Solutions to HW # 3

1. To simplify the given integral, replace $\sin^2\theta$ by $\frac{1}{2}(1-\cos 2\theta)$, with the result that

$$I = \int_0^{\pi} \frac{d\theta}{(a+\frac{1}{2}b) - \frac{1}{2}b\cos 2\theta}.$$

Changing the variable, we may write

$$I = \int_0^{2\pi} \frac{(1/2)\,d\theta'}{(a+\frac{1}{2}b) - \frac{1}{2}b\cos\theta'}.$$

Now, follow the procedure as in Example 7.2.1 of Arfken, to get

$$I = \frac{(1/2)\cdot 2\pi}{\sqrt{(a+\frac{1}{2}b)^2 - (\frac{1}{2}b)^2}} = \frac{\pi}{\sqrt{a(a+b)}}. \checkmark$$

<u>Restrictions on a & b</u>: $\quad |a+\frac{1}{2}b| > |\frac{1}{2}b|$,

which boils down to the requirement that $a(a+b) > 0$. ✓

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If one does not replace $\sin^2\theta$ by $\frac{1}{2}(1-\cos 2\theta)$ and keeps it as it is, one has to work a bit harder!

Substituting $z = e^{i\theta}$, etc., the given integral assumes the form

$$I = \oint_C \frac{(1/2)\,(dz/iz)}{a + b\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2} = \oint_C \frac{2iz\,dz}{bz^4 - 2(2a+b)z^2 + b},$$

where C is a unit circle centered at the origin.

To simplify this integral, substitute $\underline{\underline{z^2 = z'}}$, to get

$$I = \oint_{C'} \frac{i\,dz'}{bz'^2 - 2(2a+b)z' + b},$$

where C' is $\underline{\text{again}}$ a unit circle centered at the origin but the contour now goes around the origin $\boxed{\text{TWICE}}$!!
The integrand has simple poles at

$$z' = \left(2\frac{a}{b} + 1\right) \pm 2\sqrt{\frac{a^2}{b^2} + \frac{a}{b}} \quad ; \quad z'_1 z'_2 = 1.$$

Assuming that a & b have the same sign (so that the ratio a/b is +ve), we find that z'_1 is outside the circle C' and z'_2 inside. Now, the residue at z'_2 is

$$\frac{i}{b(z'_2 - z'_1)} = \frac{(i/b)}{-4\sqrt{\frac{a^2}{b^2} + \frac{a}{b}}} = \frac{-i}{4\sqrt{a(a+b)}}.$$

It follows that

$$I = \boxed{2} \cdot 2\pi i \frac{-i}{4\sqrt{a(a+b)}} = \frac{\pi}{\sqrt{a(a+b)}}.$$ ✓ Continued →

If we stick to the integral at the top of the last page, then we have 4 simple poles at

$$z_{1,2} = \pm\left[\sqrt{\frac{a}{b}+1} + \sqrt{\frac{a}{b}}\right] \quad \& \quad z_{3,4} = \pm\left[\sqrt{\frac{a}{b}+1} - \sqrt{\frac{a}{b}}\right].$$

Assuming that $(a/b) > 0$, the poles z_1 & z_2 are outside the contour & the poles z_3 & z_4 inside.

Writing the integrand as

$$\frac{2iz}{b\,(z-z_1)(z-z_2)(z-z_3)(z-z_4)},$$

the residue at z_3 is

$$\frac{2iz_3}{b\,(z_3-z_1)(z_3-z_2)(z_3-z_4)}.$$

Since $z_3 = -z_4$, $z_3/(z_3-z_4) = 1/2$. At the same time $(z_3 - z_1) = -2\sqrt{\frac{a}{b}}$ whereas $(z_3-z_2) = 2\sqrt{\frac{a}{b}+1}$.

Res. at z_3, therefore, is $\dfrac{1}{4i\sqrt{a(a+b)}}$.

Res. at z_4 turns out to be the same.

$$\therefore\ I = 2\pi i\left[\frac{1}{4i\sqrt{a(a+b)}} + \frac{1}{4i\sqrt{a(a+b)}}\right] = \frac{\pi}{\sqrt{a(a+b)}}$$ ✓

2. $f(z) = \dfrac{z^2 - z + 2}{(z^2+1)(z^2+9)} = \dfrac{z^2 - z + 2}{(z+i)(z-i)(z+3i)(z-3i)}$, with

simple poles at $z = \pm i, \pm 3i$.

The integral here is of the type I_2, with $|f(z)|$ going to zero faster than $\dfrac{1}{|z|}$ as $|z| \longrightarrow \infty$. Hence,

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$$

$$= 2\pi i \left(\text{res. at } z = i + \text{res. at } z = 3i \right)$$

$$= 2\pi i \left[\frac{-1-i+2}{2i \cdot 4i \cdot -2i} + \frac{-9-3i+2}{4i \cdot 2i \cdot 6i} \right]$$

$$= 2\pi i \left[\frac{1-i}{16i} + \frac{-7-3i}{-48i} \right]$$

$$= \frac{2\pi i}{48i} \left[(3-3i) - (-7-3i) \right]$$

$$= \frac{5\pi}{12}.$$

3. $$I = \int_0^\infty \frac{\ln x}{x^\beta(1+x)}\,dx = -\frac{\partial}{\partial\beta}\int_0^\infty \frac{x^{-\beta}}{1+x}\,dx.$$

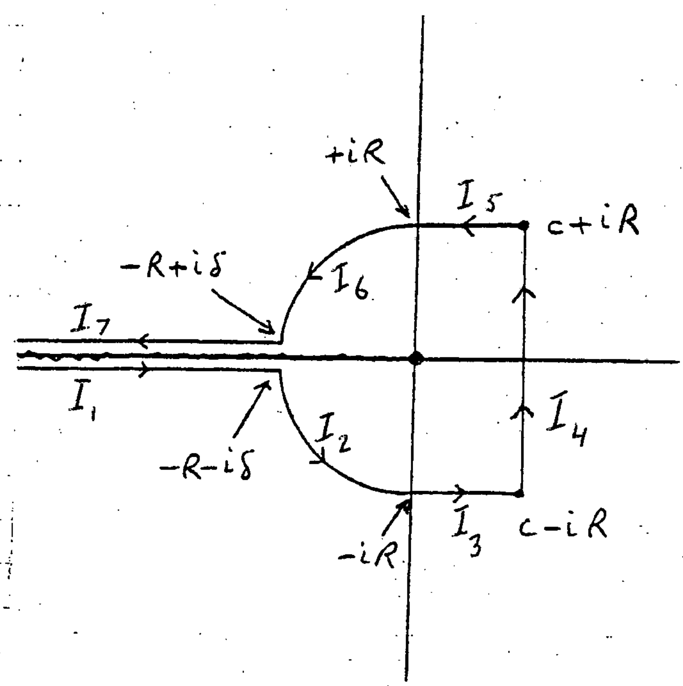
The latter integral is precisely $\int_0^\infty \frac{x^{\mu-1}}{1+x}\,dx$, with $\mu = 1-\beta$. It follows that

$$\int_0^\infty \frac{x^{-\beta}}{1+x}\,dx = \left(\frac{\pi}{\sin(\pi\mu)}\right)_{\mu=1-\beta} = \frac{\pi}{\sin(\pi\beta)} \qquad (0<\beta<1).$$

$$\therefore\ I = \frac{\pi^2\cos(\pi\beta)}{\sin^2(\pi\beta)}.$$

4. (a) Modifying Hankel's loop as shown below (with c a real and positive constant, $\delta \to 0$ and $R \to \infty$), deduce that for $\mathrm{Re}\, z > 0$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^t\, t^{-z}\, dt \qquad \text{(Laplace)}.$$



Our $I_c = \dfrac{1}{2\pi i} \displaystyle\int_C e^t\, t^{-z}\, dt = \sum_{i=1}^{7} I_i$.

As $R \to \infty$, I_1 & I_7 simply disappear.

I_2 & $I_6 \to 0$ because (writing $t = \tau e^{i\varphi}$) the magnitude of these integrals is controlled by the factor $e^{\tau \cos\varphi}$, with $\tau \to \infty$ and $\cos\varphi$ negative.

I_3 & $I_5 \to 0$ because these integrals are of the form $\int_{(0,c)} e^{x \pm iR} \, (x \pm iR)^{-z} \, dx$, with the controlling factor

$(R^2+x^2)^{-\frac{1}{2}\operatorname{Re} z}$ and $\operatorname{Re} z$ positive.

We are thus left with only I_4, which leads to the desired result.

(b) Show that, with a suitable substitution, the above result can be expressed in the form

$$\frac{1}{\Gamma(z)} = \frac{e^c c^{1-z}}{\pi} \int_0^{\pi/2} \cos(c \tan\theta - z\theta) \cos^{z-2}\theta \, d\theta .$$

Looking closely at the limits of integration in the Laplace integral (and keeping an eye on the integrand of the final result), we try the substitution

$$t = \underline{c + ic\tan\theta} = \underline{c \sec\theta \, e^{i\theta}}, \quad \text{to get}$$

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} e^{c + ic\tan\theta} \left(c \sec\theta \, e^{i\theta}\right)^{-z} ic \sec^2\theta \, d\theta$$

$$= \frac{e^c c^{1-z}}{2\pi} \int_{-\pi/2}^{\pi/2} e^{i(c\tan\theta - z\theta)} (\sec\theta)^{2-z} \, d\theta .$$

Writing $e^{i(\cdots)}$ as $\cos(\cdots) + i\sin(\cdots)$, we readily see that the sine-part vanishes and the cosine-part yields the desired result.

5. $$I = \int_{-\infty}^{\infty} \frac{\cos(bx)}{x^2+a^2}\,dx = \mathrm{Re} \int_{-\infty}^{\infty} \frac{e^{ibx}}{x^2+a^2}\,dx.$$

So, we consider the contour integral

$$I_C = \oint_C \frac{e^{ibz}}{z^2+a^2}.$$

{Assuming $b>0$, choose C as shown in Fig. 7.6 of Arfken.

As $|z| \longrightarrow \infty$, $f(z) \longrightarrow 0$ fast enough to make $I_R \longrightarrow 0$.
[Jordan's lemma need not be invoked] !!!

Our contour integral is, therefore, exactly equal to

$$\int_{-\infty}^{\infty} \frac{e^{ibx}}{x^2+a^2}\,dx.$$

We have two simple poles at $z = ia$ & $-ia$; assuming $a>0$, only the former one counts. The residue at $z = ia$ is

$$\frac{e^{ib(ia)}}{(2z)_{z=ia}} = \frac{e^{-ab}}{2ia}.$$

It follows that $I_C = 2\pi i \cdot \dfrac{e^{-ab}}{2ia} = \dfrac{\pi}{a} e^{-ab}$ ✓ $(a, b > 0)$.

{ If b were $-$ve, you have to choose the "mirror contour" C', etc.
The form of the given integral, containing $\cos(bx)$, shows that the integral is insensitive to the sign of b. Hence the result.

We now consider $I_C = \oint_C \frac{e^{ibz}}{z^2 - a^2}$ $(b>0)$.

We now have <u>two simple poles</u>, $x = a$ & $x = -a$, right <u>on</u> the contour. Using Cauchy's principal value, we have

$$\mathcal{P} I_C = \pi i \left[\frac{e^{iba}}{2a} + \frac{e^{-iba}}{-2a} \right]$$

$$= \frac{\pi i}{2a} \cdot 2i \sin(ab) = -\frac{\pi}{a} \sin(ab).$$

If b were $-$ve, the result would still be the same — with b replaced by $|b|$.

Finally, the integral $\int_{-\infty}^{\infty} \frac{\cos(bx) - \cos(cx)}{x^2} dx$ is given by

$$\lim_{a \to 0} \left[\frac{\pi}{a} e^{-a|b|} - \frac{\pi}{a} e^{-a|c|} \right]$$

or by

$$\lim_{a \to 0} \left[\frac{-\pi}{a} \sin(a|b|) - \frac{-\pi}{a} \sin(a|c|) \right].$$

Either way, we get $[-\pi|b|] - [-\pi|c|] = \pi[|c| - |b|]$.

6. The function $f(z) = e^{iz}$ is analytic in the upper-half plane and on the real axis. Being equal to $e^{-y}(\cos x + i\sin x)$, it clearly goes to zero as $y \to +\infty$; accordingly, I_R will vanish as $R \to \infty$. Conditions conforming to dispersion relations are thus satisfied!

We now want to verify that the real part, $\cos x$, and the imaginary part, $\sin x$, of e^{ix} do satisfy the dispersion relations, that is,

$$\cos x_0 = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin x}{x - x_0}\, dx \quad \text{or} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin x - \sin x_0}{x - x_0}\, dx \qquad (1,2)$$

AND

$$\sin x_0 = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\cos x}{x - x_0}\, dx \quad \text{or} \quad \frac{-1}{\pi} \int_{-\infty}^{\infty} \frac{\cos x - \cos x_0}{x - x_0}\, dx. \qquad (3,4)$$

Substituting $x = x_0 + \xi$, the right-hand side of (1) becomes

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\sin x_0 \cos\xi + \cos x_0 \sin\xi}{\xi}\, d\xi .$$

Now,

$$P \int_{-\infty}^{\infty} \frac{\cos\xi}{\xi}\, d\xi = 0 \quad (\text{why?})$$

&

$$\int_{-\infty}^{\infty} \frac{\sin\xi}{\xi}\, d\xi = \pi \quad (\text{done in the class}).$$

Hence the result.

Similarly, the right-hand side of (3) becomes

$$-\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\cos x_0 \cos\xi - \sin x_0 \sin\xi}{\xi}\,d\xi$$

$= \sin x_0$, as desired.

This much should have been enough, but let us look at what (2) gives, namely

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin x_0(\cos\xi - 1) + \cos x_0 \sin\xi}{\xi}\,d\xi,$$

with the same result as obtained in (1). No P-value, though!

Similarly, (4) leads to the same result as (3).