sim \hbar$; $\Delta t \Delta E \sim \hbar$

Schrodinger equation: $-\frac{h^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + U(x) \Psi(x,t) = i\hbar \frac{\partial \Psi}{\partial t}$; $\Psi(x,t) = \psi(x)e^{it\hbar}$
Time – independent Schrödinger equation: \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + U(x) \psi(x) = E \psi(x) \]

∞ square well: \[ \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) ; \quad E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2} ; \quad \hbar = 3.81 eV \cdot \text{A} \quad (\text{electron}) \]

Harmonic oscillator: \[ \Psi_n(x) = H_n(x) e^{-\frac{m\omega x^2}{2\hbar}} ; \quad E_n = (n + \frac{1}{2}) \hbar \omega ; \quad E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 = \frac{1}{2} m \omega^2 A^2 ; \quad \Delta n = \pm 1 \]

Justify all your answers to all problems

Problem 1 (10 points)
A free electron (not subject to any forces) is described by the wavepacket \[ \psi(x,t) = \int dk \ a(k) e^{i(kx-\omega(k)t)} \]
with \( a(k) \) given in the figure, with \( k_1 = 299 \text{A}^{-1}, k_2 = 301 \text{A}^{-1} \).

(a) Estimate the uncertainty in the position of this electron, in A.
(b) Find the speed of this electron, approximately.
(c) Calculate \( \psi(x,t=0) \) and make a graph of it versus \( x \).

Problem 2 (10 points)
A proton and a neutron attract each other through the strong nuclear force and form a bound state, called the deuteron. Assume the attractive potential can be described by the expression \[ U(x) = C \frac{x^2}{a^2} \]
as a function of the distance \( x \) between the proton and the neutron, where \( C \) and \( a \) are constants. Assume also the size of the deuteron is \( a \), i.e. \( a \) is the (average) distance between the proton and neutron in the deuteron.

(a) The size of the deuteron is experimentally found to be approximately 1 F (1F=10^{-5} \text{A}), so \( a=10^{-5} \text{A} \) in the above formula. Estimate the value of the constant \( C \).
(b) If instead \( a \) had the value 0.529 A, meaning the size of the deuteron would be the same as the size of a hydrogen atom, what would the value of the constant \( C \) in the above formula be?

Hint: you may assume one of the nucleons is at rest, and minimize the energy of the other nucleon.

Problem 3 (10 points)
An electron is in a box of length 6A in the lowest energy state (ground state).

(a) Find its energy.
(b) How much more likely is this electron to be found at the center of the box versus at distance 1A from one (any one) of the walls?
(c) What is the largest wavelength photon that this electron can absorb in making a transition to another state? Give your answer in A. (Angstroms)

Justify all your answers to all problems