Chapter 2

Systems of Particles

2.1 Work-Energy Theorem

Consider a system of many particles, with positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \ . \tag{2.1}$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \, \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i \ . \tag{2.2}$$

Here, we've used the relation

$$\frac{d}{dt}\big(\boldsymbol{A}^2\big) = 2\,\boldsymbol{A} \cdot \frac{d\boldsymbol{A}}{dt} \ . \tag{2.3}$$

We now invoke Newton's 2nd Law, $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, to write eqn. 2.2 as $\dot{T}_i = \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i$. We integrate this equation from time $t_{\rm A}$ to $t_{\rm B}$:

$$\begin{aligned} T_i^{\rm (B)} - T_i^{\rm (A)} &= \int\limits_{t_{\rm A}}^{t_{\rm B}} \! dt \ \frac{dT_i}{dt} \\ &= \int\limits_{t_{\rm A}}^{t_{\rm B}} \! dt \, \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \equiv \sum_i W_i^{\rm (A\to B)} \ , \end{aligned} \tag{2.4}$$

where $W_i^{\rm (A\to B)}$ is the total *work done* on particle i during its motion from state A to state B, Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{\rm (A\to B)} = \sum_i W_i^{\rm (A\to B)}$. Eqn. 2.4 is known as the *work-energy theorem*. It says that

In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done: $T^{\rm (B)} - T^{\rm (A)} = W^{\rm (A\to B)}$.

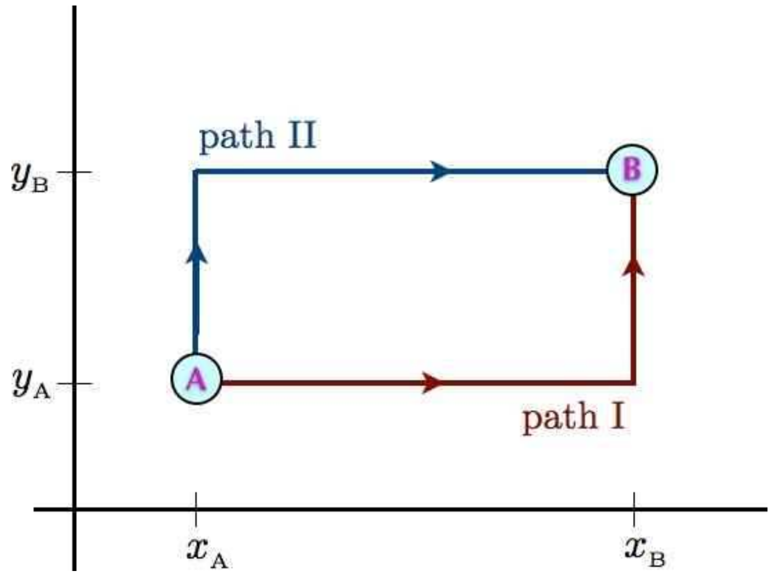

Figure 2.1: Two paths joining points A and B.

2.2 Conservative and Nonconservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{(A\to B)} = \int_{t_A}^{t_B} dt\, \boldsymbol{F} \cdot \boldsymbol{v} \ , \tag{2.5}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. This is the most general expression for the work done. If the force $\boldsymbol{F}$ depends only on the particle's position $\boldsymbol{r}$, we may write $d\boldsymbol{r} = \boldsymbol{v}\, dt$, and then

$$W^{(A\to B)} = \int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} d\boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) \ . \tag{2.6}$$

Consider now the force

$$\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}} \ , \tag{2.7}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 2.1:

$$W^{(I)} = K_1 \int_{x_A}^{x_B} dx\, y_A + K_2 \int_{y_A}^{y_B} dy\, x_B = K_1\, y_A\, (x_B - x_A) + K_2\, x_B\, (y_B - y_A) \tag{2.8}$$

$$W^{(II)} = K_1 \int_{x_A}^{x_B} dx\, y_B + K_2 \int_{y_A}^{y_B} dy\, x_A = K_1\, y_B\, (x_B - x_A) + K_2\, x_A\, (y_B - y_A) \ . \tag{2.9}$$

Note that in general $W^{(\mathrm{I})} \neq W^{(\mathrm{II})}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(\mathrm{I})} - W^{(\mathrm{II})} = (K_2 - K_1)\,(x_{\mathrm{B}} - x_{\mathrm{A}})\,(y_{\mathrm{B}} - y_{\mathrm{A}}) \ . \tag{2.10}$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 2.1. The reason for this is Stokes' theorem:

$$\oint\limits_{\partial\mathcal{C}} d\boldsymbol{\ell}\cdot\boldsymbol{F} = \int\limits_{\mathcal{C}} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F} \ . \tag{2.11}$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path[1], dS is the scalar differential area element, $\hat{\boldsymbol{n}}$ is the unit normal to that differential area element, and $\boldsymbol{\nabla}\times\boldsymbol{F}$ is the curl of $\boldsymbol{F}$:

$$\begin{aligned}
\boldsymbol{\nabla}\times\boldsymbol{F} &= \det\begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \ .
\end{aligned} \tag{2.12}$$

For the force under consideration, $\boldsymbol{F}(\boldsymbol{r}) = K_1\,y\,\hat{\boldsymbol{x}} + K_2\,x\,\hat{\boldsymbol{y}}$, the curl is

$$\boldsymbol{\nabla}\times\boldsymbol{F} = (K_2 - K_1)\,\hat{\boldsymbol{z}} \ , \tag{2.13}$$

which is a constant. The RHS of eqn. 2.11 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 2.10, we evaluate the integral $\oint\limits_{\mathcal{C}} d\boldsymbol{\ell}\cdot\boldsymbol{F}$ along the path $\gamma_{\mathrm{II}}^{-1}\circ\gamma_{\mathrm{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the integral of $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 2.10.

When $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ everywhere in space, we can always write $\boldsymbol{F} = -\boldsymbol{\nabla}U$, where $U(\boldsymbol{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$\begin{aligned}
W^{(\mathrm{A}\to\mathrm{B})} &= \int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) \\
&= -\int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r}\cdot\boldsymbol{\nabla}U \\
&= U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ .
\end{aligned} \tag{2.14}$$

[1]If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

The work-energy theorem then gives

$$T^{(\mathrm{B})} - T^{(\mathrm{A})} = U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ , \tag{2.15}$$

which says

$$E^{(\mathrm{B})} = T^{(\mathrm{B})} + U(\boldsymbol{r}_{\mathrm{B}}) = T^{(\mathrm{A})} + U(\boldsymbol{r}_{\mathrm{A}}) = E^{(\mathrm{A})} \ . \tag{2.16}$$

Thus, the total energy $E = T + U$ is conserved.

2.2.1 Example : integrating $\boldsymbol{F} = -\boldsymbol{\nabla} U$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can compute $U(\boldsymbol{r})$ by integrating, *viz.*

$$U(\boldsymbol{r}) = U(\boldsymbol{0}) - \int_{\boldsymbol{0}}^{\boldsymbol{r}} d\boldsymbol{r}' \cdot \boldsymbol{F}(\boldsymbol{r}') \ . \tag{2.17}$$

The integral does not depend on the path chosen connecting $\boldsymbol{0}$ and $\boldsymbol{r}$. For example, we can take

$$U(x,y,z) = U(0,0,0) \ - \int_{(0,0,0)}^{(x,0,0)} dx' \, F_x(x',0,0) \ - \int_{(x,0,0)}^{(x,y,0)} dy' \, F_y(x,y',0) \ - \int_{(z,y,0)}^{(x,y,z)} dz' \, F_z(x,y,z') \ . \tag{2.18}$$

The constant $U(0,0,0)$ is arbitrary and impossible to determine from $\boldsymbol{F}$ alone.

As an example, consider the force

$$\boldsymbol{F}(\boldsymbol{r}) = -ky\,\hat{\boldsymbol{x}} - kx\,\hat{\boldsymbol{y}} - 4bz^3\,\hat{\boldsymbol{z}} \ , \tag{2.19}$$

where k and b are constants. We have

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_x = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) = 0 \tag{2.20}$$

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_y = \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) = 0 \tag{2.21}$$

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_z = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0 \ , \tag{2.22}$$

so $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ and $\boldsymbol{F}$ must be expressible as $\boldsymbol{F} = -\boldsymbol{\nabla} U$. Integrating using eqn. 2.18, we have

$$U(x,y,z) = U(0,0,0) \ + \int_{(0,0,0)}^{(x,0,0)} dx' \, k \cdot 0 \ + \int_{(x,0,0)}^{(x,y,0)} dy' \, kxy' \ + \int_{(z,y,0)}^{(x,y,z)} dz' \, 4bz'^3 \tag{2.23}$$

$$= U(0,0,0) + kxy + bz^4 \ . \tag{2.24}$$

Another approach is to integrate the partial differential equation $\boldsymbol{\nabla} U = -\boldsymbol{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\boldsymbol{x}}$-component,

$$\frac{\partial U}{\partial x} = ky \ . \tag{2.25}$$

Integrating, we obtain

$$U(x,y,z) = kxy + f(y,z) \ , \tag{2.26}$$

where $f(y,z)$ is at this point an *arbitrary function* of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \quad \Longrightarrow \quad U(x,y,z) = kxy + g(x,z) \ . \tag{2.27}$$

Finally, the z-component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \quad \Longrightarrow \quad U(x,y,z) = bz^4 + h(x,y) \ . \tag{2.28}$$

We now equate the first two expressions:

$$kxy + f(y,z) = kxy + g(x,z) \ . \tag{2.29}$$

Subtracting kxy from each side, we obtain the equation $f(y,z) = g(x,z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y,z) = g(x,z) = q(z) \ , \tag{2.30}$$

where $q(z)$ is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x,y) = kxy + q(z) \ . \tag{2.31}$$

The only possible solution is $h(x,y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x,y,z) = C + kxy + bz^4 \ . \tag{2.32}$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x,y,z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z. The fact that $f(y,z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y,z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y,z)$.

2.3 Conservative Forces in Many Particle Systems

$$T = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \tag{2.33}$$

$$U = \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.34}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r}-\boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i\,\ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\text{ext})} + \boldsymbol{F}_i^{(\text{int})}\ , \tag{2.35}$$

with

$$\boldsymbol{F}_i^{(\text{ext})} = -\frac{\partial V(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \tag{2.36}$$

$$\boldsymbol{F}_i^{(\text{int})} = -\sum_j \frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\boldsymbol{r}_i} \equiv \sum_j \boldsymbol{F}_{ij}^{(\text{int})}\ . \tag{2.37}$$

Here, $\boldsymbol{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j:

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\partial \boldsymbol{r}_i} = -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\, v'\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)\ . \tag{2.38}$$

Note that $\boldsymbol{F}_{ij}^{(\text{int})} = -\boldsymbol{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$, in which case we may write the interparticle force as

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\hat{\boldsymbol{r}}_{ij}\, v'\big(r_{ij}\big)\ . \tag{2.39}$$

2.4 Linear and Angular Momentum

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$. Its rate of change is

$$\frac{d\boldsymbol{P}}{dt} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\sum_{i\neq j} \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{F}_{ij}^{(\text{int})}+\boldsymbol{F}_{ji}^{(\text{int})}=0} = \boldsymbol{F}_{\text{tot}}^{(\text{ext})}\ , \tag{2.40}$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i = M\dot{\boldsymbol{R}} \tag{2.41}$$

$$M = \sum_i m_i \quad \text{(total mass)} \tag{2.42}$$

$$\boldsymbol{R} = \frac{\sum_i m_i\,\boldsymbol{r}_i}{\sum_i m_i} \quad \text{(center-of-mass)}\ . \tag{2.43}$$

Next, consider the total angular momentum,

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i\ . \tag{2.44}$$

The rate of change of $\boldsymbol{L}$ is then

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \sum_i \big\{ m_i \dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i + m_i \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i \big\} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{ext})} + \sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}^{(\mathrm{int})} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\mathrm{ext})} + \overbrace{\tfrac{1}{2}\sum_{i\neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij}^{(\mathrm{int})}}^{\boldsymbol{r}_{ij}\times \boldsymbol{F}_{ij}^{(\mathrm{int})}=0} \\ &= \boldsymbol{N}_{\mathrm{tot}}^{(\mathrm{ext})} \ . \end{aligned} \tag{2.45}$$

Finally, it is useful to establish the result

$$T = \tfrac{1}{2}\sum_i m_i\, \dot{\boldsymbol{r}}_i^2 = \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2}\sum_i m_i \big(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{R}}\big)^2 \ , \tag{2.46}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$\begin{aligned} 0 &= \int\limits_{\mathrm{initial}}^{\mathrm{final}} dE = \int\limits_{\mathrm{initial}}^{\mathrm{final}} dT + \int\limits_{\mathrm{initial}}^{\mathrm{final}} dU \\ &= T^{(\mathrm{B})} - T^{(\mathrm{A})} - \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i \ , \end{aligned} \tag{2.47}$$

which is to say

$$\Delta T = T^{(\mathrm{B})} - T^{(\mathrm{A})} = \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i = -\Delta U \ . \tag{2.48}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.49}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i\, \phi(\boldsymbol{r}_i) \longrightarrow \int d^3\!r\, \rho(\boldsymbol{r})\, \phi(\boldsymbol{r}) \ , \tag{2.50}$$

where $\rho(\boldsymbol{r})$ is the mass density, and $\phi(\boldsymbol{r})$ is any function.

2.5 Scaling of Solutions for Homogeneous Potentials

2.5.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U\big(\lambda\,\boldsymbol{r}_1,\ldots,\lambda\,\boldsymbol{r}_N\big) = \lambda^k\,U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\ . \tag{2.51}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big) = -G\sum_{i<j}\frac{m_i\,m_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}\quad ;\quad k=-1\ , \tag{2.52}$$

and the harmonic oscillator,

$$U\big(q_1,\ldots,q_n\big) = \tfrac{1}{2}\sum_{\sigma,\sigma'} V_{\sigma\sigma'}\,q_\sigma\,q_{\sigma'}\quad ;\quad k=+2\ . \tag{2.53}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1,\ldots,x_n)$ is homogeneous:

$$H(\lambda\,x_1,\ldots,\lambda\,x_n) = \lambda^k\,H(x_1,\ldots,x_n)\ . \tag{2.54}$$

Then

$$\boxed{\frac{d}{d\lambda}\bigg|_{\lambda=1} H\big(\lambda\,x_1,\ldots,\lambda\,x_n\big) = \sum_{i=1}^n x_i\,\frac{\partial H}{\partial x_i} = k\,H} \tag{2.55}$$

2.5.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\boldsymbol{r}_i = \alpha\,\tilde{\boldsymbol{r}}_i\qquad ,\qquad t=\beta\,\tilde{t}\ . \tag{2.56}$$

Then

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\alpha}{\beta}\,\frac{d\tilde{\boldsymbol{r}}_i}{d\tilde{t}}\qquad ,\qquad \frac{d^2\boldsymbol{r}_i}{dt^2} = \frac{\alpha}{\beta^2}\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2}\ . \tag{2.57}$$

The force $\boldsymbol{F}_i$ is given by

$$\begin{aligned}\boldsymbol{F}_i &= -\frac{\partial}{\partial\boldsymbol{r}_i}\,U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\\ &= -\frac{\partial}{\partial(\alpha\tilde{\boldsymbol{r}}_i)}\,\alpha^k\,U\big(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N\big)\\ &= \alpha^{k-1}\,\tilde{\boldsymbol{F}}_i\ .\end{aligned} \tag{2.58}$$

Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2}\, m_i\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} = \alpha^{k-1}\, \tilde{\boldsymbol{F}}_i \ . \tag{2.59}$$

If we choose β such that

We now demand

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \ , \tag{2.60}$$

then the equation of motion is invariant under the rescaling transformation! This means that if $\boldsymbol{r}(t)$ is a solution to the equations of motion, then so is $\alpha\, \boldsymbol{r}\big(\alpha^{\frac{1}{2}k-1}\, t\big)$. This gives us an entire one-parameter family of solutions, for all real positive α.

If $\boldsymbol{r}(t)$ is periodic with period T, the $\boldsymbol{r}_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k}\, T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \ . \tag{2.61}$$

Here, $\alpha = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly:

$$[v] = \frac{L}{T} \qquad \Rightarrow \qquad \frac{v'}{v} = \frac{L'}{L}\bigg/\frac{T'}{T} = \alpha^{\frac{1}{2}k} \tag{2.62}$$

$$[E] = \frac{ML^2}{T^2} \qquad \Rightarrow \qquad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2 \bigg/ \left(\frac{T'}{T}\right)^2 = \alpha^k \tag{2.63}$$

$$[\boldsymbol{L}] = \frac{ML^2}{T} \qquad \Rightarrow \qquad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2 \bigg/ \frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \ . \tag{2.64}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\alpha) = \alpha\, q_\sigma(t) \ . \tag{2.65}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\alpha) = \alpha\, \boldsymbol{r}\big(\alpha^{-3/2}\, t\big) \ . \tag{2.66}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 , \tag{2.67}$$

also known as Kepler's Third Law.

2.6 Appendix I : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \ldots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \ldots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu\big(x_1, \ldots, x_d\big) \ . \tag{2.68}$$

In general these are nonlinear equations.

Let $\hat{\boldsymbol{e}}_i^0 = \hat{\boldsymbol{x}}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{\boldsymbol{e}}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\boldsymbol{s} = \sum_{i=1}^d \hat{\boldsymbol{e}}_i^0 \, dx_i = \sum_{\mu=1}^d \hat{\boldsymbol{e}}_\mu \, h_\mu(q) \, dq_\mu \ , \tag{2.69}$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν. Finding the coefficient of dq_μ then gives

$$h_\mu(q) \, \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \hat{\boldsymbol{e}}_i^0 \qquad \Rightarrow \qquad \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^d M_{\mu\, i} \, \hat{\boldsymbol{e}}_i^0 \ , \tag{2.70}$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \, \frac{\partial x_i}{\partial q_\mu} \ . \tag{2.71}$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{\boldsymbol{e}}_\mu \cdot \hat{\boldsymbol{e}}_\nu = \big(M M^{\rm t}\big)_{\mu\nu} = \frac{1}{h_\mu(q) \, h_\nu(q)} \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} \ . \tag{2.72}$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q) \, \delta_{\mu\nu} \ . \tag{2.73}$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \bigg(\frac{\partial x_i}{\partial q_\mu}\bigg)^2} \ . \tag{2.74}$$

Note that

$$(d\boldsymbol{s})^2 = \sum_{\mu=1}^d h_\mu^2(q) \, (dq_\mu)^2 \ . \tag{2.75}$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\boldsymbol{s})^2 = \sum_{\mu,\nu=1}^d g_{\mu\nu}(q) \, dq_\mu \, dq_\nu \ , \tag{2.76}$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

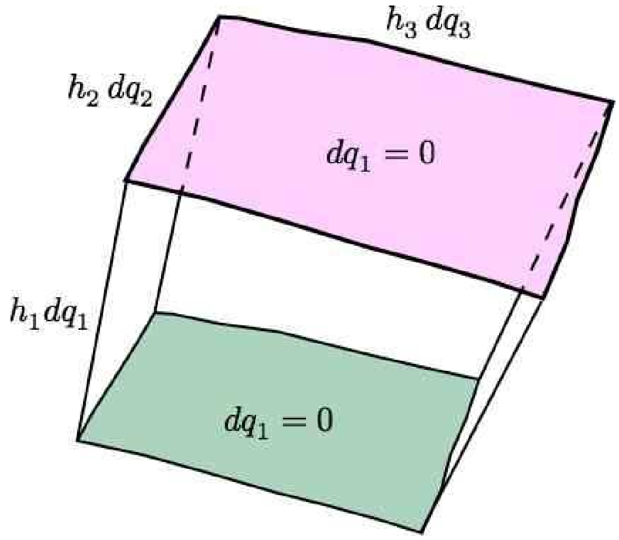

Figure 2.2: Volume element Ω for computing divergences.

2.6.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ):

$$x = \rho \sin\theta \cos\phi \quad , \quad y = \rho \sin\theta \sin\phi \quad , \quad z = \rho \cos\theta \ . \tag{2.77}$$

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2 \sin^2\!\theta \ . \tag{2.78}$$

Thus,

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\theta}}\, d\theta + \rho \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \ . \tag{2.79}$$

2.6.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient $\boldsymbol{\nabla} U$ of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1}\, dq_1 + \frac{\partial U}{\partial q_2}\, dq_2 + \frac{\partial U}{\partial q_3}\, dq_3 \\ &\equiv \boldsymbol{\nabla} U \cdot d\boldsymbol{s} \ . \end{aligned} \tag{2.80}$$

Thus,

$$\boldsymbol{\nabla} = \frac{\hat{\boldsymbol{e}}_1}{h_1(q)}\, \frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{e}}_2}{h_2(q)}\, \frac{\partial}{\partial q_2} + \frac{\hat{\boldsymbol{e}}_3}{h_3(q)}\, \frac{\partial}{\partial q_3} \ . \tag{2.81}$$

For the divergence, we use the divergence theorem, and we appeal to fig. 2.2:

$$\int\limits_{\Omega} dV\, \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int\limits_{\partial\Omega} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A} \ , \tag{2.82}$$

where $\varOmega$ is a region of three-dimensional space and $\partial\varOmega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \boldsymbol{\nabla} \cdot \boldsymbol{A} \cdot (h_1\, dq_1)\,(h_2\, dq_2)\,(h_3\, dq_3)\ . \tag{2.83}$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_1\, h_2\, h_3 \,\Big|_{q_1}^{q_1+dq_1} dq_2\, dq_3 + A_2\, h_1\, h_3 \,\Big|_{q_2}^{q_2+dq_2} dq_1\, dq_3 + A_3\, h_1\, h_2 \,\Big|_{q_3}^{q_1+dq_3} dq_1\, dq_2 \\ &= \bigg[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\bigg]\, dq_1\, dq_2\, dq_3\ . \end{aligned} \tag{2.84}$$

We therefore conclude

$$\boxed{\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{h_1\, h_2\, h_3}\bigg[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\bigg]}\ . \tag{2.85}$$

To obtain the curl $\boldsymbol{\nabla} \times \boldsymbol{A}$, we use Stokes' theorem again,

$$\int_{\varSigma} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{A} = \oint_{\partial\varSigma} d\boldsymbol{\ell} \cdot \boldsymbol{A}\ , \tag{2.86}$$

where $\varSigma$ is a two-dimensional region of space and $\partial\varSigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2\, dq_2$ and $h_3\, dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\boldsymbol{e}}_1 \cdot \boldsymbol{\nabla} \times \boldsymbol{A}\,(h_2\, dq_2)\,(h_3\, dq_3)\ . \tag{2.87}$$

The RHS is

$$\begin{aligned} \text{RHS} &= A_3\, h_3 \,\Big|_{q_2}^{q_2+dq_2} dq_3 - A_2\, h_2 \,\Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \bigg[\frac{\partial}{\partial q_2}\big(A_3\, h_3\big) - \frac{\partial}{\partial q_3}\big(A_2\, h_2\big)\bigg]\, dq_2\, dq_3\ . \end{aligned} \tag{2.88}$$

Therefore

$$(\boldsymbol{\nabla} \times \boldsymbol{A})_1 = \frac{1}{h_2\, h_3}\bigg(\frac{\partial(h_3\, A_3)}{\partial q_2} - \frac{\partial(h_2\, A_2)}{\partial q_3}\bigg)\ . \tag{2.89}$$

This is one component of the full result

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{h_1\, h_2\, h_2}\,\det\begin{pmatrix} h_1\, \hat{\boldsymbol{e}}_1 & h_2\, \hat{\boldsymbol{e}}_2 & h_3\, \hat{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\, A_1 & h_2\, A_2 & h_3\, A_3 \end{pmatrix}\ . \tag{2.90}$$

The Laplacian of a scalar function U is given by

$$\begin{aligned} \nabla^2 U &= \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} U \\ &= \frac{1}{h_1\, h_2\, h_3}\bigg\{\frac{\partial}{\partial q_1}\bigg(\frac{h_2\, h_3}{h_1}\,\frac{\partial U}{\partial q_1}\bigg) + \frac{\partial}{\partial q_2}\bigg(\frac{h_1\, h_3}{h_2}\,\frac{\partial U}{\partial q_2}\bigg) + \frac{\partial}{\partial q_3}\bigg(\frac{h_1\, h_2}{h_3}\,\frac{\partial U}{\partial q_3}\bigg)\bigg\}\ . \end{aligned} \tag{2.91}$$

2.7 Common curvilinear orthogonal systems

2.7.1 Rectangular coordinates

In *rectangular* coordinates (x, y, z), we have

$$h_x = h_y = h_z = 1 \ . \tag{2.92}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz \tag{2.93}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \ . \tag{2.94}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{x}}\, \frac{\partial U}{\partial x} + \hat{\boldsymbol{y}}\, \frac{\partial U}{\partial y} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \ . \tag{2.95}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \ . \tag{2.96}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} \ . \tag{2.97}$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.98}$$

2.7.2 Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z), we have

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\, \cos\phi + \hat{\boldsymbol{y}}\, \sin\phi \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\, \cos\phi - \hat{\boldsymbol{\phi}}\, \sin\phi \qquad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\, d\phi \tag{2.99}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\, \sin\phi + \hat{\boldsymbol{y}}\, \cos\phi \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\, \sin\phi + \hat{\boldsymbol{\phi}}\, \cos\phi \qquad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}\, d\phi \ . \tag{2.100}$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \ . \tag{2.101}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}}\, \rho\, d\phi + \hat{\boldsymbol{z}}\, dz \tag{2.102}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \ . \tag{2.103}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{\rho}} \, \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho} \, \frac{\partial U}{\partial \phi} + \hat{\boldsymbol{z}} \, \frac{\partial U}{\partial z} \ . \tag{2.104}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{\rho} \, \frac{\partial(\rho \, A_\rho)}{\partial \rho} + \frac{1}{\rho} \, \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \ . \tag{2.105}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \bigg(\frac{1}{\rho} \, \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \bigg) \hat{\boldsymbol{\rho}} + \bigg(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \bigg) \hat{\boldsymbol{\phi}} + \bigg(\frac{1}{\rho} \, \frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \, \frac{\partial A_\rho}{\partial \phi} \bigg) \hat{\boldsymbol{z}} \ . \tag{2.106}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho} \, \frac{\partial}{\partial \rho} \bigg(\rho \, \frac{\partial U}{\partial \rho} \bigg) + \frac{1}{\rho^2} \, \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.107}$$

2.7.3 Spherical coordinates

In *spherical* coordinates (r, θ, ϕ), we have

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \sin\theta \tag{2.108}$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}} \cos\theta \cos\phi + \hat{\boldsymbol{y}} \cos\theta \sin\phi - \hat{\boldsymbol{z}} \cos\theta \tag{2.109}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi \ , \tag{2.110}$$

for which

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}} \quad , \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \ . \tag{2.111}$$

The inverse is

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \tag{2.112}$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \tag{2.113}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta \ . \tag{2.114}$$

The differential relations are

$$d\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \, d\theta + \sin\theta \, \hat{\boldsymbol{\phi}} \, d\phi \tag{2.115}$$

$$d\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{r}} \, d\theta + \cos\theta \, \hat{\boldsymbol{\phi}} \, d\phi \tag{2.116}$$

$$d\hat{\boldsymbol{\phi}} = -\big(\sin\theta \, \hat{\boldsymbol{r}} + \cos\theta \, \hat{\boldsymbol{\theta}} \big) \, d\phi \tag{2.117}$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r \, \sin\theta \ . \tag{2.118}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{r}} \, dr + \hat{\boldsymbol{\theta}} \, r \, d\theta + \hat{\boldsymbol{\phi}} \, r \sin\theta \, d\phi \tag{2.119}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta \, \dot{\phi}^2 \ . \tag{2.120}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{r}} \, \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\theta}}}{r} \, \frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r \sin\theta} \, \frac{\partial U}{\partial \phi} \ . \tag{2.121}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{r^2} \, \frac{\partial (r^2 A_r)}{r} + \frac{1}{r \sin\theta} \, \frac{\partial (\sin\theta \, A_\theta)}{\partial \theta} + \frac{1}{r \sin\theta} \, \frac{\partial A_\phi}{\partial \phi} \ . \tag{2.122}$$

The curl is

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{r \sin\theta} &\left(\frac{\partial (\sin\theta \, A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right) \hat{\boldsymbol{r}} + \frac{1}{r} \left(\frac{1}{\sin\theta} \, \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right) \hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r} \left(\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right) \hat{\boldsymbol{\phi}} \ . \end{aligned} \tag{2.123}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2} \, \frac{\partial}{\partial r} \left(r^2 \, \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \, \frac{\partial}{\partial \theta} \left(\sin\theta \, \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\!\theta} \, \frac{\partial^2 U}{\partial \phi^2} \ . \tag{2.124}$$

2.7.4 Kinetic energy

Note the form of the kinetic energy of a point particle:

$$T = \tfrac{1}{2} m \left(\frac{d\boldsymbol{s}}{dt} \right)^2 = \tfrac{1}{2} m \big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \big) \qquad \text{(3D Cartesian)} \tag{2.125}$$

$$= \tfrac{1}{2} m \big(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 \big) \qquad \text{(2D polar)} \tag{2.126}$$

$$= \tfrac{1}{2} m \big(\dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \big) \qquad \text{(3D cylindrical)} \tag{2.127}$$

$$= \tfrac{1}{2} m \big(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\!\theta \, \dot{\phi}^2 \big) \qquad \text{(3D polar)} \ . \tag{2.128}$$