Physics 214 UCSD/225a UCSB

Lecture 10

- Halzen & Martin Chapter 4
  - Electron-muon scattering
  - Cross section definition
  - Decay rate definition
  - treatment of identical particles => symmetrizing
  - crossing
Electrodynamics of Spinless particles

• We replace $p^\mu$ with $p^\mu + eA^\mu$ in classical EM for a particle of charge $-e$ moving in an EM potential $A^\mu$
• In QM, this translates into: $i\partial^\mu \rightarrow i\partial^\mu + eA^\mu$
• And thus to the modified Klein Gordon Equation:

$$\left(\partial^\mu \partial_\mu + m^2\right)\phi = -V\phi$$

$$V = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu) - e^2 A^2$$

V here is the potential energy of the perturbation.
Two-by-two process Overview

- Start with general discussion of how to relate number of scatters in AB -> CD scattering to “beam & target independent” cross section in terms of $W_{fi}$.
- Calculate $W_{fi}$ for electron-muon scattering.
- Calculate cross section from that.
- Show relationship between cross section and “invariant amplitude” (or “Matrix Element”).
Reminder from last lecture

\[ j^\mu = (\rho, \vec{j}) \]

\[ \partial^\mu j_\mu = 0 \]

\[ \rho = i \left( \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) \]

\[ J = -i (\phi^* \nabla \phi - \phi \nabla \phi^*) \]

Plane wave solutions are:

\[ \phi(t, \vec{x}) = N e^{-i p_\mu x^\mu} \]

4-vector current for the plane wave solutions we find:

\[ \rho = 2E |N|^2 \]

\[ \vec{j} = 2\vec{p} |N|^2 \]

\[ J^\mu = 2p^\mu |N|^2 \]

The \( 2|N|^2 \) is an arbitrary normalization
Cross Section for AB -> CD

• Basic ideas:

\[ \text{Cross section} = \sigma = \frac{W_{fi}}{(\text{initial flux})} \times (\text{number of final states}) \]

# of scatters = (flux of beam) x (# of particles in target) x \( \sigma \)

\( W_{fi} \) = rate per unit time and volume

“Cross section” is independent of characteristics of beam and target !!!
Aside on wave function Normalization

\[ W_{fi} \]

Cross section = \[ \sigma = \frac{W_{fi}}{(\text{initial flux})} (\text{number of final states}) \]

\[ W_{fi} \propto N^4 \]

Number of final states/initial flux \( \propto N^{-4} \)

Cross section is thus independent of choice of wave function normalization

(as it should, of course!)

We will see this explicitly as we walk through this now.
Two-Two process AB -> CD

• Normalize plane wave in constant volume
  – This is obviously not covariant, so the volume normalization better cancel out before we’re done!

\[
\int \rho dV = 2E \Rightarrow N = \frac{1}{\sqrt{V}}
\]

• # of particles per volume = \(2E/V = n\)
• # of particles A crossing area per time = \(v_A \cdot n_A\)
• Flux(AB) = \(v_A \cdot n_A \cdot (2E_B/V) = v_A \cdot (2E_A/V) \cdot (2E_B/V)\)
Aside on covariant flux

- Flux = $v_A (2E_A/V) (2E_B/V)$
- Now let target (i.e. B) move collinear with beam (i.e. A): Flux = $(v_A - v_B) (2E_A/V) (2E_B/V)$
- Now take $v=p/E$: Flux = $(E_B p_A + E_A p_B) 4/V^2$
- Now a little relativistic algebra:
  \[
  (p_A^\mu p_B^\mu)^2 - m_A^2 m_B^2 = (E_A E_B - p_A p_B)^2 - m_A^2 m_B^2
  \]
  \[
  (E_A E_B)^2 = (p^2 + m^2)_A (p^2 + m^2)_B
  \]
  \[
  p_A = -p_B
  \]

**Putting the pieces together and adding some algebra:**

\[
(p_A^\mu p_B^\mu)^2 - m_A^2 m_B^2 = (p_A E_B + p_B E_A)^2
\]

Obviously covariant!

( up to $1/V^2$ normalization factor that is arbitrary, and will cancel)
Number of final states/particle

- QM restricts the number of final states that a single particle in a box of volume $V$ can have:

\[
\frac{\text{Number of final states}}{2E \text{ particles}} = \frac{Vdp^3}{(2\pi)^3 2E}
\]

This follows from Exercise 4.1 in H&M that you will do as homework exercise.
Putting the pieces together

Cross section \( \sigma = \frac{W_{fi}}{(\text{initial flux})} \) (number of final states)

\[
\sigma = \frac{W_{fi}}{v_A (2E_A/V)(2E_B/V)} \quad \frac{Vdp_C^3}{(2\pi)^3 2E_C} \quad \frac{Vdp_D^3}{(2\pi)^3 2E_D}
\]

Next we calculate \( W_{fi} \)
Electron Muon Scattering

• Use what we did last lecture
  – Electron scattering in EM field
• With the field being the one generated by the muon as source.
  – Use covariant form of maxwell’s equation in Lorentz Gauge to get $V$, the perturbation potential.
• Plug it into $T_{fi}$
In form of diagrams

Electron-muon scattering
Electron Muon scattering

$\Box^2 A^\mu = J^\mu_{(2)}$ Maxwell Equation

$J^\mu_{(2)} = -e N_B N_D (p_D + p_B)^\mu e^{i(p_D-p_B)x}$

$A^\mu = -\frac{1}{q^2} J^\mu_{(2)}$

$T_{fi} = -i \int J^{(1)}_\mu \frac{-1}{q^2} J^\mu_{(2)} d^4 x$

$T_{fi} = -i N_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_A - p_B) M$

$-iM = (ie(p_A + p_C)^\mu) \frac{-ig_{\mu\nu}}{q^2} (ie(p_D + p_B)^\nu)$

Note: $\Box^2 e^{i q x} = -q^2 e^{i q x}$

Note the symmetry: (1) <-> (2)

Note the structure: Vertex x propagator x Vertex
Reminder

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} = g_{\mu\nu}
\]

\[A^\mu B_\mu = g_{\mu\nu} A^\mu B^v\]
Reminder: $T_{fi} \rightarrow W_{fi}$

$$W_{fi} = \lim_{t \rightarrow \infty} \frac{|T_{fi}|^2}{t} \Rightarrow \frac{|T_{fi}|^2}{tV}$$

Last time we didn’t work in a covariant fashion. This time around, we want to do our integrations across both time and space, i.e. $W$ is a rate per unit time and volume.

$$T_{fi} = -iN_A N_B N_C N_D (2\pi)^4 \delta^{(4)}(p_D + p_C - p_A - p_B)M$$

As last time, we argue that one $\delta$-function remains after $||^2$ while the other gives us a $tV$ to cancel the $tV$ in the denominator.
Putting it all together for $W_{fi}$

\[ N = \frac{1}{\sqrt{V}} \quad \quad W_{fi} = \left| \frac{T_{fi}}{tV} \right|^2 \]

\[ T_{fi} = \frac{-i(2\pi)^4}{V^2} \delta^{(4)}(p_D + p_C - p_A - p_B)M \]

\[ W_{fi} = (2\pi)^4 \frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{V^4} \left| M \right|^2 \]
Putting it all together for $\sigma$

$$\sigma = \frac{W_{fi}}{\nu_A \left(2E_A/V\right) \left(2E_B/V\right)} \frac{V dp_C^3}{(2\pi)^3 2E_C} \frac{V dp_D^3}{(2\pi)^3 2E_D}$$

$$d\sigma = \frac{V^2}{4\nu_A E_A E_B} (2\pi)^4 \frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{V^4} |M|^2 \frac{V^2 dp_c^3 dp_D^3}{(2\pi)^6 4 E_c E_D}$$

$$d\sigma = \frac{1}{64\pi^2} \frac{\delta^{(4)}(p_D + p_C - p_A - p_B)}{\nu_A E_A E_B} |M|^2 \frac{dp_c^3 dp_D^3}{E_c E_D}$$
Aside on outgoing states

- While the incoming states have definite momentum, the outgoing states can have many momenta.

- The cross section is thus a differential cross section in the outgoing momenta.

\[
d\sigma = \frac{|M|^2}{4\sqrt{s}E_A E_B} \frac{1}{16\pi^2} \delta^{(4)}(p_D + p_C - p_A - p_B) \frac{dp_c^3 dp_D^3}{E_c E_D}
\]

(incoming flux is still not covariant)
It is customary to re-express

\[ d\sigma = \frac{|M|^2}{4\nu_A E_A E_B} \frac{1}{16\pi^2} \delta^{(4)}(p_D + p_C - p_A - p_B) \frac{dp_c^3 dp_D^3}{E_c E_D} \]

As:

\[ d\sigma = \frac{|M|^2}{F} dQ \]

F = flux factor:

\[ F = 4\sqrt{(p_A^\mu p_B^\mu)^2 - m_A^2 m_B^2} = 4\nu_A E_A E_B \]

dQ = Lorentz invariant phase space:

\[ dQ = \frac{1}{16\pi^2} \delta^{(4)}(p_D + p_C - p_A - p_B) \frac{dp_c^3 dp_D^3}{E_c E_D} \]
In the center-of-mass frame:

\[ F = 4p_i \sqrt{(E_A + E_B)^2} = 4p_i \sqrt{s} \]

\[ dQ = \frac{1}{4\pi^2} \frac{p_f}{4\sqrt{s}} d\Omega \]

\[ \left. \frac{d\sigma}{d\Omega} \right|_{cm} = \frac{|M|^2}{64\pi^2 s} \frac{p_f}{p_i} \]

You get to show this as homework!
Electron-electron scattering

• With identical particles in the final state, we obviously need to allow for two contributions to $M$.
  
  – Option 1:
    • C attaches at vertex with A
    • D attaches at vertex with B
  
  – Option 2:
    • C attaches at vertex with B
    • D attaches at vertex with A

• As we can’t distinguish C and D, the amplitudes add before $M$ is squared.

\[
M = -e^2 \left( \frac{(p_A + p_C)^\mu (p_B + p_D)}{(p_D - p_B)^2} + \frac{(p_A + p_D)^\mu (p_B + p_C)}{(p_C - p_B)^2} \right)
\]
Electron-positron and crossing

\[ M = -e^2 \left( \frac{(p_A + p_C) \gamma (-p_B - p_D)}{((-p_D) - (-p_B))^2} + \frac{(p_A - p_B) \gamma (-p_D + p_C)}{(p_C - (-p_D))^2} \right) \]
Electron-positron and crossing

Electron - electron

$$M = -e^2 \left( \frac{(p_A + p_C) \mu (p_B + p_D) \mu}{(p_D - p_B)^2} + \frac{(p_A + p_D) \mu (p_B + p_C) \mu}{(p_C - p_B)^2} \right)$$

Electron - positron

$$M = -e^2 \left( \frac{(p_A + p_C) \mu (-p_B - p_D) \mu}{((-p_D) - (-p_B))^2} + \frac{(p_A - p_B) \mu (-p_D + p_C) \mu}{(p_C - (-p_D))^2} \right)$$

Only difference is: $p_D \rightarrow -p_B$
$p_B \rightarrow -p_D$
Electron-Electron scattering

\[
\begin{align*}
\text{e}^- & \quad \text{p}_A \quad \text{p}_C \quad \text{e}^- \\
\text{e}^- & \quad \text{p}_B \quad \text{p}_D \quad \text{e}^- \\
\end{align*}
\]

crossing

Electron-Positron scattering

\[
\begin{align*}
\text{e}^- & \quad \text{p}_A \quad \text{p}_C \quad \text{e}^- \\
\text{e}^- & \quad -\text{p}_B \quad -\text{p}_D \quad \text{e}^- \\
\end{align*}
\]
E-mu vs e-e vs e-ebar scattering

**Electron - muon**

\[
M = -e^2 \left( \frac{(p_A + p_C)\mu (p_B + p_D)\mu}{(p_D - p_B)^2} \right)
\]

**Electron - electron**

\[
M = -e^2 \left( \frac{(p_A + p_C)\mu (p_B + p_D)\mu}{(p_D - p_B)^2} + \frac{(p_A + p_D)\mu (p_B + p_C)\mu}{(p_C - p_B)^2} \right)
\]

**Electron - positron**

\[
M = -e^2 \left( \frac{(p_A + p_C)\mu (-p_B - p_D)\mu}{((-p_D) - (-p_B))^2} + \frac{(p_A - p_B)\mu (-p_D + p_C)\mu}{(p_C - (-p_D))^2} \right)
\]