Bi-particle invariant mass distribution for a two-step decay chain of a spin-zero particle

Adish P. Vartak

1 Problem Statement

We consider a generic two-step decay chain of a spin-zero particle. The decay chain can be represented as follows

$$A \to B + C$$
$$C \to D + E$$

In the decay process mentioned above, we assume that all the particles are spinless. The objective is to evaluate the invariant mass of two massless particles B and D produced as a result of the decay.

2 Kinematic Calculation

Let the masses of particles A, C and E be m_A, m_C and m_E respectively. We start in the rest frame of the parent particle A. So the total initial momentum is zero and the total initial energy is m_A (In the calculations that follow the speed of light is set to 1). Let us orient our coorinate system such that when particle A decays, particle B is produced with momentum p_B along the negative z-axis. Therefore by momentum conservation, particle C has momentum has p_B in the positive z-direction. Energy conservation yields

$$m_A = \sqrt{p_B^2 + m_C^2} + p_B$$

$$\therefore p_B = \frac{m_A^2 - m_C^2}{2m_A} \tag{1}$$

$$E_C = \sqrt{p_B^2 + m_C^2} = \frac{m_A^2 + m_C^2}{2m_A} \tag{2}$$

Now let us consider the decay of particle C. First let us apply a boost to move into the rest frame of particle C. Suppose particle C moves with velocity v w.r.t the rest frame of A. Then in transforming to the rest frame of C we get

$$p_B{}' = \gamma \left(p_B - vE_C \right) = 0$$

This gives us an expression for v in terms of m_A and m_C

$$v = \frac{p_B}{E_C} = \frac{m_A^2 - m_C^2}{m_A^2 + m_C^2} \tag{3}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2}} = \frac{m_A^2 + m_C^2}{2m_A m_C} \tag{4}$$

Now let us consider the rest frame of particle C. Particle C decays into a massless particle D and a massive particle E. The treatment is similar to the decay of particle A discussed above. Let the momentum of particle D be ${p_D}^{'}$. Then by comparing equation (1) we get

$${p_D}^{'} = \frac{m_C^2 - m_E^2}{2m_E} = {E_D}^{'} \tag{5}$$

Suppose particle D makes an angle θ with the z-axis. Then we can decompose the momentum of particle D into two components - one parallel to the z-axis and one perpendicular to the z-axis. Let these components be ${p_{||}}^{'}$ and ${p_{\perp}}^{'}$ respectively. Then we have

$${p_{||}}^{'} = {p_D}^{'} cos\theta \tag{6}$$

$${p_{\perp}}^{'} = {p_D}^{'} sin\theta \tag{7}$$

Now let us transform back to the rest frame of particle A. The transformation equations are as follows

$$p_{||} = \gamma \left({p_{||}}^{'} + v{E_D}^{'} \right)$$

$$\therefore p_{||} = \gamma \left({p_D}^{'} cos\theta + v{p_D}^{'} \right)$$

$$\therefore p_{||} = \gamma {p_D}^{'} \left(cos\theta + v \right) \tag{8}$$

$$p_{\perp} = {p_{\perp}}^{'} \tag{9}$$

$$E_D = \gamma \left({E_D}^{'} + v{p_{||}}^{'} \right)$$

$$\therefore E_D = \gamma \left({p_D}^{'} + v{p_D}^{'} cos\theta \right)$$

$$\therefore E_D = \gamma {p_D}^{'} \left(1 + vcos\theta \right) \tag{10}$$

The invariant mass of particles B and D is given by

$$Q^2 = (E_B + E_D)^2 - \left(p_{||} - p_B \right)^2 - {p_{\perp}}^2 \tag{11}$$

$$\therefore Q^2 = E_B^2 + 2E_B E_D + E_D^2 - p_{||}^2 - p_B^2 + 2p_{||} p_B - {p_{\perp}}^2$$

$$\therefore Q^2 = 2p_B E_D + E_D^2 - p_{||}^2 + 2p_{||} p_B - {p_{\perp}}^2 \, (\because E_B = p_B)$$

$$\therefore Q^2 = 2p_B \left(E_D + p_{||} \right) + E_D^2 - p_{||}^2 - {p_{\perp}}^2$$

From equations (8) and (10) we get

$$E_D^2 - p_\parallel^2 = \gamma^2 \left(p_D{}' + vp_D{}'cos\theta\right)^2 - \gamma^2 \left(p_D{}'cos\theta + vp_D{}'\right)^2$$

$$\therefore E_D^2 - p_\parallel^2 = \gamma^2 p_D{}'^2 \left(1 + vcos\theta\right)^2 - \gamma^2 p_D{}'^2 \left(cos\theta + v\right)^2$$

$$\therefore E_D^2 - p_\parallel^2 = \gamma^2 p_D{}'^2 \left(1 + v^2cos^2\theta - v^2 - cos^2\theta\right)$$

$$\therefore E_D^2 - p_\parallel^2 = \gamma^2 p_D{}'^2 \left(1 - v^2\right)\left(1 - cos^2\theta\right)$$

$$\therefore E_D^2 - p_\parallel^2 = p_D{}'^2 \left(1 - cos^2\theta\right) \left(\because \gamma^2 = \frac{1}{1 - v^2}\right)$$

$$\therefore E_D^2 - p_\parallel^2 = p_D{}'^2 sin^2\theta = p_\perp{}^2$$

Putting this result back in the expression for Q^2 we get

$$Q^2 = 2p_B \left(E_D + p_\parallel\right)$$

$$\therefore Q^2 = 2\gamma p_D{}' p_B \left(1 + vcos\theta + v + cos\theta\right)$$

$$\therefore Q^2 = 2\gamma p_D{}' p_B \left(1 + v\right)\left(1 + cos\theta\right)$$

Using equations (1), (3), (4) and (5) we get

$$Q^2 = \frac{1}{2} m_A{}^2 \left(1 - \frac{m_C{}^2}{m_A{}^2}\right)\left(1 - \frac{m_E{}^2}{m_C{}^2}\right)\left(1 + cos\theta\right) \tag{12}$$

Q is minimum when θ is equal to π while it is maximum for θ equal to 0. Also, the minimum value of Q is zero while the maximum value is $m_A\sqrt{\left(1 - \frac{m_C{}^2}{m_A{}^2}\right)\left(1 - \frac{m_E{}^2}{m_C{}^2}\right)}$.

3 Invariant mass distribution

It is important to note that the variable θ in the kinematic equation (12) is the decay angle of particle B as seen from the rest frame of particle C. Since the particles involved in the decay chain are all spinless, the decay probability of particle C is completely isotropic. This means that $\frac{dN}{d\Omega}$ is a constant (where dN is the number of B particles produced by the decay of C particles in the solid angle $d\Omega$).

$$\therefore \frac{dN}{d\Omega} = constant$$

$$d\Omega = sin\theta d\theta d\phi$$

Since the system is rotationally symmetric around the z-axis we can integrate over $d\phi$ to get $dN = \alpha d\left(cos\theta\right)$ where α is the proportionality constant. From (12) we get

$$2QdQ = \frac{1}{2} m_A{}^2 \left(1 - \frac{m_C{}^2}{m_A{}^2}\right)\left(1 - \frac{m_E{}^2}{m_C{}^2}\right) d\left(cos\theta\right)$$

$$2QdQ = \frac{1}{2}{m_A}^2\left(1 - \frac{{m_C}^2}{{m_A}^2}\right)\left(1 - \frac{{m_E}^2}{{m_C}^2}\right)\frac{dn}{\alpha}$$

Thus we can see that dN varies linearly with Q. When we plot the invariant mass distribution of B and D we are actually plotting dN at different values of Q. In order to make this plot we divide the entire range of Q from 0 to $m_A\sqrt{\left(1 - \frac{{m_C}^2}{{m_A}^2}\right)\left(1 - \frac{{m_E}^2}{{m_C}^2}\right)}$ into small bins of width dQ and for each such bin we plot the corresponding value of dN. Since we have demonstrated that dN varies linearly with Q we can claim that the mass distribution forms a perfect triangle.

4 Conclusion

The above exercise in relativistic kinematics has applications in the phenomenology of supersymmetry. The SUSY particles form decay chains similar to the one under consideration and the triangular signature of the invariant mass distribution of decay products (two leptons) may serve as a strong indicator for discovering supersymmetry at the LHC.