Physics 214 UCSD/225a UCSB

Lecture 9

- Majorana Neutrino
- Outlook for remainder of quarter
- Halzen & Martin Chapter 3
- Start of Halzen & Martin Chapter 4

Majorana Neutrino

- I owe you an explanation from last time.
- Basically, look it up in Section 2 of hep-ph/0211134, an article by Boris Kayser that is published in a book called "Neutrino Mass".
- In a nutshell:
  - The "neutrino" from pi$^+$ decay is left handed.
  - The "anti-neutrino" from pi- decay is right handed.
  - The charged weak current violates parity 100%, and thus couples only to one of the two helicities, and a different one, depending on the charge of the W.
  - If the neutrino is its own anti-particle then you still have two helicity states, and those two are the distinction we normally look at as "neutrino" and "anti-neutrino".
- What about Lorentz invarianz ?
  - This does not cause havoc with boosted reference frames because the physical cross section for neutrino scattering on nuclei involves two currents. Both flip as you "overtake" the neutrino. See Kayser for more.

Outlook for remaining Quarter

- From now on I will follow H&M more closely.
- We'll basically cover chapters 3,4,5,6
  - A lot of this should be a review of things you have seen already either in advanced QM or intro QFT.
  - Accordingly, I'll be brief at times, and expect you to read up on it as needed !!!
- Then skip chapter 7.
- Then parts of 8,9,10, and 11, where I am not yet sure as to the order I'll do them in.
- Some of this we won't get to until next quarter.

(non-)relativistic Schroedinger Eq.

- Nonrelativistic

$$E = p^2/(2m)$$

- Relativistic

$$E^2 = p^2 + m^2$$

$$\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right)\psi = 0$$

$$-\frac{\partial^2}{\partial t^2}\phi = \left(-\nabla^2 + m^2\right)\phi$$

In both cases we replace:

$$E \to i\frac{\partial}{\partial t}$$

$$p \to -i\nabla$$

Covariant Notation

$A^\mu = (A^0, \mathbf{A}) \; ; \; A_\mu = (A^0, -\mathbf{A})$

$A^\mu B_\mu = A^0 B^0 - \mathbf{AB}$

The derivative 4-vector is given by:

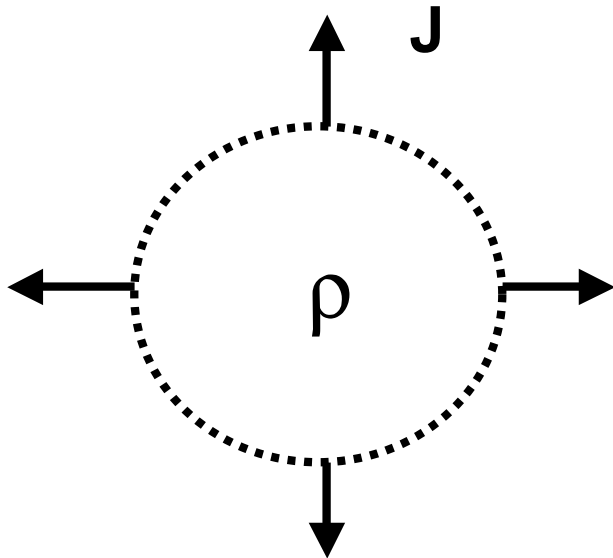$$\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla \right)$$

With: $\square^2 = \partial_\mu \, \partial^\mu$

We then get the Klein-Gordon

$$\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla \right)$$

Equation as: $(\square^2 + m^2) \, \phi = 0$

Continuity Equation

- For scattering, we need to understand the probability density flux **J**, as well as the probability density ρ.

- Conservation of probability leads to:

$$-\frac{\partial}{\partial t}\int_V \rho dV = \int_S \vec{j}\cdot\hat{n}ds = \int_V \nabla J dV$$

$$\Rightarrow \nabla J + \frac{\partial \rho}{\partial t} = 0$$

(non-)relativistic Continuity Eq.

- Nonrelativistic

$$\rho = |\psi|^2$$

$$J = \frac{-i}{2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

- Relativistic

$$\rho = i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)$$

$$J = -i\left(\phi^* \nabla \phi - \phi \nabla \phi^*\right)$$

In both cases we have plane wave solutions as:

$$\phi\left(t, \vec{x}\right) = N e^{-i p_\mu x^\mu}$$

Covariant Notation

$$j^\mu = \left(\rho, \vec{j}\right)$$

Transforms like a 4-vector

$$\partial^\mu j_\mu = 0$$

Covariant continuity equation

For the plane wave solutions we find:

$$\left.\begin{array}{l}\rho = 2E\,|N|^2 \\ \vec{j} = 2\vec{p}\,|N|^2\end{array}\right\} J^\mu = 2p^\mu\,|N|^2$$

Why $\rho \propto E$?

ρd^3x = constant under lorentz transformations

However, d³x gets lorentz contracted.
Therefore, ρ must transform time-like, i.e. dilate.

$$d^3x \rightarrow d^3x \cdot \sqrt{1-v^2}$$

$$\rho \rightarrow \rho \Big/ \sqrt{1-v^2}$$

Energy Eigenvalues of K.G. Eq.

$$(\Box^2 + m^2) \, \phi = 0$$

Or

$$E^2 = p^2 + m^2$$

$$\Downarrow$$

$$E = \pm\sqrt{p^2 + m^2}$$

Positive and negative energy solutions !

Feynman-Stueckelberg Interpretation

- Positive energy particle moving forwards in time.

- Negative energy antiparticles moving backwards in time.

$\Rightarrow$ Absorption of positron with -E is the same as emission of electron with +E.

$\Rightarrow$ In both cases charge of system increases while energy decreases.

Encourage you to read up on this in chapters 3.4 & 3.5 of H&M.

- Will get back to discussing negative energy solutions after we understand scattering in a potential.
- Will use scattering in a potential to discuss perturbation theory.
  - Assume potential is finite in space.
  - Incoming and outgoing states are free-particle solutions "far enough" away from potential.
  - Assume V is a small perturbation throughout such that free particle, i.e. plane wave starting point is a meaningful approximation.

Nonrelativistic Perturbation Theory

- Assume we know the complete set of eigenstates of the free-particle Schroedinger Equation:

$$H\phi_n = E_n\phi$$

$$\int_{Vol} \phi_n^* \phi_m \, d^3x = \delta_{nm}$$

- Now solve Schroedinger Eq. in the presence of a small perturbation V($\mathbf{x}$,t):

$$(H + V)\psi = i\frac{\partial \psi}{\partial t}$$

Any solution can be expressed as:

$$\psi = \sum_n a_n(t) \phi_n(x) e^{-iE_n t}$$

Plug this into Sch.Eq. and you get:

$$i \sum_n \frac{da_n(t)}{dt} \phi_n(x) e^{-iE_n t} = V(x,t) \sum_n a_n(t) \phi_n(x) e^{-iE_n t}$$

Multiply both sides by: $-i e^{iE_f t} \phi_f^*$

$$\frac{da_f(t)}{dt} = -i \sum_n a_n(t) e^{i(E_f - E_n)t} \int \phi_f^* V \phi_n d^3 x$$

What does this mean?

$$\frac{da_f(t)}{dt} = -i\sum_n a_n(t)e^{i(E_f - E_n)t}\int \phi_f^* V \phi_n d^3x$$

Let's think of it as scattering of an
incoming eigenstate i into an outgoing eigenstate f.

Initial and final states are eigenstates because
V is small, limited in space,
and experienced only for finite amount of time..

Assume V is small and "seen" for only a finite amount of time.

- At times long ago, the system is in eigenstate i of the free hamiltonian because it's far away from V.

- At times far in the future, the system is in eigenstate f of the free hamiltonian because it's far away from V.

- After integration over time, we thus get:

$$\frac{da_f}{dt} = -ie^{-i(E_f - E_i)t} \int d^3x \phi_f^* V \phi_i$$ <= starting point: i -> f

$$V_{fi} \equiv \int d^3x \phi_f^* V(x) \phi_i$$ <= Assume V is time independent

$$T_{fi} \equiv a_f = -iV_{fi} \int dt\, e^{-i(E_f - E_i)t} = \boxed{-2\pi i V_{fi} \delta(E_f - E_i)}$$

Result of time integration.

Meaning of $T_{fi} = -2\pi i V_{fi}\delta(E_f - E_i)$

- δ-function guarantees energy conservation.

=> Uncertainty principle guarantees that T_{fi} is meaningful only as t -> infinity.

- We thus define a more meaningful quantity W, the *"transition amplitude per unit time"* by dividing with t, and then letting t -> infinity.

$$W = \lim_{t->\infty} \frac{\left|T_{fi}\right|^2}{t} = 2\pi\left|V_{fi}\right|^2\delta(E_f - E_i)$$

Aside

$$W = \lim_{t->\infty} \frac{\left|T_{fi}\right|^2}{t} = \lim_{t->\infty} 2\pi \frac{\left|V_{fi}\right|^2}{t} \delta(E_f - E_i) \int_{-t/2}^{t/2} dt e^{-i(E_f - E_i)t}$$

The δ-function from the first integral
guarantees that the second integral is basically t,
and thus cancels with the 1/t, making the limit building trivial.

$$T_{fi}T_{fi}^* = (-iV_{fi} \int dt e^{-i(E_f - E_i)t})(-iV_{fi} \int dt e^{-i(E_f - E_i)t})^* =$$

$$\left|T_{fi}\right|^2 = 2\pi \left|V_{fi}\right|^2 \delta(E_f - E_i) \int dt e^{-i(E_f - E_i)t}$$

Physically meaningful quantities

- The transition probability per unit time, W, becomes physically meaningful once you integrate over a set of initial and final states.

- Though typically, we start with a specific initial and a set of final states:

$$W_{fi} = 2\pi \int dE_f\, \rho(E_f) \left| V_{fi} \right|^2 \delta(E_f - E_i)$$

$$W_{fi} = 2\pi \left| V_{fi} \right|^2 \rho(E) \quad \text{<= Fermi's Golden Rule}$$

Fermi's Golden Rule

- We find Fermi's Golden Rule as the leading order in perturbation theory.

- This begs the question, what's the next order, and how do we get it?

- In our lowest order approximation, we scattered from an initial state i to a final state f.

- The obvious improvement is to allow for double scattering from i to any n to f, and sum over all n.

Second Order

$$a_n = -i \int dt\, e^{i(E_n - E_i)} V_{ni}$$

$$\frac{da_f(t)}{dt} = LO - i \sum_{n \neq i} a_n(t) e^{i(E_f - E_n)t} V_{fn}$$

$$W_{fi} = 2\pi \left| V_{fi} + \sum_{n \neq i} V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni} \right|^2 \rho(E_i)$$

What have we learned?

- For each interaction vertex we get a vertex factor V_{fi} .
- For the propagation via an intermediate state we gain a "propagator" factor $1/(E_i-E_n)$.
- The intermediate state is virtual, and thus does not require energy conservation.
- However, energy is conserved between initial and final state.

$$W_{fi} = 2\pi \left| V_{fi} + \sum_{n\neq i} V_{fn} \frac{1}{E_i - E_n + i\varepsilon} V_{ni} \right|^2 \rho(E_i)$$

Let's play some games with this formalism

Photon absorption by Particle vs Antiparticle

- Particle scatter in field

$$p_i = (E_1, \vec{p_1})$$
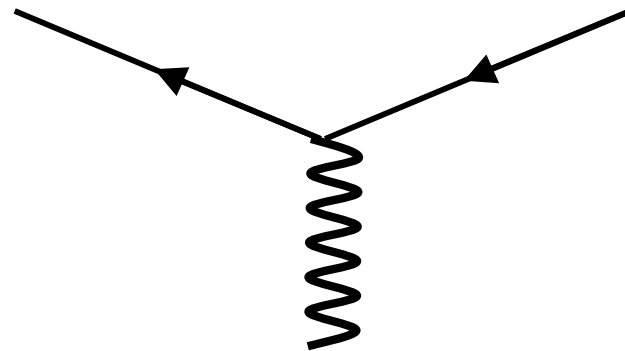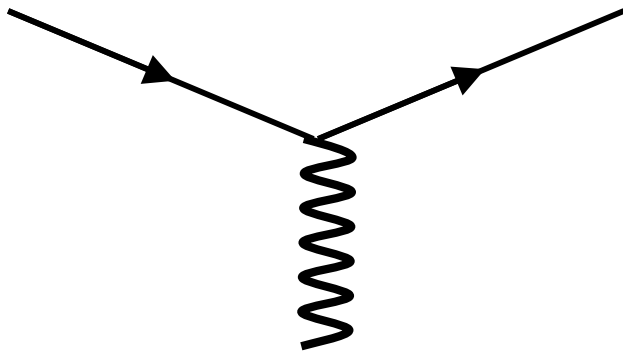
$$p_f = (E_2, \vec{p_2})$$

$$T_{fi} \approx \int dt\, \phi_f^* V \phi_i \approx \int dt\, e^{iE_2 t} e^{-i\omega t} e^{-iE_1 t}$$

$$\approx \delta(E_2 - (\omega + E_1))$$

- Antipart. scatter in field

$$p_i = (-E_2, -\vec{p_2})$$

$$p_f = (-E_1, -\vec{p_1})$$

$$T_{fi} \approx \int dt\, \phi_f^* V \phi_i \approx \int dt\, e^{i(-E_1)t} e^{-i\omega t} e^{-i(-E_2)t}$$

$$\approx \delta(E_2 - (\omega + E_1))$$

Particle and antiparticle have the same interaction with EM field.

Pair Creation from this potential

$$p_i = (-E_1, -\vec{p_1})$$

$$p_f = (E_2, \vec{p_2})$$

$$T_{fi} \approx \int dt\, \phi_f^* V \phi_i \approx \int dt\, e^{iE_2 t} e^{-i\omega t} e^{-i(-E_1)t}$$

$$\approx \delta(E_2 + E_1 - \omega)$$

Energy is conserved as it should be.

This wave function formalism is thus capable of describing particles, antiparticles, and pair production.

"Rules"

- Time goes from left to right
- Antiparticles get arrow that is backwards in time.
- "Incoming" and "outgoing" is defined by how the arrows point to the vertex.
- Antiparticles get negative energy assigned.

H&M Chapter 4

- Electrodynamics of Spinless particles
  - We replace p^μ with $p^\mu + eA^\mu$ in classical EM for a particle of charge -e moving in an EM potential A^μ
  - In QM, this translates into:

$$i\partial^\mu \rightarrow i\partial^\mu + eA^\mu$$

  - And thus to the modified Klein Gordon Equation:

$$\left(\partial^\mu\partial_\mu + m^2\right)\phi = -V\phi$$

$$V = -ie(\partial^\mu A_\mu + A^\mu\partial_\mu) - e^2A^2$$

V here is the potential energy of the perturbation.

Take results form Perturbation

$$V = \boxed{-ie(\partial^\mu A_\mu + A^\mu \partial_\mu)} - e^2 A^2$$

$$T_{fi} = -i \int \phi_f^* V(x) \phi_i d^4 x \quad \Leftarrow \text{covariant form}$$

Integrate by parts

$$T_{fi} = i \int \phi_f^* ie\left(A^\mu \partial_\mu + \boxed{\partial^\mu A_\mu}\right) \phi_i d^4 x + O(e^2)$$

$$T_{fi} = i \int ie\left(\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f^*)\phi_i\right)A^\mu d^4 x + O(e^2)$$

$$T_{fi} = -i \int J_\mu A^\mu d^4 x + O(e^2)$$

EM current for i -> f transition.

$$\boxed{J_\mu = -ie\left(\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f^*)\phi_i\right)}$$

$$J_\mu = -eN_i N_f (p_i + p_f)e^{i(p_f - p_i)x}$$

Using plane wave solutions

Aside on current

Regular current we talked about in the beginning today:

$$\left.\begin{array}{l} \rho = 2E\,|N|^2 \\ \vec{j} = 2\vec{p}\,|N|^2 \end{array}\right\} J^\mu = 2p^\mu\,|N|^2$$

Transition current from i to f:

$$J_\mu = -ie\left(\phi_f^*(\partial_\mu \phi_i) - (\partial_\mu \phi_f^*)\phi_i\right)$$

$$J_\mu = -eN_iN_f(p_i + p_f)e^{i(p_f - p_i)x}$$

The difference is that for regular current i=f , and the wave function piece (i.e. dependence on x) cancels as a result.

Now we have enough formalism together to do our first (almost) realistic scattering calculation.

(we are still talking about spinless particles only)

Electron Muon Scattering Overview

- Use what we just did
  - Electron scattering in EM field
- With the field being the one generated by the muon as source.
  - Use covariant form of maxwell's equation in Lorentz Gauge to get V, the perturbation potential.
- Plug it into T_{fi}
- Then head into more general discussion of how to express cross section in terms of invariant amplitude (or "Matrix Element").

Electron Muon scattering

$$\Box^2 A^\mu = J^\mu_{(2)} \quad \text{Maxwell Equation}$$

Note: $\Box^2 e^{iqx} = -q^2 e^{iqx}$

$$J^\mu_{(2)} = -eN_B N_D (p_D + p_B)^\mu e^{i\boxed{(p_D - p_B)}x}$$

= q

$$A^\mu = -\frac{1}{q^2} J^\mu_{(2)}$$

$$T_{fi} = -i \int J^{(1)}_\mu \frac{-1}{q^2} J^\mu_{(2)} d^4 x$$

Note the symmetry: (1) <-> (2)

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_A - p_B) M$$

$$-iM = (ie(p_A + p_C)^\mu)\frac{-ig_{\mu\nu}}{q^2}(ie(p_D + p_B)^\nu)$$

Note the structure: Vertex x propagator x Vertex