Chapter 15

Special Relativity

For an extraordinarily lucid, if characteristically brief, discussion, see chs. 1 and 2 of L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (*Course of Theoretical Physics, vol. 2*).

15.1 Introduction

All distances are relative in physics. They are measured with respect to a fixed *frame of reference*. Frames of reference in which free particles move with constant velocity are called *inertial frames*. The *principle of relativity* states that the laws of Nature are identical in all inertial frames.

15.1.1 Michelson-Morley experiment

We learned how sound waves in a fluid, such as air, obey the Helmholtz equation. Let us restrict our attention for the moment to solutions of the form $\phi(x,t)$ which do not depend on y or z. We then have a one-dimensional wave equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2}\ . \tag{15.1}$$

The fluid in which the sound propagates is assumed to be at rest. But suppose the fluid is not at rest. We can investigate this by shifting to a moving frame, defining $x' = x - ut$, with $y' = y$, $z' = z$ and of course $t' = t$. This is a Galilean transformation. In terms of the new variables, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad , \qquad \frac{\partial}{\partial t} = -u\,\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\ . \tag{15.2}$$

The wave equation is then

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial^2\phi}{\partial x'^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t'^2} - \frac{2u}{c^2}\,\frac{\partial^2\phi}{\partial x'\,\partial t'}\ . \tag{15.3}$$

Clearly the wave equation acquires a different form when expressed in the new variables (x', t'), *i.e.* in a frame in which the fluid is not at rest. The general solution is then of the modified d'Alembert form,

$$\phi(x', t') = f(x' - c_{\mathrm{R}} t') + g(x' + c_{\mathrm{L}} t') \ , \tag{15.4}$$

where $c_{\mathrm{R}} = c - u$ and $c_{\mathrm{L}} = c + u$ are the speeds of rightward and leftward propagating disturbances, respectively. Thus, there is a *preferred frame of reference* – the frame in which the fluid is at rest. In the rest frame of the fluid, sound waves travel with velocity c in either direction.

Light, as we know, is a wave phenomenon in classical physics. The propagation of light is described by Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{15.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \ , \tag{15.6}$$

where ρ and $\boldsymbol{j}$ are the local charge and current density, respectively. Taking the curl of Faraday's law, and restricting to free space where $\rho = \boldsymbol{j} = 0$, we once again have (using a Cartesian system for the fields) the wave equation,

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \ . \tag{15.7}$$

(We shall discuss below, in section 15.8, the beautiful properties of Maxwell's equations under general coordinate transformations.)

In analogy with the theory of sound, it was assumed prior to Einstein that there was in fact a preferred reference frame for electromagnetic radiation – one in which the medium which was excited during the EM wave propagation was at rest. This notional medium was called the *lumineferous ether*. Indeed, it was generally assumed during the 19^{th} century that light, electricity, magnetism, and heat (which was not understood until Boltzmann's work in the late 19^{th} century) all had separate ethers. It was Maxwell who realized that light, electricity, and magnetism were all unified phenomena, and accordingly he proposed a single ether for electromagnetism. It was believed at the time that the earth's motion through the ether would result in a drag on the earth.

In 1887, Michelson and Morley set out to measure the changes in the speed of light on earth due to the earth's movement through the ether (which was generally assumed to be at rest in the frame of the Sun). The Michelson interferometer is shown in fig. 15.1, and works as follows. Suppose the apparatus is moving with velocity $u\,\hat{\boldsymbol{x}}$ through the ether. Then the time it takes a light ray to travel from the half-silvered mirror to the mirror on the right and back again is

$$t_x = \frac{\ell}{c+u} + \frac{\ell}{c-u} = \frac{2\ell c}{c^2 - u^2} \ . \tag{15.8}$$

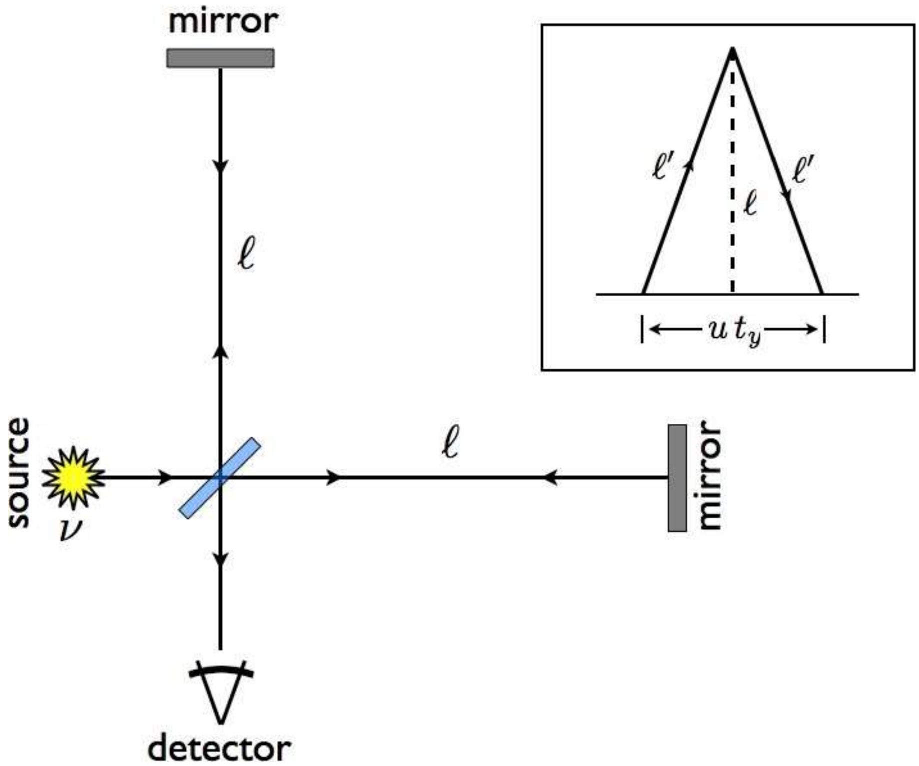

Figure 15.1: The Michelson-Morley experiment (1887) used an interferometer to effectively measure the time difference for light to travel along two different paths. Inset: analysis for the y-directed path.

For motion along the other arm of the interferometer, the geometry in the inset of fig. 15.1 shows $\ell' = \sqrt{\ell^2 + \frac{1}{4}u^2 t_y^2}$, hence

$$t_y = \frac{2\ell'}{c} = \frac{2}{c}\sqrt{\ell^2 + \tfrac{1}{4}u^2 t_y^2} \qquad \Rightarrow \qquad t_y = \frac{2\ell}{\sqrt{c^2 - u^2}} \ . \tag{15.9}$$

Thus, the difference in times along these two paths is

$$\Delta t = t_x - t_y = \frac{2\ell c}{c^2} - \frac{2\ell}{\sqrt{c^2 - u^2}} \approx \frac{\ell}{c} \cdot \frac{u^2}{c^2} \ . \tag{15.10}$$

Thus, the difference in phase between the two paths is

$$\frac{\Delta\phi}{2\pi} = \nu\,\Delta t \approx \frac{\ell}{\lambda} \cdot \frac{u^2}{c^2} \ , \tag{15.11}$$

where λ is the wavelength of the light. We take $u \approx 30\,\mathrm{km/s}$, which is the earth's orbital velocity, and $\lambda \approx 5000\,\text{Å}$. From this we find that $\Delta\phi \approx 0.02 \times 2\pi$ if $\ell = 1\,\mathrm{m}$. Michelson and Morley found that the observed fringe shift $\Delta\phi/2\pi$ was approximately 0.02 times the expected value. The inescapable conclusion was that the speed of light did not depend on the motion of the source. This was very counterintuitive!

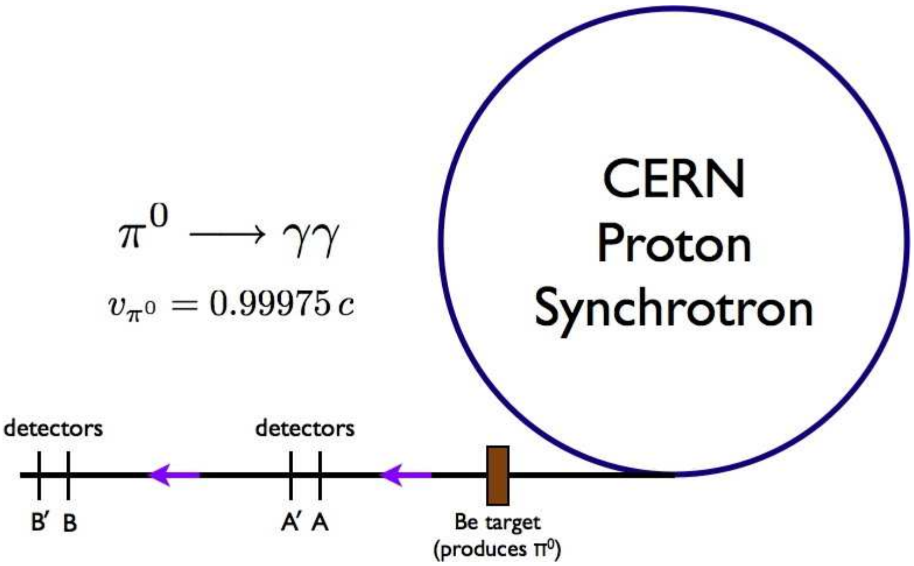

Figure 15.2: Experimental setup of Alvager *et al.* (1964), who used the decay of high energy neutral pions to test the source velocity dependence of the speed of light.

The history of the development of special relativity is quite interesting, but we shall not have time to dwell here on the many streams of scientific thought during those exciting times. Suffice it to say that the Michelson-Morley experiment, while a landmark result, was not the last word. It had been proposed that the ether could be dragged, either entirely or partially, by moving bodies. If the earth dragged the ether along with it, then there would be no ground-level 'ether wind' for the MM experiment to detect. Other experiments, however, such as stellar aberration, in which the apparent position of a distant star varies due to the earth's orbital velocity, rendered the "ether drag" theory untenable – the notional 'ether bubble' dragged by the earth could not reasonably be expected to extend to the distant stars.

A more recent test of the effect of a moving source on the speed of light was performed by T. Alvåger *et al.*, *Phys. Lett.* **12**, 260 (1964), who measured the velocity of γ-rays (photons) emitted from the decay of highly energetic neutral pions (π^0). The pion energies were in excess of 6 GeV, which translates to a velocity of $v = 0.99975\,c$, according to special relativity. Thus, photons emitted in the direction of the pions should be traveling at close to $2c$, if the source and photon velocities were to add. Instead, the velocity of the photons was found to be $c = 2.9977 \pm 0.0004 \times 10^{10}$ cm/s, which is within experimental error of the best accepted value.

15.1.2 Einsteinian and Galilean relativity

The *Principle of Relativity* states that the laws of nature are the same when expressed in any inertial frame. This principle can further be refined into two classes, depending on whether one takes the velocity of the propagation of interactions to be finite or infinite.

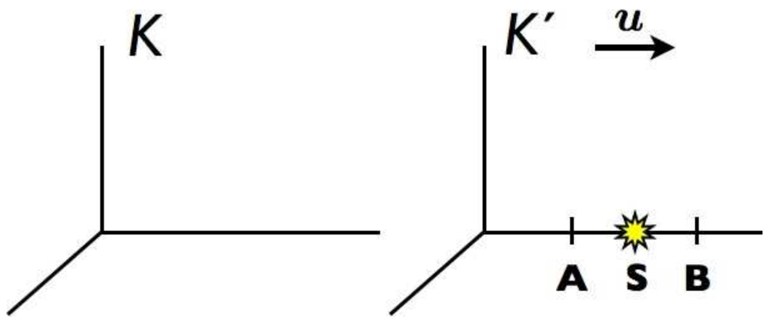

Figure 15.3: Two reference frames.

The interaction of matter in classical mechanics is described by a potential function $U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$. Typically, one has two-body interactions in which case one writes $U = \sum_{i<j} U(\boldsymbol{r}_i, \boldsymbol{r}_j)$. These interactions are thus assumed to be instantaneous, which is unphysical. The interaction of particles is mediated by the exchange of gauge bosons, such as the photon (for electromagnetic interactions), gluons (for the strong interaction, at least on scales much smaller than the 'confinement length'), or the graviton (for gravity). Their velocity of propagation, according to the principle of relativity, is the same in all reference frames, and is given by the speed of light, $c = 2.998 \times 10^8$ m/s.

Since c is so large in comparison with terrestrial velocities, and since d/c is much shorter than all other relevant time scales for typical interparticle separations d, the assumption of an instantaneous interaction is usually quite accurate. The combination of the principle of relativity with finiteness of c is known as Einsteinian relativity. When $c = \infty$, the combination comprises Galilean relativity:

$$\begin{aligned} c < \infty \quad &: \quad \text{Einsteinian relativity} \\ c = \infty \quad &: \quad \text{Galilean relativity} . \end{aligned}$$

Consider a train moving at speed u. In the rest frame of the train track, the speed of the light beam emanating from the train's headlight is $c + u$. This would contradict the principle of relativity. This leads to some very peculiar consequences, foremost among them being the fact that events which are simultaneous in one inertial frame will not in general be simultaneous in another. In Newtonian mechanics, on the other hand, time is absolute, and is independent of the frame of reference. If two events are simultaneous in one frame then they are simultaneous in all frames. This is not the case in Einsteinian relativity!

We can begin to apprehend this curious feature of simultaneity by the following *Gedankenexperiment* (a long German word meaning "thought experiment")[1]. Consider the case in fig. 15.3 in which frame K' moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to frame K. Let a source at S emit a signal (a light pulse) at $t = 0$. In the frame K' the signal's arrival at equidistant locations A and B is simultaneous. In frame K, however, A moves toward left-propagating

[1]Unfortunately, many important physicists were German and we have to put up with a legacy of long German words like *Gedankenexperiment*, *Zitterbewegung*, *Brehmsstrahlung*, *Stosszahlansatz*, *Kartoffelsalat*, *etc.*

emitted wavefront, and B moves away from the right-propagating wavefront. For classical sound, the speed of the left-moving and right-moving wavefronts is $c \mp u$, taking into account the motion of the source, and thus the relative velocities of the signal and the detectors remain at c. But according to the principle of relativity, the speed of light is c in all frames, and is so in frame K for both the left-propagating and right-propagating signals. Therefore, the relative velocity of A and the left-moving signal is $c+u$ and the relative velocity of B and the right-moving signal is $c-u$. Therefore, A 'closes in' on the signal and receives it before B, which is moving away from the signal. We might expect the arrival times to be $t^*_{\rm A} = d/(c+u)$ and $t^*_{\rm B} = d/(c-u)$, where d is the distance between the source S and either detector A or B in the K' frame. Later on we shall analyze this problem and show that

$$t^*_{\rm A} = \sqrt{\frac{c-u}{c+u}} \cdot \frac{d}{c} \qquad , \qquad t^*_{\rm B} = \sqrt{\frac{c+u}{c-u}} \cdot \frac{d}{c} \ . \tag{15.12}$$

Our naïve analysis has omitted an important detail – the *Lorentz contraction* of the distance d as seen by an observer in the K frame.

15.2 Intervals

Now let us express mathematically the constancy of c in all frames. An *event* is specified by the time and place where it occurs. Thus, an event is specified by *four* coordinates, (t, x, y, z). The four-dimensional space spanned by these coordinates is called *spacetime*. The *interval* between two events in spacetime at (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is defined to be

$$s_{12} = \sqrt{c^2(t_1-t_2)^2 - (x_1-x_2)^2 - (y_1-y_2)^2 - (z_1-z_2)^2} \ . \tag{15.13}$$

For two events separated by an infinitesimal amount, the interval ds is infinitesimal, with

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \ . \tag{15.14}$$

Now when the two events denote the emission and reception of an electromagnetic signal, we have $ds^2 = 0$. This must be true in any frame, owing to the invariance of c, hence since ds and ds' are differentials of the same order, we must have $ds'^2 = ds^2$. This last result requires homogeneity and isotropy of space as well. Finally, if infinitesimal intervals are invariant, then integrating we obtain $s = s'$, and we conclude that *the interval between two space-time events is the same in all inertial frames*.

When $s^2_{12} > 0$, the interval is said to be *time-like*. For timelike intervals, we can always find a reference frame in which the two events occur at the same *locations*. As an example, consider a passenger sitting on a train. Event #1 is the passenger yawning at time t_1. Event #2 is the passenger yawning again at some later time t_2. To an observer sitting in the train station, the two events take place at different locations, but in the frame of the passenger, they occur at the same location.

When $s^2_{12} < 0$, the interval is said to be *space-like*. Note that $s_{12} = \sqrt{s^2_{12}} \in i\mathbb{R}$ is pure imaginary, so one says that imaginary intervals are spacelike. As an example, at this

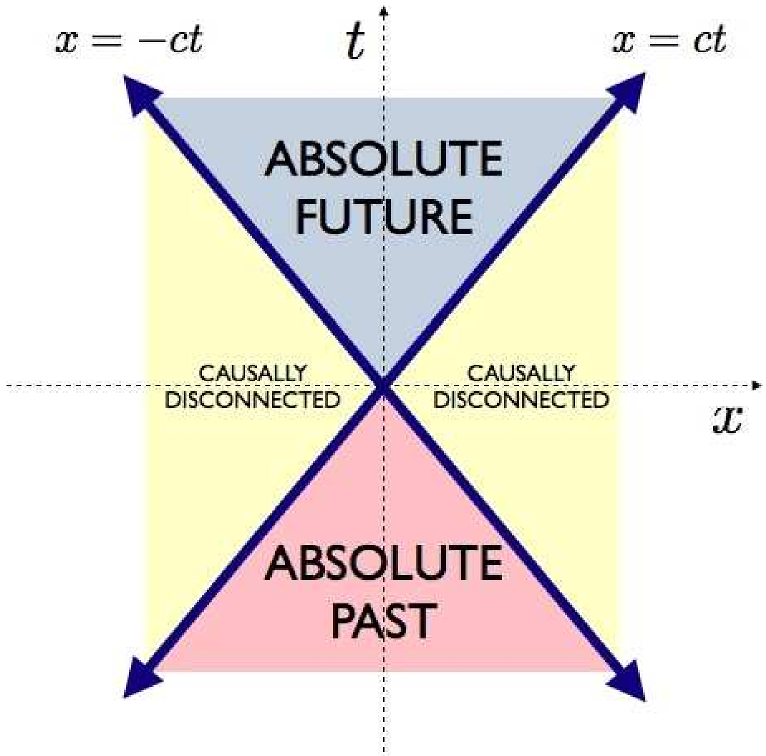

Figure 15.4: A $(1+1)$–dimensional light cone. The forward light cone consists of timelike events with $\Delta t > 0$. The backward light cone consists of timelike events with $\Delta t < 0$. The causally disconnected regions are time-like, and intervals connecting the origin to any point on the light cone itself are light-like.

moment, in the frame of the reader, the North and South poles of the earth are separated by a space-like interval. If the interval between two events is space-like, a reference frame can always be found in which the events are simultaneous.

An interval with $s_{12} = 0$ is said to be *light-like*.

This leads to the concept of the *light cone*, depicted in fig. 15.4. Consider an event E. In the frame of an inertial observer, all events with $s^2 > 0$ and $\Delta t > 0$ are in E's *forward light cone* and are part of his *absolute future*. Events with $s^2 > 0$ and $\Delta t < 0$ lie in E's *backward light cone* are are part of his *absolute past*. Events with spacelike separations $s^2 < 0$ are *causally disconnected* from E. Two events which are causally disconnected can not possible influence each other. Uniform rectilinear motion is represented by a line $t = x/v$ with constant slope. If $v < c$, this line is contained within E's light cone. E is potentially influenced by all events in its backward light cone, *i.e.* its absolute past. It is impossible to find a frame of reference which will transform past into future, or spacelike into timelike intervals.

15.2.1 Proper time

Proper time is the time read on a clock traveling with a moving observer. Consider two observers, one at rest and one in motion. If dt is the differential time elapsed in the rest frame, then

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \tag{15.15}$$

$$= c^2\,dt'^2 \ , \tag{15.16}$$

where dt' is the differential time elapsed on the moving clock. Thus,

$$dt' = dt\,\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \ , \tag{15.17}$$

and the time elapsed on the moving observer's clock is

$$t'_2 - t'_1 = \int\limits_{t_1}^{t_2} \!dt\,\sqrt{1 - \frac{\boldsymbol{v}^2(t)}{c^2}} \ . \tag{15.18}$$

Thus, *moving clocks run slower*. This is an essential feature which is key to understanding many important aspects of particle physics. A particle with a brief lifetime can, by moving at speeds close to c, appear to an observer in our frame to be long-lived. It is customary to define two dimensionless measures of a particle's velocity:

$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{v}}{c} \qquad , \qquad \gamma \equiv \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}} \ . \tag{15.19}$$

As $v \to c$, we have $\beta \to 1$ and $\gamma \to \infty$.

Suppose we wish to compare the elapsed time on two clocks. We keep one clock at rest in an inertial frame, while the other executes a closed path in space, returning to its initial location after some interval of time. When the clocks are compared, the moving clock will show a smaller elapsed time. This is often stated as the "twin paradox." The total elapsed time on a moving clock is given by

$$\tau = \frac{1}{c} \int\limits_a^b \! ds \ , \tag{15.20}$$

where the integral is taken over the *world line* of the moving clock. The elapsed time τ takes on a minimum value when the path from a to b is a straight line. To see this, one can express $\tau\big[\boldsymbol{x}(t)\big]$ as a functional of the path $\boldsymbol{x}(t)$ and extremize. This results in $\ddot{\boldsymbol{x}} = 0$.

15.2.2 Irreverent problem from Spring 2002 final exam

Flowers for Algernon – Bob's beloved hamster, Algernon, is very ill. He has only three hours to live. The veterinarian tells Bob that Algernon can be saved only through a gallbadder transplant. A suitable donor gallbladder is available from a hamster recently pronounced brain dead after a blender accident in New York (miraculously, the gallbladder was unscathed), but it will take Life Flight five hours to bring the precious rodent organ to San Diego.

Bob embarks on a bold plan to save Algernon's life. He places him in a cage, ties the cage to the end of a strong meter-long rope, and whirls the cage above his head while the Life Flight team is *en route*. Bob reasons that *if he can make time pass more slowly for Algernon*, the gallbladder will arrive in time to save his life.

(a) At how many revolutions per second must Bob rotate the cage in order that the gallbladder arrive in time for the life-saving surgery? What is Algernon's speed v_0?

Solution : We have $\beta(t) = \omega_0 R/c$ is constant, therefore, from eqn. 15.18,

$$\Delta t = \gamma\,\Delta t' \ . \tag{15.21}$$

Setting $\Delta t' = 3\,\text{hr}$ and $\Delta t = 5\,\text{hr}$, we have $\gamma = \frac{5}{3}$, which entails $\beta = \sqrt{1-\gamma^{-2}} = \frac{4}{5}$. Thus, $v_0 = \frac{4}{5}\,c$, which requires a rotation frequency of $\omega_0/2\pi = 38.2\,\text{MHz}$.

(b) Bob finds that he cannot keep up the pace! Assume Algernon's speed is given by

$$v(t) = v_0\sqrt{1-\frac{t}{T}} \tag{15.22}$$

where v_0 is the speed from part (a), and $T = 5\,\text{h}$. As the plane lands at the pet hospital's emergency runway, Bob peers into the cage to discover that Algernon is dead! In order to fill out his death report, the veterinarian needs to know: *when did Algernon die?* Assuming he died after his own hamster watch registered three hours, derive an expression for the elapsed time on the veterinarian's clock at the moment of Algernon's death.

Solution : ⟨Sniffle⟩. We have $\beta(t) = \frac{4}{5}\left(1-\frac{t}{T}\right)^{1/2}$. We set

$$T' = \int\limits_0^{T^*}\! dt\,\sqrt{1-\beta^2(t)} \tag{15.23}$$

where $T' = 3\,\text{hr}$ and T^* is the time of death in Bob's frame. We write $\beta_0 = \frac{4}{5}$ and $\gamma_0 = (1-\beta_0^2)^{-1/2} = \frac{5}{3}$. Note that $T'/T = \sqrt{1-\beta_0^2} = \gamma_0^{-1}$.

Rescaling by writing $\zeta = t/T$, we have

$$\begin{aligned}\frac{T'}{T} = \gamma_0^{-1} &= \int\limits_0^{T^*/T}\! d\zeta\,\sqrt{1-\beta_0^2+\beta_0^2\,\zeta} \\ &= \frac{2}{3\beta_0^2}\left[\left(1-\beta_0^2+\beta_0^2\,\frac{T^*}{T}\right)^{3/2} - (1-\beta_0^2)^{3/2}\right] \\ &= \frac{2}{3\gamma_0}\cdot\frac{1}{\gamma_0^2-1}\left[\left(1+(\gamma_0^2-1)\,\frac{T^*}{T}\right)^{3/2} - 1\right] \ . \end{aligned} \tag{15.24}$$

Solving for T^*/T we have

$$\frac{T^*}{T} = \frac{\left(\frac{3}{2}\,\gamma_0^2 - \frac{1}{2}\right)^{2/3} - 1}{\gamma_0^2 - 1} \ . \tag{15.25}$$

With $\gamma_0 = \frac{5}{3}$ we obtain

$$\frac{T^*}{T} = \frac{9}{16}\left[\left(\tfrac{11}{3}\right)^{2/3} - 1\right] = 0.77502\ldots \tag{15.26}$$

Thus, $T^* = 3.875\,\mathrm{hr} = 3$ hr 52 min 50.5 sec after Bob starts swinging.

(c) Identify at least three practical problems with Bob's scheme.

Solution : As you can imagine, student responses to this part were varied and generally sarcastic. *E.g.* "the atmosphere would ignite," or "Bob's arm would fall off," or "Algernon's remains would be found on the inside of the far wall of the cage, squashed flatter than a coat of semi-gloss paint," *etc.*

15.3 Four-Vectors and Lorentz Transformations

We have spoken thus far about different reference frames. So how precisely do the coordinates (t, x, y, z) transform between frames K and K'? In classical mechanics, we have $t = t'$ and $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{u}\,t$, according to fig. 15.3. This yields the *Galilean transformation*,

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_x & 1 & 0 & 0 \\ u_y & 0 & 1 & 0 \\ u_z & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \ . \tag{15.27}$$

Such a transformation does not leave intervals invariant.

Let us define the *four-vector* x^μ as

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} \ . \tag{15.28}$$

Thus, $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. In order for intervals to be invariant, the transformation between x^μ in frame K and $x'^{\,\mu}$ in frame K' must be linear:

$$x^\mu = L^\mu_{\ \nu}\, x'^{\,\nu} \ , \tag{15.29}$$

where we are using the Einstein convention of summing over repeated indices. We define the *Minkowski metric tensor* $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{15.30}$$

Clearly $g = g^{\rm t}$ is a symmetric matrix.

Note that the matrix $L^\alpha_{\ \beta}$ has one raised index and one lowered index. For the notation we are about to develop, it is very important to distinguish raised from lowered indices. To raise or lower an index, we use the metric tensor. For example,

$$x_\mu = g_{\mu\nu}\, x^\nu = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} \ . \tag{15.31}$$

The act of summing over an identical raised and lowered index is called *index contraction*. Note that

$$g^\mu_{\ \nu} = g^{\mu\rho}\, g_{\rho\nu} = \delta^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{15.32}$$

Now let's investigate the invariance of the interval. We must have $x'^{\,\mu}\, x'_\mu = x^\mu\, x_\mu$. Note that

$$\begin{aligned} x^\mu\, x_\mu &= L^\mu_{\ \alpha}\, x'^{\,\alpha}\, L_\mu^{\ \beta}\, x'_\beta \\ &= \big(L^\mu_{\ \alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta}\big)\, x'^{\,\alpha}\, x'^{\,\beta} \ , \end{aligned} \tag{15.33}$$

from which we conclude

$$L^\mu_{\ \alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta} = g_{\alpha\beta} \ . \tag{15.34}$$

This result also may be written in other ways:

$$L^{\mu\alpha}\, g_{\mu\nu}\, L^{\nu\beta} = g^{\alpha\beta} \quad , \quad L^{{\rm t}\,\mu}_{\alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta} = g_{\alpha\beta} \tag{15.35}$$

Another way to write this equation is $L^{\rm t}\, g\, L = g$. A rank-4 matrix which satisfies this constraint, with $g = {\rm diag}(+,-,-,-)$ is an element of the group ${\rm O}(3,1)$, known as the *Lorentz group*.

Let us now count the freedoms in L. As a 4×4 real matrix, it contains 16 elements. The matrix $L^{\rm t}\, g\, L$ is a symmetric 4×4 matrix, which contains 10 independent elements: 4 along the diagonal and 6 above the diagonal. Thus, there are 10 constraints on 16 elements of L, and we conclude that the group ${\rm O}(3,1)$ is 6-dimensional. This is also the dimension of the four-dimensional orthogonal group ${\rm O}(4)$, by the way. Three of these six parameters may be taken to be the Euler angles. That is, the group ${\rm O}(3)$ constitutes a three-dimensional *subgroup* of the Lorentz group ${\rm O}(3,1)$, with elements

$$L^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} \ , \tag{15.36}$$

where $R^{\rm t}R = MI$, *i.e.* $R \in {\rm O}(3)$ is a rank-3 orthogonal matrix, parameterized by the three Euler angles (ϕ, θ, ψ). The remaining three parameters form a vector $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ and define a second class of Lorentz transformations, called boosts:[2]

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\,\hat\beta_x\,\hat\beta_x & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & 1+(\gamma-1)\,\hat\beta_y\,\hat\beta_y & (\gamma-1)\,\hat\beta_y\,\hat\beta_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat\beta_x\,\hat\beta_z & (\gamma-1)\,\hat\beta_y\,\hat\beta_z & 1+(\gamma-1)\,\hat\beta_z\,\hat\beta_z \end{pmatrix} \quad , \tag{15.37}$$

where

$$\widehat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} \qquad , \qquad \gamma = \big(1-\boldsymbol{\beta}^2\big)^{-1/2} \ . \tag{15.38}$$

IMPORTANT : Since the components of $\boldsymbol{\beta}$ are not the spatial components of a four vector, we will only write these components with a lowered index, as β_i, with $i = 1, 2, 3$. We will not write β^i with a raised index, but if we did, we'd mean the same thing, *i.e.* $\beta^i = \beta_i$. Note that for the spatial components of a 4-vector like x^μ, we have $x_i = -x^i$.

Let's look at a simple example, where $\beta_x = \beta$ and $\beta_y = \beta_z = 0$. Then

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{15.39}$$

The effect of this Lorentz transformation $x^\mu = L^{\mu}{}_{\nu}\, x'^{\,\nu}$ is thus

$$ct = \gamma c t' + \gamma\beta x' \tag{15.40}$$

$$x = \gamma\beta c t' + \gamma x' \ . \tag{15.41}$$

How fast is the origin of K' moving in the K frame? We have $dx' = 0$ and thus

$$\frac{1}{c}\,\frac{dx}{dt} = \frac{\gamma\beta\, c\, dt'}{\gamma\, c\, dt'} = \beta \ . \tag{15.42}$$

Thus, $u = \beta c$, *i.e.* $\beta = u/c$.

It is convenient to take advantage of the fact that ${\rm P}^{\boldsymbol{\beta}}_{ij} \equiv \hat\beta_i\,\hat\beta_j$ is a *projection operator*, which satisfies $\big({\rm P}^{\boldsymbol{\beta}}\big)^2 = {\rm P}^{\boldsymbol{\beta}}$. The action of ${\rm P}^{\boldsymbol{\beta}}_{ij}$ on any vector $\boldsymbol{\xi}$ is to project that vector onto the $\hat{\boldsymbol{\beta}}$ direction:

$${\rm P}^{\boldsymbol{\beta}}\,\boldsymbol{\xi} = (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi})\,\hat{\boldsymbol{\beta}} \ . \tag{15.43}$$

We may now write the general Lorentz boost, with $\boldsymbol{\beta} = \boldsymbol{u}/c$, as

$$L = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\rm t} \\ \gamma\boldsymbol{\beta} & {\rm I} + (\gamma-1)\,{\rm P}^{\boldsymbol{\beta}} \end{pmatrix} \ , \tag{15.44}$$

[2] Unlike rotations, the boosts do not themselves define a subgroup of ${\rm O}(3,1)$.

where I is the 3×3 unit matrix, and where we write column and row vectors

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} \quad , \quad \boldsymbol{\beta}^{\rm t} = \begin{pmatrix} \beta_x & \beta_y & \beta_z \end{pmatrix} \tag{15.45}$$

as a mnemonic to help with matrix multiplications. We now have

$$\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\rm t} \\ \gamma\boldsymbol{\beta} & {\rm I} + (\gamma-1)\,{\rm P}^{\boldsymbol{\beta}} \end{pmatrix} \begin{pmatrix} ct' \\ \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} \gamma ct' + \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}' \\ \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,{\rm P}^{\boldsymbol{\beta}}\,\boldsymbol{x}' \end{pmatrix} . \tag{15.46}$$

Thus,

$$ct = \gamma ct' + \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}' \tag{15.47}$$

$$\boldsymbol{x} = \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{x}')\,\hat{\boldsymbol{\beta}} . \tag{15.48}$$

If we resolve $\boldsymbol{x}$ and $\boldsymbol{x}'$ into components parallel and perpendicular to $\boldsymbol{\beta}$, writing

$$x_{\parallel} = \hat{\boldsymbol{\beta}}\cdot\boldsymbol{x} \quad , \quad \boldsymbol{x}_{\perp} = \boldsymbol{x} - (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{x})\,\hat{\boldsymbol{\beta}} , \tag{15.49}$$

with corresponding definitions for $x'_{\parallel}$ and $\boldsymbol{x}'_{\perp}$, the general Lorentz boost may be written as

$$ct = \gamma ct' + \gamma\beta x'_{\parallel} \tag{15.50}$$

$$x_{\parallel} = \gamma\beta ct' + \gamma x'_{\parallel} \tag{15.51}$$

$$\boldsymbol{x}_{\perp} = \boldsymbol{x}'_{\perp} . \tag{15.52}$$

Thus, the components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are parallel to $\boldsymbol{\beta}$ enter into a one-dimensional Lorentz boost along with t and t', as described by eqn. 15.41. The components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are perpendicular to $\boldsymbol{\beta}$ are unaffected by the boost.

Finally, the Lorentz group $\mathrm{O}(3,1)$ is a group under multiplication, which means that if $L_{\rm a}$ and $L_{\rm b}$ are elements, then so is the product $L_{\rm a}\,L_{\rm b}$. Explicitly, we have

$$(L_{\rm a}\,L_{\rm b})^{\rm t}\,g\,L_{\rm a}\,L_{\rm b} = L_{\rm b}^{\rm t}\,(L_{\rm a}^{\rm t}\,g\,L_{\rm a})\,L_{\rm b} = L_{\rm b}^{\rm t}\,g\,L_{\rm b} = g . \tag{15.53}$$

15.3.1 Covariance and contravariance

Note that

$$\begin{aligned} L^{{\rm t}\,\mu}_{\alpha}\; g_{\mu\nu}\, L^{\nu}{}_{\beta} &= \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\alpha\beta} , \end{aligned} \tag{15.54}$$

since $\gamma^2\,(1-\beta^2)=1$. This is in fact the general way that tensors transform under a Lorentz transformation:

$$\text{covariant vectors}:\ \ x^\mu = L^\mu_{\ \nu}\, x'^{\,\nu} \tag{15.55}$$

$$\text{covariant tensors}:\ \ F^{\mu\nu} = L^\mu_{\ \alpha}\, L^\nu_{\ \beta}\, F'^{\,\alpha\beta} = L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} \tag{15.56}$$

Note how index contractions always involve one raised index and one lowered index. Raised indices are called *contravariant indices* and lowered indiced are called *covariant indices*. The transformation rules for contravariant vectors and tensors are

$$\text{contravariant vectors}:\ \ x_\mu = L_\mu^{\ \nu}\, x'_\nu \tag{15.57}$$

$$\text{contravariant tensors}:\ \ F_{\mu\nu} = L_\mu^{\ \alpha}\, L_\nu^{\ \beta}\, F'_{\ \alpha\beta} = L_\mu^{\ \alpha}\, F'_{\ \alpha\beta}\, L^{\mathrm{t}\beta}_{\ \ \nu} \tag{15.58}$$

A *Lorentz scalar* has no indices at all. For example,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu\ , \tag{15.59}$$

is a Lorentz scalar. In this case, we have contracted a tensor with two four-vectors. The dot product of two four-vectors is also a Lorentz scalar:

$$\begin{aligned} a\cdot b \equiv a^\mu\, b_\mu &= g_{\mu\nu}\, a^\mu\, b^\nu \\ &= a^0\, b^0 - a^1\, b^1 - a^2\, b^2 - a^3\, b^3 \\ &= a^0\, b^0 - \boldsymbol{a}\cdot\boldsymbol{b}\ . \end{aligned} \tag{15.60}$$

Note that the dot product $a\cdot b$ of four-vectors is invariant under a simultaneous Lorentz transformation of both a^μ and b^μ, *i.e.* $a\cdot b = a'\cdot b'$. Indeed, this invariance is the very definition of what it means for something to be a Lorentz scalar. Derivatives with respect to covariant vectors yield contravariant vectors:

$$\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f \qquad , \qquad \frac{\partial A^\mu}{\partial x^\nu} = \partial_\nu A^\mu \equiv B^\mu_{\ \nu} \qquad , \qquad \frac{\partial A^\mu_{\ \nu}}{\partial x^\lambda} = \partial_\lambda B^\mu_{\ \nu} \equiv C^\mu_{\ \nu\lambda}$$

et cetera. Note that differentiation with respect to the *co*variant vector x^μ is expressed by the *contra*variant differential operator ∂_μ:

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\ \frac{\partial}{\partial x}\,,\ \frac{\partial}{\partial y}\,,\ \frac{\partial}{\partial z}\right) \tag{15.61}$$

$$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\ -\frac{\partial}{\partial x}\,,\ -\frac{\partial}{\partial y}\,,\ -\frac{\partial}{\partial z}\right)\ . \tag{15.62}$$

The contraction $\Box \equiv \partial^\mu \partial_\mu$ is a Lorentz scalar differential operator, called the *D'Alembertian*:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\ . \tag{15.63}$$

The Helmholtz equation for scalar waves propagating with speed c can thus be written in compact form as $\Box\phi = 0$.

15.3.2 What to do if you hate raised and lowered indices

Admittedly, this covariant and contravariant business takes some getting used to. Ultimately, it helps to keep straight which indices transform according to L (covariantly) and which transform according to $L^{\rm t}$ (contravariantly). If you find all this irksome, the raising and lowering can be safely ignored. We define the position four-vector as before, but with no difference between raised and lowered indices. In fact, we can just represent all vectors and tensors with lowered indices exclusively, writing *e.g.* $\mathrm{x}_\mu = (ct, x, y, z)$. The metric tensor is $\mathrm{g} = \mathrm{diag}(+,-,-,-)$ as before. The dot product of two four-vectors is

$$\mathrm{x}\cdot\mathrm{y} = \mathrm{g}_{\mu\nu}\,\mathrm{x}_\mu\,\mathrm{y}_\nu \ . \tag{15.64}$$

The Lorentz transformation is

$$\mathrm{x}_\mu = \mathrm{L}_{\mu\nu}\,\mathrm{x}'_\nu \ . \tag{15.65}$$

Since this preserves intervals, we must have

$$\begin{aligned}\mathrm{g}_{\mu\nu}\,\mathrm{x}_\mu\,\mathrm{y}_\nu &= \mathrm{g}_{\mu\nu}\,\mathrm{L}_{\mu\alpha}\,\mathrm{x}'_\alpha\,\mathrm{L}_{\nu\beta}\,\mathrm{y}'_\beta \\ &= \big(\mathrm{L}^{\rm t}_{\alpha\mu}\,\mathrm{g}_{\mu\nu}\,\mathrm{L}_{\nu\beta}\big)\,\mathrm{x}'_\alpha\,\mathrm{y}'_\beta \ ,\end{aligned} \tag{15.66}$$

which entails

$$\mathrm{L}^{\rm t}_{\alpha\mu}\,\mathrm{g}_{\mu\nu}\,\mathrm{L}_{\nu\beta} = \mathrm{g}_{\alpha\beta} \ . \tag{15.67}$$

In terms of the quantity $L^\mu{}_\nu$ defined above, we have $\mathrm{L}_{\mu\nu} = L^\mu{}_\nu$. In this convention, we could completely avoid raised indices, or we could simply make no distinction, taking $\mathrm{x}^\mu = \mathrm{x}_\mu$ and $\mathrm{L}_{\mu\nu} = \mathrm{L}^\mu{}_\nu = \mathrm{L}^{\mu\nu}$, *etc.*

15.3.3 Comparing frames

Suppose in the K frame we have a measuring rod which is at rest. What is its length as measured in the K' frame? Recall K' moves with velocity $\boldsymbol{u} = u\,\hat{\boldsymbol{x}}$ with respect to K. From the Lorentz transformation in eqn. 15.41, we have

$$x_1 = \gamma(x'_1 + \beta c\,t'_1) \tag{15.68}$$

$$x_2 = \gamma(x'_2 + \beta c\,t'_2) \ , \tag{15.69}$$

where $x_{1,2}$ are the positions of the ends of the rod in frame K. The rod's length in any frame is the instantaneous spatial separation of its ends. Thus, we set $t'_1 = t'_2$ and compute the separation $\Delta x' = x'_2 - x'_1$:

$$\Delta x = \gamma\,\Delta x' \qquad \Longrightarrow \qquad \Delta x' = \gamma^{-1}\Delta x = \big(1-\beta^2\big)^{1/2}\,\Delta x \ . \tag{15.70}$$

The *proper length* ℓ_0 of a rod is its instantaneous end-to-end separation in its rest frame. We see that

$$\ell(\beta) = \big(1-\beta^2\big)^{1/2}\,\ell_0 \ , \tag{15.71}$$

so the length is always greatest in the rest frame. This is an example of a *Lorentz-Fitzgerald contraction*. Note that the *transverse* dimensions do not contract:

$$\Delta y' = \Delta y \qquad , \qquad \Delta z' = \Delta z \ . \tag{15.72}$$

Thus, the *volume contraction* of a bulk object is given by its length contraction: $\mathcal{V}' = \gamma^{-1}\,\mathcal{V}$.

A striking example of relativistic issues of length, time, and simultaneity is the famous 'pole and the barn' paradox, described in the Appendix (section). Here we illustrate some essential features via two examples.

15.3.4 Example I

Next, let's analyze the situation depicted in fig. 15.3. In the K' frame, we'll denote the following spacetime points:

$$A' = \begin{pmatrix} ct' \\ -d \end{pmatrix} \quad , \quad B' = \begin{pmatrix} ct' \\ +d \end{pmatrix} \quad , \quad S'_- = \begin{pmatrix} ct' \\ -ct' \end{pmatrix} \quad , \quad S'_- = \begin{pmatrix} ct' \\ +ct' \end{pmatrix} \ . \tag{15.73}$$

Note that the origin in K' is given by $O' = (ct', 0)$. Here we are setting $y = y' = z = z' = 0$ and dealing only with one spatial dimension. The points $S'_\pm$ denote the left-moving (S'_-) and right-moving (S'_+) wavefronts. We now use the Lorentz transformation

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta \\ \gamma\,\beta & \gamma \end{pmatrix} \tag{15.74}$$

to transform to the K frame. Thus,

$$S_- = LS'_- = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ -ct' \end{pmatrix} = \gamma\,(1-\beta) \begin{pmatrix} ct' \\ -ct' \end{pmatrix} \tag{15.75}$$

$$S_+ = LS'_+ = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ +ct' \end{pmatrix} = \gamma\,(1+\beta) \begin{pmatrix} ct' \\ +ct' \end{pmatrix} \ . \tag{15.76}$$

We also have

$$A = LA' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ -d \end{pmatrix} = \gamma \begin{pmatrix} ct' - \beta d \\ \beta ct' - d \end{pmatrix} \tag{15.77}$$

$$B = LB' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} ct' \\ +d \end{pmatrix} = \gamma \begin{pmatrix} ct' + \beta d \\ \beta ct' + d \end{pmatrix} \ . \tag{15.78}$$

The signal arrives at A in the K frame when $A = S_-$. The solution is

$$S_- = A = \begin{pmatrix} ct^*_A \\ x^*_A \end{pmatrix} = \begin{pmatrix} \gamma\,(1-\beta)\,d \\ -(1-\beta)\,d \end{pmatrix} \ . \tag{15.79}$$

For the signal to arrive at B, we set

$$S_+ = B = \begin{pmatrix} ct^*_B \\ x^*_B \end{pmatrix} = \begin{pmatrix} \gamma\,(1+\beta)\,d \\ (1+\beta)\,d \end{pmatrix} \ . \tag{15.80}$$

Thus, $t^*_A = \gamma(1-\beta)d/c$ and $t^*_B = \gamma(1+\beta)d/c$. Thus, the two events are *not* simultaneous in K. The arrival at A is first.

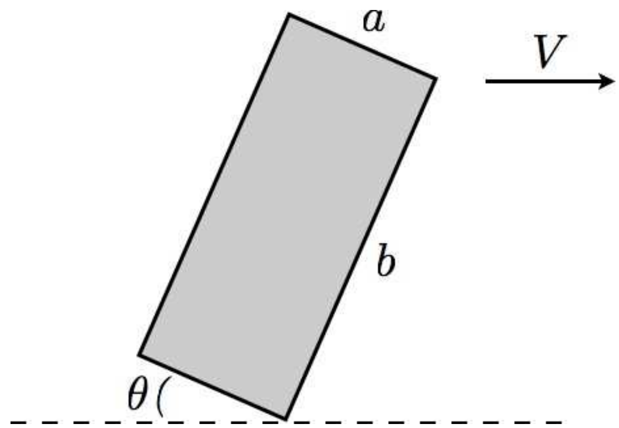

Figure 15.5: A rectangular plate moving at velocity $\boldsymbol{V} = V\,\hat{\boldsymbol{x}}$.

15.3.5 Example II

Consider a rod of length ℓ_0 extending from the origin to the point $\ell_0\,\hat{\boldsymbol{x}}$ at rest in frame K. In the frame K, the two ends of the rod are located at spacetime coordinates

$$A = \begin{pmatrix} ct \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ct \\ \ell_0 \end{pmatrix} \ , \tag{15.81}$$

respectively. Now consider the origin in frame K'. Its spacetime coordinates are

$$C' = \begin{pmatrix} ct' \\ 0 \end{pmatrix} \ . \tag{15.82}$$

To an observer in the K frame, we have

$$C = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct' \\ \gamma\beta ct' \end{pmatrix} \ . \tag{15.83}$$

Now consider two events. The first event is the coincidence of A with C, *i.e.* the origin of K' instantaneously coincides with the origin of K. Setting $A = C$ we obtain $t = t' = 0$. The second event is the coincidence of B with C. Setting $B = C$ we obtain $t = l_0/\beta c$ and $t' = \ell_0/\gamma\beta c$. Note that $t = \ell(\beta)/\beta c$, *i.e.* due to the Lorentz-Fitzgerald contraction of the rod as seen in the K' frame, where $\ell(\beta) = \ell_0/\gamma$.

15.3.6 Deformation of a rectangular plate

Problem: A rectangular plate of dimensions $a \times b$ moves at relativistic velocity $\boldsymbol{V} = V\hat{\boldsymbol{x}}$ as shown in fig. 15.5. In the rest frame of the rectangle, the a side makes an angle θ with respect to the $\hat{\boldsymbol{x}}$ axis. Describe in detail and sketch the shape of the plate as measured by an observer in the laboratory frame. Indicate the lengths of all sides and the values of all interior angles. Evaluate your expressions for the case $\theta = \frac{1}{4}\pi$ and $V = \sqrt{\frac{2}{3}}\,c$.

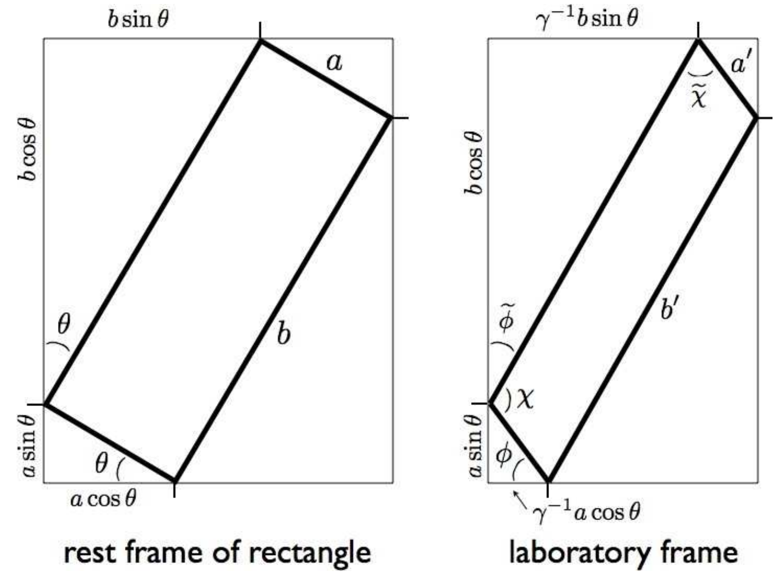

Figure 15.6: Relativistic deformation of the rectangular plate.

Solution: An observer in the laboratory frame will measure lengths parallel to $\hat{\boldsymbol{x}}$ to be Lorentz contracted by a factor γ^{-1}, where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = V/c$. Lengths perpendicular to $\hat{\boldsymbol{x}}$ remain unaffected. Thus, we have the situation depicted in fig. 15.6. Simple trigonometry then says

$$\tan\phi = \gamma\tan\theta \quad , \quad \tan\tilde{\phi} = \gamma^{-1}\tan\theta \ ,$$

as well as

$$\begin{aligned} a' &= a\sqrt{\gamma^{-2}\cos^2\!\theta + \sin^2\!\theta} = a\sqrt{1-\beta^2\cos^2\!\theta} \\ b' &= b\sqrt{\gamma^{-2}\sin^2\!\theta + \cos^2\!\theta} = b\sqrt{1-\beta^2\sin^2\!\theta} \ . \end{aligned}$$

The plate deforms to a parallelogram, with internal angles

$$\begin{aligned} \chi &= \tfrac{1}{2}\pi + \tan^{-1}(\gamma\tan\theta) - \tan^{-1}(\gamma^{-1}\tan\theta) \\ \tilde{\chi} &= \tfrac{1}{2}\pi - \tan^{-1}(\gamma\tan\theta) + \tan^{-1}(\gamma^{-1}\tan\theta) \ . \end{aligned}$$

Note that the area of the plate as measured in the laboratory frame is

$$\begin{aligned} \Omega' &= a'\,b'\,\sin\chi = a'\,b'\,\cos(\phi - \tilde{\phi}) \\ &= \gamma^{-1}\,\Omega \ , \end{aligned}$$

where $\Omega = ab$ is the proper area. The area contraction factor is γ^{-1} and not γ^{-2} (or γ^{-3} in a three-dimensional system) because only the parallel dimension gets contracted.

Setting $V = \sqrt{\frac{2}{3}}\,c$ gives $\gamma = \sqrt{3}$, and with $\theta = \frac{1}{4}\pi$ we have $\phi = \frac{1}{3}\pi$ and $\tilde{\phi} = \frac{1}{6}\pi$. The interior angles are then $\chi = \frac{2}{3}\pi$ and $\tilde{\chi} = \frac{1}{3}\pi$. The side lengths are $a' = \sqrt{\frac{2}{3}}\,a$ and $b' = \sqrt{\frac{2}{3}}\,b$.

15.3.7 Transformation of velocities

Let K' move at velocity $\boldsymbol{u} = c\boldsymbol{\beta}$ relative to K. The transformation from K' to K is given by the Lorentz boost,

$$L^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_x & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & 1+(\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z & 1+(\gamma-1)\,\hat{\beta}_z\,\hat{\beta}_z \end{pmatrix} . \tag{15.84}$$

Applying this, we have

$$dx^{\mu} = L^{\mu}_{\ \nu}\,dx'^{\,\nu} \ . \tag{15.85}$$

This yields

$$dx^0 = \gamma\,dx'^{\,0} + \gamma\,\boldsymbol{\beta}\cdot d\boldsymbol{x}' \tag{15.86}$$

$$d\boldsymbol{x} = \gamma\,\boldsymbol{\beta}\,dx'^{\,0} + d\boldsymbol{x}' + (\gamma-1)\,\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\!\cdot\! d\boldsymbol{x}' \ . \tag{15.87}$$

We then have

$$\begin{aligned} \boldsymbol{V} = c\,\frac{d\boldsymbol{x}}{dx^0} &= \frac{c\,\gamma\,\boldsymbol{\beta}\,dx'^{\,0} + c\,d\boldsymbol{x}' + c\,(\gamma-1)\,\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\!\cdot\! d\boldsymbol{x}'}{\gamma\,dx'^{\,0} + \gamma\,\boldsymbol{\beta}\!\cdot\! d\boldsymbol{x}'} \\ &= \frac{\boldsymbol{u} + \gamma^{-1}\,\boldsymbol{V}' + (1-\gamma^{-1})\,\hat{\boldsymbol{u}}\,\hat{\boldsymbol{u}}\!\cdot\!\boldsymbol{V}'}{1+\boldsymbol{u}\!\cdot\!\boldsymbol{V}'/c^2} \ . \end{aligned} \tag{15.88}$$

The second line is obtained by dividing both numerator and denominator by $dx'^{\,0}$, and then writing $\boldsymbol{V}' = d\boldsymbol{x}'/dx'^{\,0}$. There are two special limiting cases:

$$\text{velocities parallel } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 1\big) \quad \Longrightarrow \quad \boldsymbol{V} = \frac{(u+V')\,\hat{\boldsymbol{u}}}{1+u\,V'/c^2} \tag{15.89}$$

$$\text{velocities perpendicular } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 0\big) \quad \Longrightarrow \quad \boldsymbol{V} = \boldsymbol{u} + \gamma^{-1}\boldsymbol{V}' \ . \tag{15.90}$$

Note that if either u or V' is equal to c, the resultant expression has $|\boldsymbol{V}| = c$ as well. One can't boost the speed of light!

Let's revisit briefly the example in section 15.3.4. For an observer, in the K frame, the relative velocity of S and A is $c-u$, because even though we must boost the velocity $-c\,\hat{\boldsymbol{x}}$ of the left-moving light wave by $u\,\hat{\boldsymbol{x}}$, the result is still $-c\,\hat{\boldsymbol{x}}$, according to our velocity addition formula. Thus, the relative speed of A and S is $c-u$, which means that

$$t^*_A = \frac{d(\beta)}{c+u} = \frac{d}{\gamma}\cdot\frac{1}{c+u} = \frac{d}{\gamma c}\cdot\frac{1-\beta}{1-\beta^2} = \gamma\,(1-\beta)\,\frac{d}{c} \ , \tag{15.91}$$

since $d(\beta) = \gamma^{-1}d$. This result is exactly as found in section 15.3.4 by other means. A corresponding analysis yields $t^*_B = \gamma\,(1+\beta)\,d/c$. again in agreement with the earlier result. Here, it is crucial to account for the Lorentz contraction of the distance between the source S and the observers A and B as measured in the K frame.

15.3.8 Four-velocity and four-acceleration

In nonrelativistic mechanics, the velocity $\boldsymbol{V} = \frac{d\boldsymbol{x}}{dt}$ is locally tangent to a particle's trajectory. In relativistic mechanics, one defines the *four-velocity*,

$$u^\alpha \equiv \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{\sqrt{1-\boldsymbol{\beta}^2}\, c\, dt} = \begin{pmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{pmatrix} \ , \tag{15.92}$$

which is locally tangent to the world line of a particle. Note that

$$g_{\alpha\beta}\, u^\alpha\, u^\beta = 1 \ . \tag{15.93}$$

The four-acceleration is defined as

$$w^\nu \equiv \frac{du^\nu}{ds} = \frac{d^2x^\nu}{ds^2} \ . \tag{15.94}$$

Note that $u \cdot w = 0$, so the 4-velocity and 4-acceleration are orthogonal with respect to the Minkowski metric.

15.4 Three Kinds of Relativistic Rockets

15.4.1 Constant acceleration model

Consider a rocket which undergoes constant acceleration along $\hat{\boldsymbol{x}}$. Clearly the rocket has no rest frame *per se*, because its velocity is changing. However, this poses no serious obstacle to discussing its relativistic motion. We consider a frame K' in which the rocket is *instantaneously* at rest. In such a frame, the rocket's 4-acceleration is $w'^{\,\alpha} = (0, a/c^2)$, where we suppress the transverse coordinates y and z. In an inertial frame K, we have

$$w^\alpha = \frac{d}{ds}\begin{pmatrix} \gamma \\ \gamma\beta \end{pmatrix} = \frac{\gamma}{c}\begin{pmatrix} \dot\gamma \\ \gamma\dot\beta + \dot\gamma\beta \end{pmatrix} \ . \tag{15.95}$$

Transforming $w'^{\,\alpha}$ into the K frame, we have

$$w^\alpha = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ a/c^2 \end{pmatrix} = \begin{pmatrix} \gamma\beta a/c^2 \\ \gamma a/c^2 \end{pmatrix} \ . \tag{15.96}$$

Taking the upper component, we obtain the equation

$$\dot\gamma = \frac{\beta a}{c} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{a}{c} \ , \tag{15.97}$$

the solution of which, with $\beta(0) = 0$, is

$$\beta(t) = \frac{at}{\sqrt{c^2 + a^2t^2}} \qquad , \qquad \gamma(t) = \sqrt{1 + \left(\frac{at}{c}\right)^2} \ . \tag{15.98}$$

The proper time for an observer moving with the rocket is thus

$$\tau = \int_0^t \frac{c\,dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a}\,\sinh^{-1}\left(\frac{at}{c}\right) .$$

For large times $t \gg c/a$, the proper time grows logarithmically in t, which is parametrically slower. To find the position of the rocket, we integrate $\dot{x} = c\beta$, and obtain, with $x(0) = 0$,

$$x(t) = \int_0^t \frac{a\,ct_1\,dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a}\left(\sqrt{c^2 + a^2\,t^2}\; - \; c\right) . \tag{15.99}$$

It is interesting to consider the situation in the frame K'. We then have

$$\beta(\tau) = \tanh(a\tau/c) \quad , \quad \gamma(\tau) = \cosh(a\tau/c) . \tag{15.100}$$

For an observer in the frame K', the distance he has traveled is $\Delta x'(\tau) = \Delta x(\tau)/\gamma(\tau)$, as we found in eqn. 15.70. Now $x(\tau) = (c^2/a)\big(\cosh(a\tau/c) - 1\big)$, hence

$$\Delta x'(\tau) = \frac{c^2}{a}\Big(1 - \mathrm{sech}(a\tau/c)\Big) . \tag{15.101}$$

For $\tau \ll c/a$, we expand $\mathrm{sech}(a\tau/c) \approx 1 - \frac{1}{2}(a\tau/c)^2$ and find $x'(\tau) = \frac{1}{2}a\tau^2$, which clearly is the nonrelativistic limit. For $\tau \to \infty$, however, we have $\Delta x'(\tau) \to c^2/a$ is *finite*! Thus, while the entire Universe is falling behind the accelerating observer, it all piles up at a *horizon* a distance c^2/a behind it, in the frame of the observer. The light from these receding objects is increasingly red-shifted (see section 15.6 below), until it is no longer visible. Thus, as John Baez describes it, the horizon is "a dark plane that appears to be swallowing the entire Universe!" In the frame of the inertial observer, however, nothing strange appears to be happening at all!

15.4.2 Constant force with decreasing mass

Suppose instead the rocket is subjected to a constant force F_0 in its instantaneous rest frame, and furthermore that the rocket's mass satisfies $m(\tau) = m_0(1 - \alpha\tau)$, where τ is the proper time for an observer moving with the rocket. Then from eqn. 15.97, we have

$$\begin{aligned}\frac{F_0}{m_0(1-\alpha\tau)} &= \frac{d(\gamma\beta)}{dt} = \gamma^{-1}\,\frac{d(\gamma\beta)}{d\tau} \\ &= \frac{1}{1-\beta^2}\,\frac{d\beta}{d\tau} = \frac{d}{d\tau}\,\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \quad ,\end{aligned} \tag{15.102}$$

after using the chain rule, and with $d\tau/dt = \gamma^{-1}$. Integrating, we find

$$\ln\left(\frac{1+\beta}{1-\beta}\right) = \frac{2F_0}{\alpha m_0 c}\,\ln\big(1 - \alpha\tau\big) \quad \Longrightarrow \quad \beta(\tau) = \frac{1-(1-\alpha\tau)^r}{1+(1-\alpha\tau)^r} \quad , \tag{15.103}$$

with $r = 2F_0/\alpha m_0 c$. As $\tau \to \alpha^{-1}$, the rocket loses all its mass, and it asymptotically approaches the speed of light.

It is convenient to write

$$\beta(\tau) = \tanh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] , \tag{15.104}$$

in which case

$$\gamma = \frac{dt}{d\tau} = \cosh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] \tag{15.105}$$

$$\frac{1}{c}\,\frac{dx}{d\tau} = \sinh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] . \tag{15.106}$$

Integrating the first of these from $\tau = 0$ to $\tau = \alpha^{-1}$, we find $t^* \equiv t\big(\tau = \alpha^{-1}\big)$ is

$$t^* = \frac{1}{2\alpha}\int_0^1 \!d\sigma\,\Big(\sigma^{-r/2} + \sigma^{r/2}\Big) = \begin{cases} \Big[\alpha^2 - \big(\frac{F_0}{mc}\big)^2\Big]^{-1}\alpha & \text{if } \alpha > \frac{F_0}{mc} \\ \infty & \text{if } \alpha \le \frac{F_0}{mc}\ . \end{cases} \tag{15.107}$$

Since $\beta(\tau = \alpha^{-1}) = 1$, this is the time in the K frame when the rocket reaches the speed of light.

15.4.3 Constant *ejecta* velocity

Our third relativistic rocket model is a generalization of what is commonly known as the *rocket equation* in classical physics. The model is one of a rocket which is continually ejecting burnt fuel at a velocity $-u$ in the instantaneous rest frame of the rocket. The nonrelativistic rocket equation follows from overall momentum conservation:

$$dp_{\rm rocket} + dp_{\rm fuel} = d(mv) + (v-u)\,(-dm) = 0\ , \tag{15.108}$$

since if $dm < 0$ is the differential change in rocket mass, the differential *ejecta* mass is $-dm$. This immediately gives

$$m\,dv + u\,dm = 0 \qquad \Longrightarrow \qquad v = u\,\ln\left(\frac{m_0}{m}\right) , \tag{15.109}$$

where the rocket is assumed to begin at rest, and where m_0 is the initial mass of the rocket. Note that as $m \to 0$ the rocket's speed increases without bound, which of course violates special relativity.

In relativistic mechanics, as we shall see in section 15.5, the rocket's momentum, as described by an inertial observer, is $p = \gamma m v$, and its energy is $\gamma m c^2$. We now write two equations for overall conservation of momentum and energy:

$$d(\gamma m v) + \gamma_{\rm e} v_{\rm e}\, dm_{\rm e} = 0 \tag{15.110}$$

$$d(\gamma m c^2) + \gamma_{\rm e} (dm_{\rm e}\, c^2) = 0 \ , \tag{15.111}$$

where $v_{\rm e}$ is the velocity of the *ejecta* in the inertial frame, $dm_{\rm e}$ is the differential mass of the *ejecta*, and $\gamma_{\rm e} = \big(1 - \frac{v_{\rm e}^2}{c^2}\big)^{-1/2}$. From the second of these equations, we have

$$\gamma_{\rm e}\, dm_{\rm e} = -d(\gamma m) \ , \tag{15.112}$$

which we can plug into the first equation to obtain

$$(v - v_{\rm e})\, d(\gamma m) + \gamma m\, dv = 0 \ . \tag{15.113}$$

Before solving this, we remark that eqn. 15.112 implies that $dm_{\rm e} < |dm|$ – the differential mass of the *ejecta* is less than the mass lost by the rocket! This is Einstein's famous equation $E = mc^2$ at work – more on this later.

To proceed, we need to use the parallel velocity addition formula of eqn. 15.89 to find $v_{\rm e}$:

$$v_{\rm e} = \frac{v - u}{1 - \frac{uv}{c^2}} \qquad \Longrightarrow \qquad v - v_{\rm e} = \frac{u\big(1 - \frac{v^2}{c^2}\big)}{\big(1 - \frac{uv}{c^2}\big)} \ . \tag{15.114}$$

We now define $\beta_u = u/c$, in which case eqn, 15.113 becomes

$$\beta_u \, (1 - \beta^2)\, d(\gamma m) + (1 - \beta \beta_u)\, \gamma m\, d\beta = 0 \ . \tag{15.115}$$

Using $d\gamma = \gamma^3 \beta\, d\beta$, we observe a felicitous cancellation of terms, leaving

$$\beta_u \, \frac{dm}{m} + \frac{d\beta}{1 - \beta^2} = 0 \ . \tag{15.116}$$

Integrating, we obtain

$$\beta = \tanh\left(\beta_u \, \ln \frac{m_0}{m}\right) \ . \tag{15.117}$$

Note that this agrees with the result of eqn. 15.104, if we take $\beta_u = F_0/\alpha m c$.

15.5 Relativistic Mechanics

Relativistic particle dynamics follows from an appropriately extended version of Hamilton's principle $\delta S = 0$. The action S must be a Lorentz scalar. The action for a free particle is

$$S\big[\boldsymbol{x}(t)\big] = -mc \int\limits_a^b ds = -mc^2 \int\limits_{t_a}^{t_b} dt\, \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \ . \tag{15.118}$$

Thus, the free particle Lagrangian is

$$L = -mc^2\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} = -mc^2 + \tfrac{1}{2}m\boldsymbol{v}^2 + \tfrac{1}{8}mc^2\left(\frac{\boldsymbol{v}^2}{c^2}\right)^2 + \ldots \ . \tag{15.119}$$

Thus, L can be written as an expansion in powers of $\boldsymbol{v}^2/c^2$. Note that $L(\boldsymbol{v}=0) = -mc^2$. We interpret this as $-U_0$, where $U_0 = mc^2$ is the *rest energy* of the particle. As a constant, it has no consequence for the equations of motion. The next term in L is the familiar nonrelativistic kinetic energy, $\frac{1}{2}m\boldsymbol{v}^2$. Higher order terms are smaller by increasing factors of $\beta^2 = v^2/c^2$.

We can add a potential $U(\boldsymbol{x},t)$ to obtain

$$L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = -mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}} - U(\boldsymbol{x},t) \ . \tag{15.120}$$

The momentum of the particle is

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m \dot{\boldsymbol{x}} \ . \tag{15.121}$$

The force is $\boldsymbol{F} = -\boldsymbol{\nabla} U$ as usual, and Newton's Second Law still reads $\dot{\boldsymbol{p}} = \boldsymbol{F}$. Note that

$$\dot{\boldsymbol{p}} = \gamma m \left(\dot{\boldsymbol{v}} + \frac{v\dot{v}}{c^2}\,\gamma^2 \boldsymbol{v}\right) \ . \tag{15.122}$$

Thus, the force $\boldsymbol{F}$ is not necessarily in the direction of the acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$. The Hamiltonian, recall, is a function of coordinates and momenta, and is given by

$$H = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2} + U(\boldsymbol{x},t) \ . \tag{15.123}$$

Since $\partial L/\partial t = 0$ for our case, H is conserved by the motion of the particle. There are two limits of note:

$$|\boldsymbol{p}| \ll mc \quad \text{(non-relativistic)} \quad : \ H = mc^2 + \frac{\boldsymbol{p}^2}{2m} + U + \mathcal{O}(p^4/m^4c^4) \tag{15.124}$$

$$|\boldsymbol{p}| \gg mc \quad \text{(ultra-relativistic)} \quad : \ H = c|\boldsymbol{p}| + U + \mathcal{O}(mc/p) \ . \tag{15.125}$$

Expressed in terms of the coordinates and velocities, we have $H = E$, the total energy, with

$$E = \gamma mc^2 + U \ . \tag{15.126}$$

In particle physics applications, one often defines the kinetic energy T as

$$T = E - U - mc^2 = (\gamma - 1)mc^2 \ . \tag{15.127}$$

When electromagnetic fields are included,

$$\begin{aligned} L(\boldsymbol{x},\dot{\boldsymbol{x}},t) &= -mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}} - q\,\phi + \frac{q}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{x}} \\ &= -\gamma mc^2 - \frac{q}{c}\,A_\mu\,\frac{dx^\mu}{dt} \ , \end{aligned} \tag{15.128}$$

where the electromagnetic *4-potential* is $A^\mu = (\phi\,,\,\boldsymbol{A})$. Recall $A_\mu = g_{\mu\nu}\,A^\nu$ has the sign of its spatial components reversed. One the has

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m \dot{\boldsymbol{x}} + \frac{q}{c}\,\boldsymbol{A}\ , \tag{15.129}$$

and the Hamiltonian is

$$H = \sqrt{m^2c^4 + \Big(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\Big)^2} + q\,\phi\ . \tag{15.130}$$

15.5.1 Relativistic harmonic oscillator

From $E = \gamma mc^2 + U$, we have

$$\dot{x}^2 = c^2\left[1 - \left(\frac{mc^2}{E - U(x)}\right)^2\right]\ . \tag{15.131}$$

Consider the one-dimensional harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$. We define the turning points as $x = \pm b$, satisfying

$$E - mc^2 = U(\pm b) = \tfrac{1}{2}kb^2\ . \tag{15.132}$$

Now define the angle θ via $x \equiv b\cos\theta$, and further define the dimensionless parameter $\epsilon = kb^2/4mc^2$. Then, after some manipulations, one obtains

$$\dot{\theta} = \omega_0\,\frac{\sqrt{1 + \epsilon\sin^2\!\theta}}{1 + 2\epsilon\sin^2\!\theta}\ , \tag{15.133}$$

with $\omega_0 = \sqrt{k/m}$ as in the nonrelativistic case. Hence, the problem is reduced to quadratures (a quaint way of saying 'doing an an integral'):

$$t(\theta) - t_0 = \omega_0^{-1}\int\limits_{\theta_0}^{\theta} d\vartheta\ \frac{1 + 2\epsilon\sin^2\!\vartheta}{\sqrt{1 + \epsilon\sin^2\!\vartheta}}\ . \tag{15.134}$$

While the result can be expressed in terms of elliptic integrals, such an expression is not particularly illuminating. Here we will content ourselves with computing the period $T(\epsilon)$:

$$T(\epsilon) = \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}} d\vartheta\ \frac{1 + 2\epsilon\sin^2\!\vartheta}{\sqrt{1 + \epsilon\sin^2\!\vartheta}} \tag{15.135}$$

$$= \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}} d\vartheta\,\Big(1 + \tfrac{3}{2}\epsilon\sin^2\!\vartheta - \tfrac{5}{8}\epsilon^2\sin^4\!\vartheta + \ldots\Big)$$

$$= \frac{2\pi}{\omega_0}\cdot\Big\{1 + \tfrac{3}{4}\epsilon - \tfrac{15}{64}\epsilon^2 + \ldots\Big\}\ . \tag{15.136}$$

Thus, for the relativistic harmonic oscillator, the period does depend on the amplitude, unlike the nonrelativistic case.

15.5.2 Energy-momentum 4-vector

Let's focus on the case where $U(\boldsymbol{x}) = 0$. This is in fact a realistic assumption for subatomic particles, which propagate freely between collision events.

The differential proper time for a particle is given by

$$d\tau = \frac{ds}{c} = \gamma^{-1}\, dt \ , \tag{15.137}$$

where $x^\mu = (ct, \boldsymbol{x})$ are coordinates for the particle in an inertial frame. Thus,

$$\boldsymbol{p} = \gamma m \dot{\boldsymbol{x}} = m\, \frac{d\boldsymbol{x}}{d\tau} \qquad , \qquad \frac{E}{c} = mc\gamma = m\, \frac{dx^0}{d\tau} \ , \tag{15.138}$$

with $x^0 = ct$. Thus, we can write the *energy-momentum 4-vector* as

$$p^\mu = m\, \frac{dx^\mu}{d\tau} = \begin{pmatrix} E/c \\ p^x \\ p^y \\ p^z \end{pmatrix} \ . \tag{15.139}$$

Note that $p^\nu = mcu^\nu$, where u^ν is the 4-velocity of eqn. 15.92. The four-momentum satisfies the relation

$$p^\mu\, p_\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2 \ . \tag{15.140}$$

The relativistic generalization of force is

$$f^\mu = \frac{dp^\mu}{d\tau} = \big(\gamma \boldsymbol{F}\!\cdot\!\boldsymbol{v}/c\,,\, \gamma\boldsymbol{F}\big) \ , \tag{15.141}$$

where $\boldsymbol{F} = d\boldsymbol{p}/dt$ as usual.

The energy-momentum four-vector transforms covariantly under a Lorentz transformation. This means

$$p^\mu = L^\mu_{\ \nu}\, p'^{\,\nu} \ . \tag{15.142}$$

If frame K' moves with velocity $\boldsymbol{u} = c\beta\, \hat{\boldsymbol{x}}$ relative to frame K, then

$$\frac{E}{c} = \frac{c^{-1}E' + \beta\, p'^x}{\sqrt{1-\beta^2}} \quad , \quad p^x = \frac{p'^x + \beta c^{-1} E'}{\sqrt{1-\beta^2}} \quad , \quad p^y = p'^y \quad , \quad p^z = p'^z \ . \tag{15.143}$$

In general, from eqns. 15.50, 15.51, and 15.52, we have

$$\frac{E}{c} = \gamma \frac{E'}{c} + \gamma\beta p'_\parallel \tag{15.144}$$

$$p_\parallel = \gamma\beta \frac{E}{c} + \gamma p'_\parallel \tag{15.145}$$

$$\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp \tag{15.146}$$

where $p_\parallel = \hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{p}$ and $\boldsymbol{p}_\perp = \boldsymbol{p} - (\hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{p})\,\hat{\boldsymbol{\beta}}$.

15.5.3 4-momentum for massless particles

For a massless particle, such as a photon, we have $p^\mu\, p_\mu = 0$, which means $E^2 = \boldsymbol{p}^2\, c^2$. The 4-momentum may then be written $p^\mu = \big(|\boldsymbol{p}|\,,\,\boldsymbol{p}\big)$. We define the *4-wavevector* k^μ by the relation $p^\mu = \hbar k^\mu$, where $\hbar = h/2\pi$ and h is Planck's constant. We also write $\omega = ck$, with $E = \hbar\omega$.

15.6 Relativistic Doppler Effect

The 4-wavevector $k^\mu = \big(\omega/c\,,\,\boldsymbol{k}\big)$ for electromagnetic radiation satisfies $k^\mu\, k_\mu = 0$. The energy-momentum 4-vector is $p^\mu = \hbar k^\mu$. The phase $\phi(x^\mu) = -k_\mu\, x^\mu = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ of a plane wave is a Lorentz scalar. This means that the total number of wave crests (*i.e.* $\phi = 2\pi n$) emitted by a source will be the total number observed by a detector.

Suppose a moving source emits radiation of angular frequency ω' in its rest frame. Then

$$\begin{aligned} k'^{\,\mu} &= L^\mu_{\ \nu}(-\boldsymbol{\beta})\, k^\nu \\ &= \begin{pmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\,\hat\beta_x\,\hat\beta_x & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_z \\ -\gamma\,\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & 1+(\gamma-1)\,\hat\beta_y\,\hat\beta_y & (\gamma-1)\,\hat\beta_y\,\hat\beta_z \\ -\gamma\,\beta_z & (\gamma-1)\,\hat\beta_x\,\hat\beta_z & (\gamma-1)\,\hat\beta_y\,\hat\beta_z & 1+(\gamma-1)\,\hat\beta_z\,\hat\beta_z \end{pmatrix} \begin{pmatrix} \omega/c \\ k^x \\ k^y \\ k^z \end{pmatrix} . \end{aligned} \tag{15.147}$$

This gives

$$\frac{\omega'}{c} = \gamma\,\frac{\omega}{c} - \gamma\,\boldsymbol{\beta}\cdot\boldsymbol{k} = \gamma\,\frac{\omega}{c}\,(1-\beta\cos\theta)\ , \tag{15.148}$$

where $\theta = \cos^{-1}(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{k}})$ is the angle measured in K between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{k}}$. Solving for ω, we have

$$\omega = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\ \omega_0\ , \tag{15.149}$$

where $\omega_0 = \omega'$ is the angular frequency in the rest frame of the moving source. Thus,

$$\theta = 0 \quad \Rightarrow \quad \text{source approaching} \quad \Rightarrow \quad \omega = \sqrt{\frac{1+\beta}{1-\beta}}\ \omega_0 \tag{15.150}$$

$$\theta = \tfrac{1}{2}\pi \quad \Rightarrow \quad \text{source perpendicular} \quad \Rightarrow \quad \omega = \sqrt{1-\beta^2}\ \omega_0 \tag{15.151}$$

$$\theta = \pi \quad \Rightarrow \quad \text{source receding} \quad \Rightarrow \quad \omega = \sqrt{\frac{1-\beta}{1+\beta}}\ \omega_0\ . \tag{15.152}$$

Recall the non-relativistic Doppler effect:

$$\omega = \frac{\omega_0}{1-(V/c)\cos\theta}\ . \tag{15.153}$$

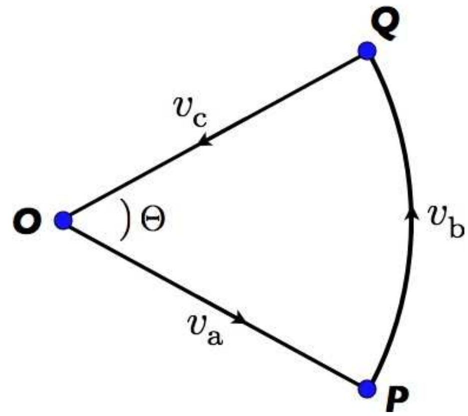

Figure 15.7: Alice's big adventure.

We see that approaching sources have their frequencies shifted higher; this is called the *blue shift*, since blue light is on the high frequency (short wavelength) end of the optical spectrum. By the same token, receding sources are *red-shifted* to lower frequencies.

15.6.1 Romantic example

Alice and Bob have a "May-December" thang going on. Bob is May and Alice December, if you get my drift. The social stigma is too much to bear! To rectify this, they decide that Alice should take a ride in a space ship. Alice's itinerary takes her along a sector of a circle of radius R and angular span of $\Theta = 1$ radian, as depicted in fig. 15.7. Define $\mathrm{O} \equiv (r = 0)$, $\mathrm{P} \equiv (r = R, \phi = -\frac{1}{2}\Theta)$, and $\mathrm{Q} \equiv (r = R, \phi = \frac{1}{2}\Theta)$. Alice's speed along the first leg (straight from O to P) is $v_\mathrm{a} = \frac{3}{5}\,c$. Her speed along the second leg (an arc from P to Q) is $v_\mathrm{b} = \frac{12}{13}\,c$. The final leg (straight from Q to O) she travels at speed $v_\mathrm{c} = \frac{4}{5}\,c$. Remember that the length of an circular arc of radius R and angular spread α (radians) is $\ell = \alpha R$.

(a) Alice and Bob synchronize watches at the moment of Alice's departure. What is the elapsed time on Bob's watch when Alice returns? What is the elapsed time on Alice's watch? What must R be in order for them to erase their initial 30 year age difference?

Solution : In Bob's frame, Alice's trip takes a time

$$\begin{aligned}\Delta t &= \frac{R}{c\beta_\mathrm{a}} + \frac{R\Theta}{c\beta_\mathrm{b}} + \frac{R}{c\beta_\mathrm{c}} \\ &= \frac{R}{c}\left(\tfrac{5}{3} + \tfrac{13}{12} + \tfrac{5}{4}\right) = \frac{4R}{c}\ .\end{aligned} \tag{15.154}$$

The elapsed time on Alice's watch is

$$\begin{aligned}\Delta t' &= \frac{R}{c\gamma_\mathrm{a}\beta_\mathrm{a}} + \frac{R\Theta}{c\gamma_\mathrm{b}\beta_\mathrm{b}} + \frac{R}{c\gamma_\mathrm{c}\beta_\mathrm{c}} \\ &= \frac{R}{c}\left(\tfrac{5}{3}\cdot\tfrac{4}{5} + \tfrac{13}{12}\cdot\tfrac{5}{13} + \tfrac{5}{4}\cdot\tfrac{3}{5}\right) = \frac{5R}{2c}\ .\end{aligned} \tag{15.155}$$

Thus, $\Delta T = \Delta t - \Delta t' = 3R/2c$ and setting $\Delta T = 30\,\mathrm{yr}$, we find $R = 20$ ly. So Bob will have aged 80 years and Alice 50 years upon her return. (Maybe this isn't such a good plan after all.)

(b) As a signal of her undying love for Bob, Alice continually shines a beacon throughout her trip. The beacon produces monochromatic light at wavelength $\lambda_0 = 6000\,\text{Å}$ (frequency $f_0 = c/\lambda_0 = 5 \times 10^{14}$ Hz). Every night, Bob peers into the sky (with a radiotelescope), hopefully looking for Alice's signal. What frequencies f_a, f_b, and f_c does Bob see?

Solution : Using the relativistic Doppler formula, we have

$$\begin{aligned} f_\mathrm{a} &= \sqrt{\frac{1-\beta_\mathrm{a}}{1+\beta_\mathrm{a}}} \times f_0 = \tfrac{1}{2} f_0 \\ f_\mathrm{b} &= \sqrt{1-\beta_\mathrm{b}^2} \times f_0 = \tfrac{5}{13} f_0 \\ f_\mathrm{c} &= \sqrt{\frac{1+\beta_\mathrm{c}}{1-\beta_\mathrm{c}}} \times f_0 = 3 f_0 \ . \end{aligned} \tag{15.156}$$

(c) Show that the total number of wave crests counted by Bob is the same as the number emitted by Alice, over the entire trip.

Solution : Consider first the O–P leg of Alice's trip. The proper time elapsed on Alice's watch during this leg is $\Delta t'_a = R/c\gamma_a\beta_a$, hence she emits $N'_a = Rf_0/c\gamma_a\beta_a$ wavefronts during this leg. Similar considerations hold for the P–Q and Q–O legs, so $N'_b = R\Theta f_0/c\gamma_b\beta_b$ and $N'_c = Rf_0/c\gamma_c\beta_c$.

Although the duration of the O–P segment of Alice's trip takes a time $\Delta t_\mathrm{a} = R/c\beta_\mathrm{a}$ in Bob's frame, he keeps receiving the signal at the Doppler-shifted frequency f_a until the wavefront emitted when Alice arrives at P makes its way back to Bob. That takes an extra time R/c, hence the number of crests emitted for Alice's O–P leg is

$$N_\mathrm{a} = \left(\frac{R}{c\beta_\mathrm{a}} + \frac{R}{c}\right)\sqrt{\frac{1-\beta_\mathrm{a}}{1+\beta_\mathrm{a}}} \times f_0 = \frac{Rf_0}{c\gamma_a\beta_a} = N'_\mathrm{a} \ , \tag{15.157}$$

since the source is receding from the observer.

During the P–Q leg, we have $\theta = \frac{1}{2}\pi$, and Alice's velocity is orthogonal to the wavevector $\boldsymbol{k}$, which is directed radially inward. Bob's first signal at frequency f_b arrives a time R/c after Alice passes P, and his last signal at this frequency arrives a time R/c after Alice passes Q. Thus, the total time during which Bob receives the signal at the Doppler-shifted frequency f_b is $\Delta t_\mathrm{b} = R\Theta/c$, and

$$N_\mathrm{b} = \frac{R\Theta}{c\beta_\mathrm{b}} \cdot \sqrt{1-\beta_\mathrm{b}^2} \times f_0 = \frac{R\Theta f_0}{c\gamma_b\beta_b} = N'_\mathrm{b} \ . \tag{15.158}$$

Finally, during the Q–O home stretch, Bob first starts to receive the signal at the Doppler-shifted frequency f_c a time R/c after Alice passes Q, and he continues to receive the signal until the moment Alice rushes into his open and very flabby old arms when she makes it back to O. Thus, Bob receives the frequency f_c signal for a duration $\Delta t_c - R/c$, where $\Delta t_c = R/c\beta_c$. Thus,

$$N_a = \left(\frac{R}{c\beta_c} - \frac{R}{c}\right)\sqrt{\frac{1+\beta_c}{1-\beta_c}} \times f_0 = \frac{Rf_0}{c\gamma_c\beta_c} = N'_c \ , \tag{15.159}$$

since the source is approaching.

Therefore, the number of wavelengths emitted by Alice will be precisely equal to the number received by Bob – none of the waves gets lost.

15.7 Relativistic Kinematics of Particle Collisions

As should be expected, special relativity is essential toward the understanding of subatomic particle collisions, where the particles themselves are moving at close to the speed of light. In our analysis of the kinematics of collisions, we shall find it convenient to adopt the standard convention on units, where we set $c \equiv 1$. Energies will typically be given in GeV, where $1\,\text{GeV} = 10^9\,\text{eV} = 1.602 \times 10^{-10}\,\text{J}$. Momenta will then be in units of GeV/c, and masses in units of GeV/c^2. With $c \equiv 1$, it is then customary to quote masses in energy units. For example, the mass of the proton in these units is $m_p = 938\,\text{MeV}$, and $m_{\pi^-} = 140\,\text{MeV}$.

For a particle of mass M, its 4-momentum satisfies $P_\mu\, P^\mu = M^2$ (remember $c = 1$). Consider now an observer with 4-velocity U^μ. The energy of the particle, in the rest frame of the observer is $E = P^\mu\, U_\mu$. For example, if $P^\mu = (M, 0, 0, 0)$ is its rest frame, and $U^\mu = (\gamma\, , \gamma\boldsymbol{\beta})$, then $E = \gamma M$, as we have already seen.

Consider next the emission of a photon of 4-momentum $P^\mu = (\hbar\omega/c, \hbar\boldsymbol{k})$ from an object with 4-velocity V^μ, and detected in a frame with 4-velocity U^μ. In the frame of the detector, the photon energy is $E = P^\mu\, U_\mu$, while in the frame of the emitter its energy is $E' = P^\mu\, V_\mu$. If $U^\mu = (1, 0, 0, 0)$ and $V^\mu = (\gamma\, , \gamma\boldsymbol{\beta})$, then $E = \hbar\omega$ and $E' = \hbar\omega' = \gamma\hbar(\omega - \boldsymbol{\beta}\cdot\boldsymbol{k}) = \gamma\hbar\omega(1 - \beta\cos\theta)$, where $\theta = \cos^{-1}\big(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{k}}\big)$. Thus, $\omega = \gamma^{-1}\omega'/(1 - \beta\cos\theta)$. This recapitulates our earlier derivation in eqn. 15.148.

Consider next the interaction of several particles. If in a given frame the 4-momenta of the reactants are P_i^μ, where n labels the reactant 'species', and the 4-momenta of the products are Q_j^μ, then if the collision is elastic, we have that total 4-momentum is conserved, *i.e.*

$$\sum_{i=1}^{N} P_i^\mu = \sum_{j=1}^{\bar{N}} Q_j^\mu \ , \tag{15.160}$$

where there are N reactants and $\bar{N}$ products. For massive particles, we can write

$$P_i^\mu = \gamma_i\, m_i\, (1\, , \boldsymbol{v}_i) \quad , \quad Q_j^\mu = \bar{\gamma}_j\, \bar{m}_j\, (1\, , \bar{v}_j) \ , \tag{15.161}$$

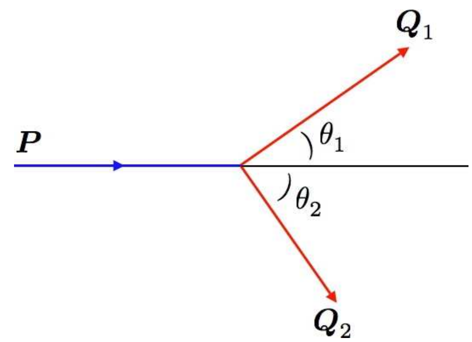

Figure 15.8: Spontaneous decay of a single reactant into two products.

while for massless particles,

$$P_i^\mu = \hbar k_i \left(1\,,\,\hat{\boldsymbol{k}}\right) \quad , \quad Q_j^\mu = \hbar \bar{k}_j \left(1\,,\,\hat{\bar{\boldsymbol{k}}}\right) \; . \tag{15.162}$$

15.7.1 Spontaneous particle decay into two products

Consider first the decay of a particle of mass M into two particles. We have $P^\mu = Q_1^\mu + Q_2^\mu$, hence in the rest frame of the (sole) reactant, which is also called the 'center of mass' (CM) frame since the total 3-momentum vanishes therein, we have $M = E_1 + E_2$. Since $E_i^{\rm CM} = \gamma^{\rm CM} m_i$, and $\gamma_i \geq 1$, clearly we must have $M > m_1 + m_2$, or else the decay cannot possibly conserve energy. To analyze further, write $P^\mu - Q_1^\mu = Q_2^\mu$. Squaring, we obtain

$$M^2 + m_1^2 - 2P_\mu\, Q_1^\mu = m_2^2 \; . \tag{15.163}$$

The dot-product $P \cdot Q_1$ is a Lorentz scalar, and hence may be evaluated in any frame.

Let us first consider the CM frame, where $P^\mu = M(1,0,0,0)$, and $P_\mu\, Q_1^\mu = M E_1^{\rm CM}$, where $E_1^{\rm CM}$ is the energy of $n = 1$ product in the rest frame of the reactant. Thus,

$$E_1^{\rm CM} = \frac{M^2 + m_1^2 - m_2^2}{2M} \qquad , \qquad E_2^{\rm CM} = \frac{M^2 + m_2^2 - m_1^2}{2M} \; , \tag{15.164}$$

where the second result follows merely from switching the product labels. We may now write $Q_1^\mu = (E_1^{\rm CM}, \boldsymbol{p}^{\rm CM})$ and $Q_2^\mu = (E_2^{\rm CM}, -\boldsymbol{p}^{\rm CM})$, with

$$\begin{aligned} (\boldsymbol{p}^{\rm CM})^2 &= (E_1^{\rm CM})^2 - m_1^2 = (E_2^{\rm CM})^2 - m_2^2 \\ &= \left(\frac{M^2 - m_1^2 - m_2^2}{2M}\right)^2 - \left(\frac{m_1\, m_2}{M}\right)^2 \; . \end{aligned} \tag{15.165}$$

In the laboratory frame, we have $P^\mu = \gamma M\,(1\,,\,\boldsymbol{V})$ and $Q_i^\mu = \gamma_i\, m_i\,(1\,,\,\boldsymbol{V}_i)$. Energy and momentum conservation then provide four equations for the six unknowns $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Thus, there is a two-parameter family of solutions, assuming we regard the reactant velocity $\boldsymbol{V}^{\rm K}$ as fixed, corresponding to the freedom to choose $\hat{\boldsymbol{p}}^{\rm CM}$ in the CM frame solution above. Clearly the three vectors $\boldsymbol{V}$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$ must lie in the same plane, and with $\boldsymbol{V}$ fixed, only one additional parameter is required to fix this plane. The other free parameter may be taken to be the relative angle $\theta_1 = \cos^{-1}\big(\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{V}}_1\big)$ (see fig. 15.8). The angle θ_2 as well as the speed V_2 are then completely determined. We can use eqn. 15.163 to relate θ_1 and V_1:

$$M^2 + m_1^2 - m_2^2 = 2Mm_1\gamma\gamma_1\big(1 - VV_1\cos\theta_1\big) \ . \tag{15.166}$$

It is convenient to express both γ_1 and V_1 in terms of the energy E_1:

$$\gamma_1 = \frac{E_1}{m_1} \qquad , \qquad V_1 = \sqrt{1-\gamma_1^{-2}} = \sqrt{1-\frac{m_1^2}{E_1^2}} \ . \tag{15.167}$$

This results in a quadratic equation for E_1, which may be expressed as

$$(1 - V^2\cos^2\!\theta_1)E_1^2 - 2\sqrt{1-V^2}\,E_1^{\rm CM}E_1 + (1-V^2)(E_1^{\rm CM})^2 + m_1^2\,V^2\cos^2\!\theta_1 = 0 \ , \tag{15.168}$$

the solutions of which are

$$E_1 = \frac{\sqrt{1-V^2}\,E_1^{\rm CM} \pm V\cos\theta_1\sqrt{(1-V^2)(E_1^{\rm CM})^2 - (1-V^2\cos^2\!\theta_1)m_1^2}}{1-V^2\cos^2\!\theta_1} \ . \tag{15.169}$$

The discriminant is positive provided

$$\left(\frac{E_1^{\rm CM}}{m_1}\right)^2 > \frac{1-V^2\cos^2\!\theta_1}{1-V^2} \ , \tag{15.170}$$

which means

$$\sin^2\!\theta_1 < \frac{V^{-2}-1}{(V_1^{\rm CM})^{-2}-1} \equiv \sin^2\!\theta_1^* \ , \tag{15.171}$$

where

$$V_1^{\rm CM} = \sqrt{1-\left(\frac{m_1}{E_1^{\rm CM}}\right)^2} \tag{15.172}$$

is the speed of product 1 in the CM frame. Thus, for $V < V_1^{\rm CM} < 1$, the scattering angle θ_1 may take on any value, while for larger reactant speeds $V_1^{\rm CM} < V < 1$ the quantity $\sin^2\!\theta_1$ cannot exceed a critical value.

15.7.2 Miscellaneous examples of particle decays

Let us now consider some applications of the formulae in eqn. 15.164:

- Consider the decay $\pi^0 \to \gamma\gamma$, for which $m_1 = m_2 = 0$. We then have $E_1^{\rm CM} = E_2^{\rm CM} = \frac{1}{2}M$. Thus, with $M = m_{\pi^0} = 135\,{\rm MeV}$, we have $E_1^{\rm CM} = E_2^{\rm CM} = 67.5\,{\rm MeV}$ for the photon energies in the CM frame.

- For the reaction $K^+ \longrightarrow \mu^+ + \nu_\mu$ we have $M = m_{K^+} = 494\,\text{MeV}$ and $m_1 = m_{\mu^-} = 106\,\text{MeV}$. The neutrino mass is $m_2 \approx 0$, hence $E_2^{\text{CM}} = 236\,\text{MeV}$ is the emitted neutrino's energy in the CM frame.

- A Λ^0 hyperon with a mass $M = m_{\Lambda^0} = 1116\,\text{MeV}$ decays into a proton ($m_1 = m_{\text{p}} = 938\,\text{MeV}$) and a pion $m_2 = m_{\pi^-} = 140\,\text{MeV}$). The CM energy of the emitted proton is $E_1^{\text{CM}} = 943\,\text{MeV}$ and that of the emitted pion is $E_2^{\text{CM}} = 173\,\text{MeV}$.

15.7.3 Threshold particle production with a stationary target

Consider now a particle of mass M_1 moving with velocity $\boldsymbol{V}_1 = V_1\,\hat{\boldsymbol{x}}$, incident upon a stationary target particle of mass M_2, as indicated in fig. 15.9. Let the product masses be m_1, m_2, …, $m_{N'}$. The 4-momenta of the reactants and products are

$$P_1^\mu = \big(E_1\,,\,\boldsymbol{P}_1\big) \quad , \quad P_2^\mu = M_2\,\big(1\,,\,\boldsymbol{0}\big) \quad , \quad Q_j^\mu = \big(\varepsilon_j\,,\,\boldsymbol{p}_j\big)\ . \tag{15.173}$$

Note that $E_1^2 - \boldsymbol{P}_1^2 = M_1^2$ and $\varepsilon_j^2 - \boldsymbol{p}_j^2 = m_j^2$, with $j \in \{1, 2, \ldots, N'\}$.

Conservation of momentum means that

$$P_1^\mu + P_2^\mu = \sum_{j=1}^{N'} Q_j^\mu\ . \tag{15.174}$$

In particular, taking the $\mu = 0$ component, we have

$$E_1 + M_2 = \sum_{j=1}^{N'} \varepsilon_j\ , \tag{15.175}$$

which certainly entails

$$E_1 \geq \sum_{j=1}^{N'} m_j - M_2 \tag{15.176}$$

since $\varepsilon_j = \gamma_j\, m_j \geq m_j$. But can the equality ever be achieved? This would only be the case if $\gamma_j = 1$ for all j, *i.e.* the final velocities are all zero. But this itself is quite impossible, since the initial state momentum is $\boldsymbol{P}$.

To determine the threshold energy E_1^{thr}, we compare the length of the total momentum vector in the LAB and CM frames:

$$(P_1 + P_2)^2 = M_1^2 + M_2^2 + 2E_1\, M_2 \qquad \text{(LAB)} \tag{15.177}$$

$$= \left(\sum_{j=1}^{N'} \varepsilon_j^{\text{CM}}\right)^2 \qquad \text{(CM)}\ . \tag{15.178}$$

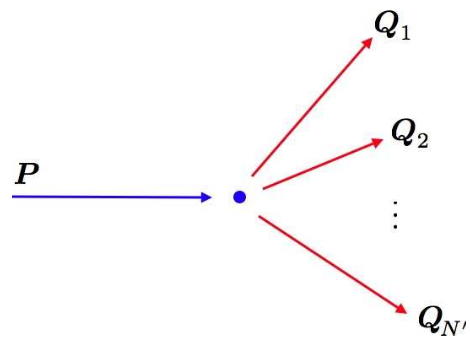

Figure 15.9: A two-particle initial state, with a stationary target in the LAB frame, and an N'-particle final state.

Thus,

$$E_1 = \frac{\left(\sum_{j=1}^{N'} \varepsilon_j^{\rm CM}\right)^2 - M_1^2 - M_2^2}{2M_2} \tag{15.179}$$

and we conclude

$$E_1 \geq E_1^{\rm THR} = \frac{\left(\sum_{j=1}^{N'} m_j\right)^2 - M_1^2 - M_2^2}{2M_2} \ . \tag{15.180}$$

Note that in the CM frame it *is* possible for each $\varepsilon_j^{\rm CM} = m_j$.

Finally, we must have $E_1^{\rm THR} \geq \sum_{j=1}^{N'} m_j - M_2$. This then requires

$$M_1 + M_2 \leq \sum_{j=1}^{N'} m_j \ . \tag{15.181}$$

15.7.4 Transformation between frames

Consider a particle with 4-velocity u^μ in frame K and consider a Lorentz transformation between this frame and a frame K' moving relative to K with velocity $\boldsymbol{V}$. We may write

$$u^\mu = \big(\gamma\,,\,\gamma v\cos\theta\,,\,\gamma v\sin\theta\,\hat{\boldsymbol{n}}_\perp\big) \quad , \quad u'^\mu = \big(\gamma'\,,\,\gamma' v'\cos\theta'\,,\,\gamma' v'\sin\theta'\,\hat{\boldsymbol{n}}'_\perp\big) \ . \tag{15.182}$$

According to the general transformation rules of eqns. 15.50, 15.51, and 15.52, we may write

$$\gamma = \Gamma\,\gamma' + \Gamma V\,\gamma' v'\cos\theta' \tag{15.183}$$

$$\gamma v\cos\theta = \Gamma V\,\gamma' + \Gamma\,\gamma' v'\cos\theta' \tag{15.184}$$

$$\gamma v\sin\theta = \gamma' v'\sin\theta' \tag{15.185}$$

$$\hat{\boldsymbol{n}}_\perp = \hat{\boldsymbol{n}}'_\perp \ , \tag{15.186}$$

where the $\hat{\boldsymbol{x}}$ axis is taken to be $\hat{\boldsymbol{V}}$, and where $\Gamma \equiv (1-V^2)^{-1/2}$. Note that the last two of these equations may be written as a single vector equation for the transverse components.

Dividing the eqn. 15.185 by eqn. 15.184, we obtain the result

$$\tan\theta = \frac{\sin\theta'}{\Gamma\Big(\frac{V}{v'} + \cos\theta'\Big)} \ . \tag{15.187}$$

We can then use eqn. 15.183 to relate v' and $\cos\theta'$:

$$\gamma'^{-1} = \sqrt{1-v'^2} = \frac{\Gamma}{\gamma}\,\big(1 + V v' \cos\theta'\big) \ . \tag{15.188}$$

Squaring both sides, we obtain a quadratic equation whose roots are

$$v' = \frac{-\Gamma^2\,V\cos\theta' \pm \sqrt{\gamma^4 - \Gamma^2\,\gamma^2\,(1-V^2\cos^2\!\theta')}}{\gamma^2 + \Gamma^2\,V^2\cos^2\!\theta'} \ . \tag{15.189}$$

CM frame mass and velocity

To find the velocity of the CM frame, simply write

$$P^\mu_{\rm tot} = \sum_{i=1}^N P^\mu_i = \left(\sum_{i=1}^N \gamma_i\,m_i\,,\,\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i\right) \tag{15.190}$$

$$\equiv \Gamma\,M\,(1\,,\,\boldsymbol{V}) \ . \tag{15.191}$$

Then

$$M^2 = \left(\sum_{i=1}^N \gamma_i\,m_i\right)^2 - \left(\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i\right)^2 \tag{15.192}$$

and

$$\boldsymbol{V} = \frac{\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i}{\sum_{i=1}^N \gamma_i\,m_i} \ . \tag{15.193}$$

15.7.5 Compton scattering

An extremely important example of relativistic scattering occurs when a photon scatters off an electron: $e^- + \gamma \longrightarrow e^- + \gamma$ (see fig. 15.10). Let us work in the rest frame of the reactant electron. Then we have

$$P^\mu_e = m_{\rm e}\,(1, 0) \qquad , \qquad \widetilde{P}^\mu_e = m_{\rm e}\,(\gamma\,,\,\gamma\boldsymbol{V}) \tag{15.194}$$

for the initial and final 4-momenta of the electron. For the photon, we have

$$P^\mu_\gamma = (\omega\,,\,\boldsymbol{k}) \qquad , \qquad \widetilde{P}^\mu_\gamma = (\widetilde{\omega}\,,\,\widetilde{\boldsymbol{k}}) \ , \tag{15.195}$$

where we've set $\hbar = 1$ as well. Conservation of 4-momentum entails

$$P^\mu_\gamma - \widetilde{P}^\mu_\gamma = \widetilde{P}^\mu_e - P^\mu_e \ . \tag{15.196}$$

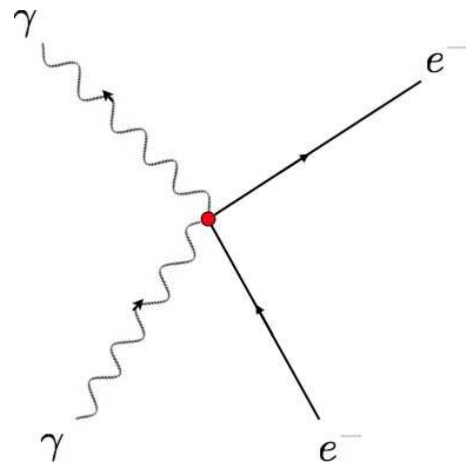

Figure 15.10: Compton scattering of a photon and an electron.

Thus,

$$\big(\omega - \tilde{\omega}\,,\,\boldsymbol{k} - \tilde{\boldsymbol{k}}\big) = m_{\rm e}\,\big(\gamma - 1\,,\gamma\boldsymbol{V}\big)\ . \tag{15.197}$$

Squaring each side, we obtain

$$\begin{aligned}\big(\omega - \tilde{\omega}\big)^2 - \big(\boldsymbol{k} - \tilde{\boldsymbol{k}}\big)^2 &= 2\omega\,\tilde{\omega}\,(\cos\theta - 1)\\ &= m_{\rm e}^2\Big((\gamma-1)^2 - \gamma^2\boldsymbol{V}^2\Big)\\ &= 2m_{\rm e}^2(1-\gamma)\\ &= 2m_{\rm e}\,\big(\tilde{\omega} - \omega\big)\ .\end{aligned} \tag{15.198}$$

Here we have used $|\boldsymbol{k}| = \omega$ for photons, and also $(\gamma - 1)\,m_{\rm e} = \omega - \tilde{\omega}$, from eqn. 15.197.

Restoring the units $\hbar$ and c, we find the Compton formula

$$\frac{1}{\tilde{\omega}} - \frac{1}{\omega} = \frac{\hbar}{m_{\rm e}c^2}\,\big(1 - \cos\theta\big)\ . \tag{15.199}$$

This is often expressed in terms of the photon wavelengths, as

$$\tilde{\lambda} - \lambda = \frac{4\pi\hbar}{m_{\rm e}c}\,\sin^2\!\big(\tfrac{1}{2}\theta\big)\ , \tag{15.200}$$

showing that the wavelength of the scattered light increases with the scattering angle in the rest frame of the target electron.

15.8 Covariant Electrodynamics

We begin with the following expression for the Lagrangian density of charged particles coupled to an electromagnetic field, and then show that the Euler-Lagrange equations recapitulate Maxwell's equations. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{c}\,j_\mu\,A^\mu\ . \tag{15.201}$$

Here, $A^\mu = (\phi\,, \boldsymbol{A})$ is the *electromagnetic 4-potential*, which combines the scalar field ϕ and the vector field $\boldsymbol{A}$ into a single 4-vector. The quantity $F_{\mu\nu}$ is the *electromagnetic field strength tensor* and is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{15.202}$$

Note that as defined $F_{\mu\nu} = -F_{\nu\mu}$ is antisymmetric. Note that, if $i = 1, 2, 3$ is a spatial index, then

$$F_{0i} = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial A^0}{\partial x^i} = E_i \tag{15.203}$$

$$F_{ij} = \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} = -\,\epsilon_{ijk}\, B_k \ . \tag{15.204}$$

Here we have used $A^\mu = (A^0\,, \boldsymbol{A})$ and $A_\mu = (A^0\,, -\boldsymbol{A})$, as well as $\partial_\mu = (c^{-1}\partial_t\,, \boldsymbol{\nabla})$.

IMPORTANT : Since the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are not part of a 4-vector, we do not use covariant / contravariant notation for their components. Thus, E_i is the $i^{\rm th}$ component of the vector $\boldsymbol{E}$. We will not write E^i with a raised index, but if we did, we'd mean the same thing: $E^i = E_i$. By contrast, for the spatial components of a four-vector like A^μ, we have $A_i = -A^i$.

Explicitly, then, we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{15.205}$$

where $F^{\mu\nu} = g^{\mu\alpha}\, g^{\nu\beta}\, F_{\alpha\beta}$. Note that when comparing $F^{\mu\nu}$ and $F_{\mu\nu}$, the components with one space and one time index differ by a minus sign. Thus,

$$-\frac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} = \frac{\boldsymbol{E}^2 - \boldsymbol{B}^2}{8\pi} \ , \tag{15.206}$$

which is the electromagnetic Lagrangian density. The $j \cdot A$ term accounts for the interaction between matter and electromagnetic degrees of freedom. We have

$$\frac{1}{c}\, j_\mu A^\mu = \varrho\, \phi - \frac{1}{c}\, \boldsymbol{j} \cdot \boldsymbol{A} \ , \tag{15.207}$$

where

$$j^\mu = \begin{pmatrix} c\varrho \\ \boldsymbol{j} \end{pmatrix} \quad , \quad A^\mu = \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} \ , \tag{15.208}$$

where ϱ is the charge density and $\boldsymbol{j}$ is the current density. Charge conservation requires

$$\partial_\mu\, j^\mu = \frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \ . \tag{15.209}$$

We shall have more to say about this further on below.

Let us now derive the Euler-Lagrange equations for the action functional,

$$S = -c^{-1}\int d^4\!x \left(\frac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} + c^{-1}\, j_\mu\, A^\mu\right) . \tag{15.210}$$

We first vary with respect to A_μ. Clearly

$$\delta F_{\mu\nu} = \partial_\mu\, \delta A_\nu - \partial_\nu\, \delta A_\mu \ . \tag{15.211}$$

We then have

$$\delta\mathcal{L} = \left(\frac{1}{4\pi}\,\partial_\mu F^{\mu\nu} - c^{-1} j^\nu\right) \delta A_\nu - \partial_\mu\left(\frac{1}{4\pi}\, F^{\mu\nu}\, \delta A_\nu\right) . \tag{15.212}$$

Ignoring the boundary term, we obtain Maxwell's equations,

$$\partial_\mu\, F^{\mu\nu} = 4\pi c^{-1}\, j^\nu \tag{15.213}$$

The $\nu = k$ component of these equations yields

$$\partial_0\, F^{0k} + \partial_i\, F^{jk} = -\partial_0\, E_k - \epsilon_{jkl}\, \partial_j\, B_l = 4\pi c^{-1}\, j^k \ , \tag{15.214}$$

which is the k component of the Maxwell-Ampère law,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\, \boldsymbol{j} + \frac{1}{c}\,\frac{\partial \boldsymbol{E}}{\partial t} \ . \tag{15.215}$$

The $\nu = 0$ component reads

$$\partial_i\, F^{i0} = \frac{4\pi}{c}\, j^0 \quad \Rightarrow \quad \boldsymbol{\nabla}\!\cdot\! \boldsymbol{E} = 4\pi\varrho \quad , \tag{15.216}$$

which is Gauss's law. The remaining two Maxwell equations come 'for free' from the very definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} = -\boldsymbol{\nabla} A^0 - \frac{1}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} \tag{15.217}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad , \tag{15.218}$$

which imply

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{B}}{\partial t} \tag{15.219}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad . \tag{15.220}$$

15.8.1 Lorentz force law

This has already been worked out in chapter 7. Here we reiterate our earlier derivation. The 4-current may be written as

$$j^\mu(\boldsymbol{x}, t) = c \sum_n q_n \int d\tau\, \frac{dX^\mu_n}{d\tau}\, \delta^{(4)}(x - X) \ . \tag{15.221}$$

Thus, writing $X_n^\mu = \big(ct\,, \boldsymbol{X}_n(t)\big)$, we have

$$j^0(\boldsymbol{x},t) = \sum_n q_n\, c\, \delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big) \tag{15.222}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \sum_n q_n\, \dot{\boldsymbol{X}}_n(t)\, \delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big)\ . \tag{15.223}$$

The Lagrangian for the matter-field interaction term is then

$$\begin{aligned} L &= -c^{-1}\!\int\! d^3\!x\, \big(j^0 A^0 - \boldsymbol{j}\cdot\boldsymbol{A}\big) \\ &= -\sum_n \bigg[q_n\, \phi(\boldsymbol{X}_n,t) - \frac{q_n}{c}\, \boldsymbol{A}(\boldsymbol{X}_n,t)\cdot\dot{\boldsymbol{X}}_n\bigg]\ , \end{aligned} \tag{15.224}$$

where $\phi = A^0$. For each charge q_n, this is equivalent to a particle with velocity-dependent potential energy

$$U(\boldsymbol{x},t) = q\, \phi(\boldsymbol{x},t) - \frac{q}{c}\, \boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{x}}\ , \tag{15.225}$$

where $\boldsymbol{x} = \boldsymbol{X}_n$.

Let's work out the equations of motion. We assume a kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{x}}^2$ for the charge. We then have

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\bigg) = \frac{\partial L}{\partial\boldsymbol{x}} \tag{15.226}$$

with $L = T - U$, which gives

$$m\,\ddot{\boldsymbol{x}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{x}})\ , \tag{15.227}$$

or, in component notation,

$$m\,\ddot{x}^i + \frac{q}{c}\,\frac{\partial A^i}{\partial x^j}\,\dot{x}^j + \frac{q}{c}\,\frac{\partial A^i}{\partial t} = -q\,\frac{\partial\phi}{\partial x^i} + \frac{q}{c}\,\frac{\partial A^j}{\partial x^i}\,\dot{x}^j\ , \tag{15.228}$$

which is to say

$$m\,\ddot{x}^i = -q\,\frac{\partial\phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\bigg(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\bigg)\,\dot{x}^j\ . \tag{15.229}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A^k}{\partial x^j}\ , \tag{15.230}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}\ , \tag{15.231}$$

we have $\epsilon_{ijk}\,B_i = \partial^j A^k - \partial^k A^j$, and

$$m\,\ddot{x}^i = -q\,\frac{\partial\phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}^j\,B_k\ , \tag{15.232}$$

Figure 15.11: Homer celebrates the manifest gauge invariance of classical electromagnetic theory.

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{x}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times \boldsymbol{B}\ , \end{aligned} \tag{15.233}$$

which is, of course, the Lorentz force law.

15.8.2 Gauge invariance

The action $S = c^{-1}\!\int\! d^4\!x\,\mathcal{L}$ admits a *gauge invariance*. Let $A^\mu \to A^\mu + \partial^\mu \Lambda$, where $\Lambda(\boldsymbol{x}, t)$ is an arbitrary scalar function of spacetime coordinates. Clearly

$$F_{\mu\nu} \to F_{\mu\nu} + \big(\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda\big) = F_{\mu\nu}\ , \tag{15.234}$$

and hence the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ remain *invariant* under the gauge transformation, even though the 4-potential itself changes. What about the matter term? Clearly

$$\begin{aligned} -c^{-1}\, j^\mu\, A_\mu \to &- c^{-1}\, j^\mu\, A_\mu - c^{-1}\, j^\mu\, \partial_\mu \Lambda \\ &= -c^{-1}\, j^\mu\, A_\mu + c^{-1} \Lambda\, \partial_\mu\, j^\mu - \partial_\mu\big(c^{-1} \Lambda\, j^\mu\big)\ . \end{aligned} \tag{15.235}$$

Once again we ignore the boundary term. We may now invoke charge conservation to write $\partial_\mu\, j^\mu = 0$, and we conclude that the action is invariant! Woo hoo! Note also the very deep connection

$$\text{gauge invariance} \quad \longleftrightarrow \quad \text{charge conservation}\ . \tag{15.236}$$

15.8.3 Transformations of fields

One last detail remains, and that is to exhibit explicitly the Lorentz transformation properties of the electromagnetic field. For the case of vectors like A^μ, we have

$$A^\mu = L^\mu_{\ \nu} \, A'^{\,\nu} \ . \tag{15.237}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, however, appear as elements in the field strength tensor $F^{\mu\nu}$. Clearly this must transform as a tensor:

$$F^{\mu\nu} = L^\mu_{\ \alpha} \, L^\nu_{\ \beta} \, F'^{\,\alpha\beta} = L^\mu_{\ \alpha} \, F'^{\,\alpha\beta} \, L^{\mathrm{t}\ \nu}_{\beta} \ . \tag{15.238}$$

We can write a general Lorentz transformation as a product of a rotation L_{rot} and a boost L_{boost}. Let's first see how rotations act on the field strength tensor. We take

$$L = L_{\mathrm{rot}} = \begin{pmatrix} 1_{1\times1} & 0_{1\times3} \\ 0_{3\times1} & R_{3\times3} \end{pmatrix} \ , \tag{15.239}$$

where $R^{\mathrm{t}} R = \mathbb{I}$, *i.e.* $R \in \mathrm{O}(3)$ is an orthogonal matrix. We must compute

$$\begin{aligned} L^\mu_{\ \alpha} \, F'^{\,\alpha\beta} \, L^{\mathrm{t}\ \nu}_{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix} \begin{pmatrix} 0 & -E'_k \\ E'_j & -\epsilon_{jkm} \, B'_m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R^{\mathrm{t}}_{kl} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E'_k \, R^{\mathrm{t}}_{kl} \\ R_{ij} \, E'_j & -\epsilon_{jkm} \, R_{ij} \, R_{lk} \, B'_m \end{pmatrix} \ . \end{aligned} \tag{15.240}$$

Thus, we conclude

$$E_l = R_{lk} \, E'_k \tag{15.241}$$

$$\epsilon_{iln} \, B_n = \epsilon_{jkm} \, R_{ij} \, R_{lk} \, B'_m \ . \tag{15.242}$$

Now for any 3×3 matrix R we have

$$\epsilon_{jks} \, R_{ij} \, R_{lk} \, R_{rs} = \det(R) \, \epsilon_{ilr} \ , \tag{15.243}$$

and therefore

$$\begin{aligned} \epsilon_{jkm} \, R_{ij} \, R_{lk} \, B'_m &= \epsilon_{jkm} \, R_{ij} \, R_{lk} \, R_{nm} \, R_{ns} \, B'_s \\ &= \det(R) \, \epsilon_{iln} \, R_{ns} \, B'_s \ , \end{aligned} \tag{15.244}$$

Therefore,

$$E_i = R_{ij} \, E'_j \qquad , \qquad B_i = \det(R) \cdot R_{ij} \, B'_j \ . \tag{15.245}$$

For any orthogonal matrix, $R^{\mathrm{t}} R = \mathbb{I}$ gives that $\det(R) = \pm 1$. The extra factor of $\det(R)$ in the transformation properties of $\boldsymbol{B}$ is due to the fact that the electric field transforms as a *vector*, while the magnetic field transforms as a *pseudovector*. Under space inversion, for example, where $R = -\mathbb{I}$, the electric field is *odd* under this transformation ($\boldsymbol{E} \to -\boldsymbol{E}$) while the magnetic field is *even* ($\boldsymbol{B} \to +\boldsymbol{B}$). Similar considerations hold in particle mechanics for the linear momentum, $\boldsymbol{p}$ (a vector) and the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ (a pseudovector). The analogy is not complete, however, because while both $\boldsymbol{p}$ and $\boldsymbol{L}$ are odd under the operation of time-reversal, $\boldsymbol{E}$ is even while $\boldsymbol{B}$ is odd.

OK, so how about boosts? We can write the general boost, from eqn. 15.37, as

$$L = \begin{pmatrix} \gamma & \gamma\hat{\boldsymbol{\beta}} \\ \gamma\hat{\boldsymbol{\beta}} & \mathrm{I} + (\gamma-1)\mathrm{P}^{\boldsymbol{\beta}} \end{pmatrix} \tag{15.246}$$

where $\mathrm{P}^{\boldsymbol{\beta}}_{ij} = \hat{\beta}_i\,\hat{\beta}_j$ is the projector onto the direction of $\boldsymbol{\beta}$. We now compute

$$L^{\mu}_{\ \alpha}\,F'^{\,\alpha\beta}\,L^{\mathrm{t}\ \nu}_{\beta} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\mathrm{P} \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{E}'^{\,\mathrm{t}} \\ \boldsymbol{E}' & -\,\epsilon_{jkm}\,B'_m \end{pmatrix} \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\mathrm{P} \end{pmatrix} \ . \tag{15.247}$$

Carrying out the matrix multiplications, we obtain

$$\boldsymbol{E} = \gamma(\boldsymbol{E}' - \boldsymbol{\beta} \times \boldsymbol{B}') - (\gamma-1)(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{E}')\hat{\boldsymbol{\beta}} \tag{15.248}$$

$$\boldsymbol{B} = \gamma(\boldsymbol{B}' + \boldsymbol{\beta} \times \boldsymbol{E}') - (\gamma-1)(\hat{\boldsymbol{\beta}} \cdot \boldsymbol{B}')\hat{\boldsymbol{\beta}} \ . \tag{15.249}$$

Expressed in terms of the components $E_{\|}$, $\boldsymbol{E}_{\perp}$, $B_{\|}$, and $\boldsymbol{B}_{\perp}$, one has

$$E_{\|} = E'_{\|} \qquad , \qquad \boldsymbol{E}_{\perp} = \gamma\big(\boldsymbol{E}'_{\perp} - \boldsymbol{\beta} \times \boldsymbol{B}'_{\perp}\big) \tag{15.250}$$

$$B_{\|} = B'_{\|} \qquad , \qquad \boldsymbol{B}_{\perp} = \gamma\big(\boldsymbol{B}'_{\perp} + \boldsymbol{\beta} \times \boldsymbol{E}'_{\perp}\big) \ . \tag{15.251}$$

Recall that for any vector $\boldsymbol{\xi}$, we write

$$\xi_{\|} = \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\xi} \tag{15.252}$$

$$\boldsymbol{\xi}_{\perp} = \boldsymbol{\xi} - (\hat{\boldsymbol{\beta}} \cdot \boldsymbol{\xi})\,\hat{\boldsymbol{\beta}} \ , \tag{15.253}$$

so that $\hat{\boldsymbol{\beta}} \cdot \boldsymbol{\xi}_{\perp} = 0$.

15.8.4 Invariance *versus* covariance

We saw that the laws of electromagnetism were *gauge* ***in****variant*. That is, the solutions to the field equations did not change under a gauge transformation $A^\mu \to A^\mu + \partial^\mu \Lambda$. With respect to Lorentz transformations, however, the theory is *Lorentz* ***co****variant*. This means that Maxwell's equations in different inertial frames take the exact same form, $\partial_\mu F^{\mu\nu} = 4\pi c^{-1} j^\nu$, but that both the fields and the sources transform appropriately under a change in reference frames. The sources are described by the current 4-vector $j^\mu = (c\varrho\,,\,\boldsymbol{j})$ and transform as

$$c\varrho = \gamma c\varrho' + \gamma\beta j'_{\|} \tag{15.254}$$

$$j_{\|} = \gamma\beta c\varrho' + \gamma j'_{\|} \tag{15.255}$$

$$\boldsymbol{j}_{\perp} = \boldsymbol{j}'_{\perp} \quad . \tag{15.256}$$

The fields transform according to eqns. 15.250 and 15.251.

Consider, for example, a static point charge q located at the origin in the frame K', which moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to K. An observer in K' measures a charge density $\varrho'(\boldsymbol{x}',t') = q\,\delta(\boldsymbol{x}')$. The electric and magnetic fields in the K' frame are then $\boldsymbol{E}' = q\,\hat{\boldsymbol{r}}'/r'^{\,2}$ and $\boldsymbol{B}' = 0$. For an observer in the K frame, the coordinates transform as

$$ct = \gamma ct' + \gamma\beta x' \qquad\qquad ct' = \gamma ct - \gamma\beta x \tag{15.257}$$

$$x = \gamma\beta ct' + \gamma x' \qquad\qquad x' = -\gamma\beta ct + \gamma x \ , \tag{15.258}$$

as well as $y = y'$ and $z = z'$. The observer in the K frame sees instead a charge at $x^\mu = (ct\,,\,ut\,,\,0\,,\,0)$ and both a charge density as well as a current density:

$$\varrho(\boldsymbol{x},t) = \gamma\varrho(\boldsymbol{x}',t') = q\,\delta(x-ut)\,\delta(y)\,\delta(z) \tag{15.259}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \gamma\beta c\,\varrho(\boldsymbol{x}',t')\,\hat{\boldsymbol{x}} = u\,q\,\delta(x-ut)\,\delta(y)\,\delta(z)\,\hat{\boldsymbol{x}} \ . \tag{15.260}$$

OK, so much for the sources. How about the fields? Expressed in terms of Cartesian coordinates, the electric field in K' is given by

$$\boldsymbol{E}'(\boldsymbol{x}',t') = q\,\frac{x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}}}{\big(x'^{\,2}+y'^{\,2}+z'^{\,2}\big)^{3/2}} \ . \tag{15.261}$$

From eqns. 15.250 and 15.251, we have $E_x = E'_x$ and $B_x = B'_x = 0$. Furthermore, we have $E_y = \gamma E'_y$, $E_z = \gamma E'_z$, $B_y = -\gamma\beta E'_z$, and $B_z = \gamma\beta E'_y$. Thus,

$$\boldsymbol{E}(\boldsymbol{x},t) = \gamma q\,\frac{(x-ut)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\big[\gamma^2(x-ut)^2+y^2+z^2\big]^{3/2}} \tag{15.262}$$

$$\boldsymbol{B}(\boldsymbol{x},t) = \frac{\gamma u}{c}\,q\,\frac{y\hat{\boldsymbol{z}} - z\hat{\boldsymbol{y}}}{\big[\gamma^2(x-ut)^2+y^2+z^2\big]^{3/2}} \ . \tag{15.263}$$

Let us define

$$\boldsymbol{R}(t) = (x-ut)\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} \ . \tag{15.264}$$

We further define the angle $\theta \equiv \cos^{-1}\!\big(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}\big)$. We may then write

$$\boldsymbol{E}(x,t) = \frac{q\,\boldsymbol{R}}{R^3}\cdot\frac{1-\beta^2}{\big(1-\beta^2\sin^2\!\theta\big)^{3/2}}$$

$$\boldsymbol{B}(x,t) = \frac{q\,\hat{\boldsymbol{\beta}}\times\boldsymbol{R}}{R^3}\cdot\frac{1-\beta^2}{\big(1-\beta^2\sin^2\!\theta\big)^{3/2}} \ . \tag{15.265}$$

The fields are therefore enhanced in the transverse directions: $E_\perp/E_\parallel = \gamma^3$.

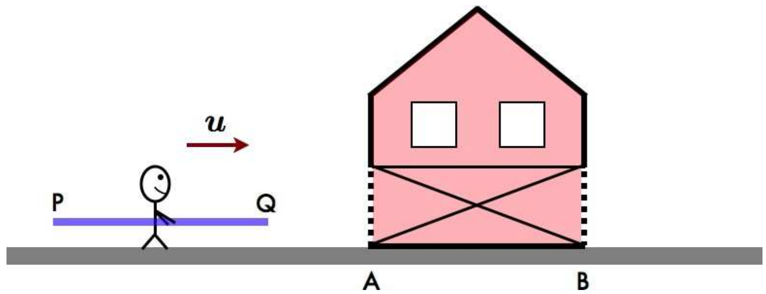

Figure 15.12: A relativistic runner carries a pole of proper length ℓ and runs into a barn of proper length ℓ.

15.9 Appendix I : The Pole, the Barn, and *Rashoman*

Akira Kurosawa's 1950 cinematic masterpiece, *Rashoman*, describes a rape, murder, and battle from four different and often contradictory points of view. It poses deep questions regarding the nature of truth. Psychologists sometimes refer to problems of subjective perception as the *Rashoman effect*. In literature, William Faulkner's 1929 novel, *The Sound and the Fury*, which describes the tormented incestuous life of a Mississippi family, also is told from four points of view. Perhaps Faulkner would be a more apt comparison with Einstein, since time plays an essential role in his novel. For example, Quentin's watch, given to him by his father, represents time and the sweep of life's arc (*"Quentin, I give you the mausoleum of all hope and desire..."*). By breaking the watch, Quentin symbolically attempts to escape time and fate. One could draw an analogy to Einstein, inheriting a watch from those who came before him, which he too broke – and refashioned. Did Faulkner know of Einstein? But I digress.

Consider a relativistic runner carrying a pole of proper length ℓ, as depicted in fig. 15.12. He runs toward a barn of proper length ℓ at velocity $u = c\beta$. Let the frame of the barn be K and the frame of the runner be K'. Recall that the Lorentz transformations between frames K and K' are given by

$$ct = \gamma ct' + \gamma x' \qquad ct' = \gamma ct - \gamma\beta x \tag{15.266}$$

$$x = \gamma\beta ct' + \gamma x' \qquad x' = -\gamma\beta ct + \gamma x \; . \tag{15.267}$$

We define the following points. Let A denote the left door of the barn and B the right door. Furthermore, let P denote the left end of the pole and Q its right end. The spacetime coordinates for these points in the two frames are clearly .

$$A = (ct\,,\,0) \qquad P' = (ct'\,,\,0) \tag{15.268}$$

$$B = (ct\,,\,\ell) \qquad Q' = (ct'\,,\,\ell) \tag{15.269}$$

We now compute A' and B' in frame K', as well as P and Q in frame K:

$$A' = (\gamma ct\,,\,-\gamma\beta ct) \qquad B' = (\gamma ct - \gamma\beta\ell\,,\,-\gamma\beta ct + \gamma\ell) \tag{15.270}$$

$$\equiv (ct'\,,\,-\beta ct') \qquad \equiv (ct'\,,\,-\beta ct' + \gamma^{-1}\ell) \; . \tag{15.271}$$

Similarly,

$$P = (\gamma ct' \, , \, \gamma\beta ct') \qquad\qquad Q = (\gamma ct' + \gamma\beta\ell \, , \, \gamma\beta ct' + \gamma\ell) \tag{15.272}$$

$$\equiv (ct \, , \, \beta ct) \qquad\qquad \equiv (ct \, , \, \beta ct + \gamma^{-1}\ell) \; . \tag{15.273}$$

We now define four events, by the coincidences of A and B with P and Q:

- Event I : The right end of the pole enters the left door of the barn. This is described by $Q = A$ in frame K and by $Q' = A'$ in frame K'.
- Event II : The right end of the pole exits the right door of the barn. This is described by $Q = B$ in frame K and by $Q' = B'$ in frame K'.
- Event III : The left end of the pole enters the left door of the barn. This is described by $P = A$ in frame K and by $P' = A'$ in frame K'.
- Event IV : The left end of the pole exits the right door of the barn. This is described by $P = B$ in frame K and by $P' = B'$ in frame K'.

Mathematically, we have in frame K that

$$\text{I}: \quad Q = A \qquad \Rightarrow \qquad t_{\text{I}} = -\frac{\ell}{\gamma u} \tag{15.274}$$

$$\text{II}: \quad Q = B \qquad \Rightarrow \qquad t_{\text{II}} = (\gamma - 1)\,\frac{\ell}{\gamma u} \tag{15.275}$$

$$\text{III}: \quad P = A \qquad \Rightarrow \qquad t_{\text{III}} = 0 \tag{15.276}$$

$$\text{IV}: \quad P = B \qquad \Rightarrow \qquad t_{\text{IV}} = \frac{\ell}{u} \tag{15.277}$$

In frame K', however

$$\text{I}: \quad Q' = A' \qquad \Rightarrow \qquad t'_{\text{I}} = -\frac{\ell}{u} \tag{15.278}$$

$$\text{II}: \quad Q' = B' \qquad \Rightarrow \qquad t'_{\text{II}} = -(\gamma - 1)\,\frac{\ell}{\gamma u} \tag{15.279}$$

$$\text{III}: \quad P' = A' \qquad \Rightarrow \qquad t'_{\text{III}} = 0 \tag{15.280}$$

$$\text{IV}: \quad P' = B' \qquad \Rightarrow \qquad t'_{\text{IV}} = \frac{\ell}{\gamma u} \tag{15.281}$$

Thus, to an observer in frame K, the order of events is I, III, II, and IV, because

$$t_{\text{I}} \; < \; t_{\text{III}} \; < \; t_{\text{II}} \; < \; t_{\text{IV}} \; . \tag{15.282}$$

For $t_{\text{III}} < t < t_{\text{II}}$, he observes that *the pole is entirely in the barn*. Indeed, the right door can start shut and the left door open, and sensors can automatically and, for the purposes of argument, instantaneously trigger the closing of the left door immediately following event

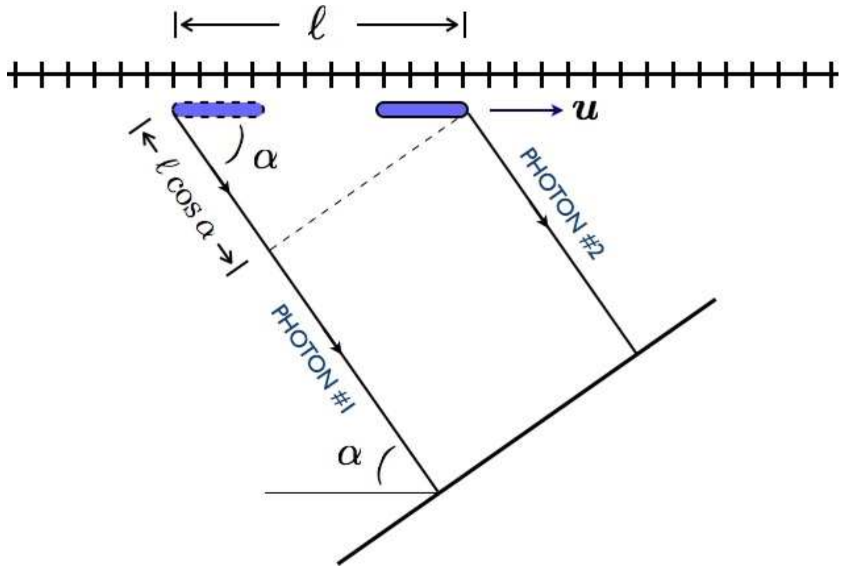

Figure 15.13: An object of proper length ℓ and moving with velocity $\boldsymbol{u}$, when photographed from an angle α, appears to have a length $\tilde{\ell}$.

III and the opening of the right door immediately prior to event II. So the pole can be inside the barn with both doors shut!

But now for the *Rashoman effect*: according to the runner, the order of events is I, II, III, and IV, because

$$t'_{\rm I} \;<\; t'_{\rm II} \;<\; t'_{\rm III} \;<\; t'_{\rm IV} \;. \qquad (15.283)$$

At no time does the runner observe the pole to be entirely within the barn. Indeed, for $t'_{\rm II} < t' < t'_{\rm III}$, both ends of the pole are sticking outside of the barn!

15.10 Appendix II : Photographing a Moving Pole

What is the length ℓ of a moving pole of proper length ℓ_0 as measured by an observer at rest? The answer would appear to be $\gamma^{-1}\ell_0$, as we computed in eqn. 15.71. However, we should be more precise when we we speak of 'length'. The relation $\ell(\beta) = \gamma^{-1}\ell_0$ tells us the *instantaneous end-to-end distance as measured in the observer's rest frame* K. But an actual experiment might not measure this quantity.

For example, suppose a relativistic runner carrying a pole of proper length ℓ_0 runs past a measuring rod which is at rest in the rest frame K of an observer. The observer *takes a photograph* of the moving pole as it passes by. Suppose further that the angle between the observer's line of sight and the velocity $\boldsymbol{u}$ of the pole is α, as shown in fig. 15.13. What is the apparent length $\ell(\alpha, u)$ of the pole as observed in the photograph? (*I.e.* the pole will appear to cover a portion of the measuring rod which is of length ℓ.)

The point here is that the shutter of the camera is very fast (otherwise the image will appear blurry). In our analysis we will assume the shutter opens and closes instantaneously. Let's define two events:

- Event 1 : photon γ_1 is emitted by the rear end of the pole.
- Event 2 : photon γ_2 is emitted by the front end of the pole.

Both photons must arrive at the camera's lens simultaneously. Since, as shown in the figure, the path of photon #1 is longer by a distance $\ell \cos\alpha$, where ℓ is the apparent length of the pole, γ_2 must be emitted a time $\Delta t = c^{-1} \ell \cos\alpha$ after γ_1. Now if we Lorentz transform from frame K to frame K', we have

$$\Delta x' = \gamma \Delta x - \gamma\beta\Delta t \ . \tag{15.284}$$

But $\Delta x' = \ell_0$ is the proper length of the pole, and $\Delta x = \ell$ is the apparent length. With $c\Delta t = \ell \cos\alpha$, then, we have

$$\ell = \frac{\gamma^{-1}\, \ell_0}{1 - \beta\cos\alpha} \ . \tag{15.285}$$

When $\alpha = 90^\circ$, we recover the familiar Lorentz-Fitzgerald contraction $\ell(\beta) = \gamma^{-1}\, \ell_0$. This is because the photons γ_1 and γ_2 are then emitted simultaneously, and the photograph measures the instantaneous end-to-end distance of the pole as measured in the observer's rest frame K. When $\cos\alpha \neq 0$, however, the two photons are not emitted simultaneously, and the apparent length is given by eqn. 15.285.