

Fluctuations in Plasma and the Test Particle Model

Contents

- i) Basic Ideas of Fluctuation and Dissipation in Low Collisionality Plasma
- ii) Test Particle Model of Thermal Equilibrium Spectrum
 - a) \rightarrow Calculation
 - b) \rightarrow key assumptions
- iii) Weakly Non-Equilibrium Relaxation \leftrightarrow Lenard-Balescu Equation
- iv) Dynamic Relaxation

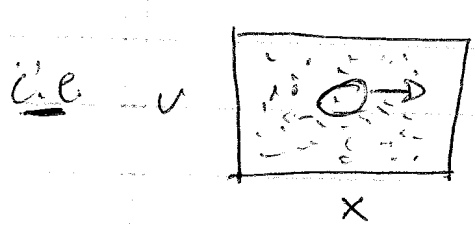
a) Basic Ideas - Equilibrium Fluctuations

→ plasma:

- $1/n\lambda_D^3 \ll 1 \Rightarrow$ many particles in Debye sphere
- $k_B T \gg e^2/n \Rightarrow$ thermal energy dominates electrostatic energy

→ Equilibrium: Balances

- (equivalent) → absorption vs. emission
- fluctuation vs. dissipation



① emission: discrete particle in plasma fluid emits waves?

$$\nabla \cdot \underline{D} = \nabla \cdot \underline{\epsilon} \underline{E} = 4\pi n_0 q \delta(\underline{x} - \underline{x}(t))$$

i.e. → a boat wake in water
→ Cerenkov emission

• discrete emission ⇔ fluctuation

i.e. - particle kinetic energy coupled to wave energy
- Cerenkov emission ⇔ particle slowing down

- ② Cerenkov emitted waves damp
 - via \rightarrow Landau damping
 - \rightarrow in/on vlasov fluid (i.e. Landau damping for $v \rightarrow 0$)



emitted waves damp \rightarrow { absorption
dissipation } $>$ heats particles

So, conceptual picture of thermal equilibrium Fluctuations is detailed balance of:

- \rightarrow Cerenkov emission of waves from individual discrete particles
- \rightarrow absorption of waves via Landau damping on Vlasov fluid

NB: ① Here, assume periodic B.C.'s \rightarrow no radiative damping, outgoing waves, etc.

Note in this picture, each particle plays a dual role (i.e. "double agent"):

\rightarrow ② In general, take damping length finite
 i.e. $\lambda_{damp} \sim \frac{\omega}{k} / |v_{th}| < L_{system}$

As:

- "emitter": a discrete particle moving along some specified (unperturbed) orbit



an identifiable 'pea' in a 'pea soup' composed of other peas.

- "absorber": an element of the Vlasov Fluid responding to and Landau damping emission from (other) discrete particles



a "crushed pea" element of the "pea soup" of the Vlasov fluid.

so

→ equilibrium plasma = soup/gas of 'dressed' test particles



Vlasov fluid
screening

⇒ "test particle model"

→ every pea in the soup acts like soup for all the other peas ----

Note: Useful Analogy

	Brownian Motion	Equilibrium Plasma
Fluctuation	$\langle v^2 \rangle_w$	$\langle E^2 \rangle_{k, \omega}$
Exciton	ω -mode	$k, \omega(k) \rightarrow$ plasma waves
Emission/Source	$\langle \tilde{f}^2 \rangle$ - fluid thermal force	particle discreteness
Absorption/Damping	$\beta \rightarrow$ Stokes Drag on Particle	EM \leftrightarrow Landau Damping of Collective Modes

Prob. : "Brownian Motion"

$$\frac{d\tilde{v}}{dt} + \beta \tilde{v} = \frac{\tilde{f}}{m}$$

(b) Test Particle Model - Fluctuation Spectrum

→ As noted before, basic idea is that:

- each particle both a 'discrete emitter' and participant in laser fluid screening cloud

- fluctuations weak → unperturbed orbits valid.

if consider stationary cond:

$$\delta F = F^c + \tilde{F}$$

\downarrow coherent laser response (screening) \downarrow discrete particle source

Calc' Debye
Calculation)

$$\delta F = \frac{|e| \tilde{E}_{k, \omega} \langle \delta F \rangle}{-i(\omega - kv)} + |e| \delta(x - x(t)) \delta(v - v(t))$$

\downarrow
discrete source

\downarrow
coherent response

$$\nabla^2 \phi = 4\pi n_0 |e| \int dV \delta F = 4\pi n_0 |e| \int dV F^c + 4\pi n_0 |e| \int dV \tilde{F}$$

$$\Rightarrow \vec{\phi}_{k,\omega} = \frac{4\pi n_0 e |\vec{v}|}{k^2} \int dV \vec{F} / \epsilon(k,\omega)$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k} \int dV \frac{\partial \langle F \rangle / \partial V}{\omega - kv}$$

using u.p.o. :

$$\begin{aligned} \int dV \vec{F}_k &= \int dx e^{-ckx} |\vec{v}| \delta(x - x(t)) \\ &= |\vec{v}| e^{-ckvt} \end{aligned}$$

so

$$\vec{\phi}_k(t) = \underline{\underline{\epsilon^{-1}(k,t)}} \frac{4\pi n_0 e |\vec{v}|}{k^2} e^{-ckvt}$$

Note: strictly speaking, have :

$$\underline{\underline{\epsilon(k,t)}} \vec{\phi}_k(t) = \frac{4\pi n_0 e |\vec{v}|}{k^2} e^{-ckvt}$$

so

$$\vec{\phi}_k(t) = \underline{\underline{\epsilon^{-1}(k,t)}} \frac{4\pi n_0 e |\vec{v}|}{k^2} e^{-ckvt} + \vec{\phi}_k^{\text{homog.}} e^{-ckvt}$$

↓
driven solution
(discreteness)

↓
homogeneous solution

$$\omega_n = \omega_r(k) + i\omega_i(k) \quad \Rightarrow \text{eigenmode freq.}$$

Now, - time asymptotically

- for $\omega_i(k) < 0 \Rightarrow$ collective modes damped

\Rightarrow only discretener driven solutions persist

Catch: \Rightarrow For $\omega_i \lesssim 0 \Rightarrow$ i) need wait quite a long time.
 ii) for sufficient source strength, amplification to nonlinearity occurs ...

n.b. moving toward, but not to, marginal stability $\Rightarrow \text{Im} \omega \rightarrow \infty$

\rightarrow if unstable modes, require ultimate nonlinear damping to balance noise
 i.e. $\epsilon_{IM} = \epsilon_{IM}(k, \omega, \langle \phi^2 \rangle)$

"noise" = thermal + nonlinear, in that case

Proceeding, then test particle model \Rightarrow

$$\langle \phi^2 \rangle_{k, \omega} = \left(\frac{4\pi n_0 |e|}{k^2} \right)^2 \int d\mathbf{v}_1 \int d\mathbf{v}_2 \frac{\langle \tilde{f}(\mathbf{v}_1) \tilde{f}(\mathbf{v}_2) \rangle_{k, \omega}}{|\epsilon(k, \omega)|^2}$$

\therefore all content $\rightarrow \langle \tilde{f}^2 \rangle_{k, \omega}$ (abbreviation)
 $\rightarrow \epsilon(k, \omega)$

Now, for discreteness noise:

$$\tilde{f} = \frac{1}{n} \sum_{i=1}^N \delta(x - X_i(t)) \delta(v - v_i(t))$$

\rightarrow

$$\left. \begin{aligned} X_i(t) &= X_{i0} + v_i t \\ v_i(t) &= \text{const} \end{aligned} \right\} \text{c.p.o.}$$

\rightarrow assume (discrete) uncorrelated test particles, so:

so $\langle \rangle = n \int d\mathbf{x}_0 \int d\mathbf{v}_0 \langle f(\mathbf{v}_0) \rangle$
 @ Maxwellian

i.e. simple avg. over equilibrium distribution

$$(k_B T \gg e^2 / \bar{n})$$

so

$$\langle \tilde{f}(t) \tilde{f}(t) \rangle = n \int dx_i \int dv_i \left[\frac{1}{n} \sum_{i=1}^N \delta(\underline{x}_i - \underline{x}_i(t)) \delta(\underline{v}_i - \underline{v}_i(t)) \right] * \left[\frac{1}{n} \sum_{j=1}^N \delta(\underline{x}_j - \underline{x}_j(t)) \delta(\underline{v}_j - \underline{v}_j(t)) \right] \langle f \rangle$$

$$= \frac{1}{n} \delta(\underline{x}_1 - \underline{x}_2) \delta(\underline{v}_1 - \underline{v}_2) \langle f \rangle$$

as avg. vanishes unless $\begin{pmatrix} \underline{x}_i \\ \underline{v}_i \end{pmatrix} = \begin{pmatrix} \underline{x}_j \\ \underline{v}_j \end{pmatrix}$

N.B. \ddagger Uncorrelated test particles can only correlate with themselves...

so

$$\langle \tilde{f}(t) \tilde{f}(t) \rangle = \frac{\langle f \rangle}{n} \delta(\underline{x}_1 - \underline{x}_2) \delta(\underline{v}_1 - \underline{v}_2)$$

Discreteness
Correlation

See Pg. 71 for further details ...

Details of TPM - Discreteness Correlation

1.) Discreteness Correlation Function

$$\tilde{f}(t) = \frac{1}{n} \sum_{i=1}^N \delta(\underline{x}_i - \underline{x}_i(t)) \delta(\underline{v}_i - \underline{v}_i(t))$$

$$\begin{cases} \underline{1} = \underline{x}_1, \underline{v}_1, t \\ \underline{2} = \underline{x}_2, \underline{v}_2, t \end{cases}$$

$$\langle \rangle := \int d\underline{x}_i \int d\underline{v}_i n \langle f \rangle$$

$$\langle \rangle = \int d\underline{x}_i \int d\underline{v}_i n \langle f \rangle$$

$$\langle \tilde{f}(t) \tilde{f}(t') \rangle =$$

$$= \int d\underline{x}_i \int d\underline{v}_i n \left(\frac{1}{n} \sum_{i=1}^N \delta(\underline{x}_i - \underline{x}_i(t)) \delta(\underline{v}_i - \underline{v}_i(t)) \right) \left(\frac{1}{n} \sum_{j=1}^N \delta(\underline{x}_j - \underline{x}_j(t')) \delta(\underline{v}_j - \underline{v}_j(t')) \right)$$

$$= \int d\underline{x}_i \int d\underline{v}_i \frac{\langle f \rangle}{n} \sum_{\substack{i,j \\ i=1}}^N \left[\delta(\underline{x}_i - \underline{x}_j(t)) \delta(\underline{x}_j - \underline{x}_i(t')) \delta(\underline{v}_i - \underline{v}_j(t)) \delta(\underline{v}_j - \underline{v}_i(t')) \right]$$

only $\neq 0$ if arguments interchangeable

$$= \int d\underline{x}_i \int d\underline{v}_i \frac{\langle f \rangle}{n} \left[\delta(\underline{x}_i - \underline{x}_i) \delta(\underline{x}_i - \underline{x}_i) \delta(\underline{v}_i - \underline{v}_i) \delta(\underline{v}_i - \underline{v}_i) \right]$$

$$= \frac{\langle f \rangle}{n} \delta(\underline{x}_1 - \underline{x}_2) \delta(\underline{v}_1 - \underline{v}_2)$$

particles uncorrelated unless same

- memory
- order
erhöht \rightarrow ensemble, time

$$\langle \tilde{f}(t) \tilde{f}(z) \rangle = \langle \tilde{f}(0) \tilde{f}(z-1) \rangle \quad \text{stat homog.}$$

$$\langle \tilde{f}(1) \tilde{f}(z) \rangle_k = \int e^{-ik(x_2-x_1)} \langle \tilde{f}(1) \tilde{f}(z) \rangle dx_-$$

2 'ahead' 1 \Rightarrow

$$\langle \tilde{f}(1) \tilde{f}(z) \rangle_{z=0} = \int_0^{\infty} dt e^{i\omega t} \int_b^{\infty} dx e^{-ik(x_2-x_1)} \langle \tilde{f}(1) \tilde{f}(z) \rangle dx_- \quad \textcircled{2}$$

② forward in time

$$+ \int_0^{\infty} dt e^{i\omega t} \int_b^{\infty} dx e^{-ik(x_2-x_1)} \langle \tilde{f}(1) \tilde{f}(z) \rangle dx_- \quad \textcircled{1}$$

① backward

$$\textcircled{2} \Rightarrow x_2 \rightarrow x_2 + v_2 \tau$$

$$\begin{aligned} \textcircled{2} &= \int_0^{\infty} dt \int_b^{\infty} dx e^{-ik(x_2-x_1)} e^{i(\omega - kv_2)\tau} \langle \tilde{f}(1) \tilde{f}(z) \rangle dx_- \\ &= \int_0^{\infty} dt e^{i(\omega - kv_2)t} \langle \tilde{f} \tilde{f} \rangle_k \end{aligned}$$

$ic_2 \rightarrow \omega - v_2 k$

$$\textcircled{1} = \int_0^{\infty} dt e^{i(\omega - kv_2)t} \langle \tilde{f} \tilde{f} \rangle_k$$

$$\textcircled{1} = \frac{-1}{i(\omega - kv_2)} \langle \tilde{f} \tilde{f} \rangle_k$$

$$\textcircled{2} = \frac{-1}{i(\omega - kv_2)} \langle \tilde{f} \tilde{f} \rangle_k$$

$$\langle F^2 \rangle = \frac{1}{N} \langle F \rangle \delta(x_0) \delta(V_0)$$

$$\langle \tilde{F}^2 \rangle_N = \frac{1}{N} \langle F \rangle \delta(V_-) \rightarrow \text{rec}$$

$$V_i = V_j$$

$$\textcircled{1} + \textcircled{2} = \left(\frac{0}{\omega - kv} + \frac{i}{\omega - kv} \right) \langle \tilde{F}^2 \rangle_4$$

$$\text{re}(\textcircled{1} + \textcircled{2}) = 2\pi \delta(\omega - kv) \frac{\langle F \rangle}{N} \delta(V_0)$$