Lecture Notes on Classical Mechanics
(A Work in Progress)

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Contents

0.1 Preface

These lecture notes are based on material presented in both graduate and undergraduate mechanics classes which I have taught on several occasions during the past 20 years at UCSD (Physics 110A-B and Physics 200A-B).

The level of these notes is appropriate for an advanced undergraduate or a first year graduate course in classical mechanics. In some instances, I've tried to collect the discussion of more advanced material into separate sections, but in many cases this proves inconvenient, and so the level of the presentation fluctuates.

I have included many worked examples within the notes, as well as in the final chapter, which contains solutions from Physics 110A and 110B midterm and final exams. In my view, problem solving is essential toward learning basic physics. The geniuses among us might apprehend the fundamentals through deep contemplation after reading texts and attending lectures. The vast majority of us, however, acquire physical intuition much more slowly, and it is through problem solving that one gains experience in patches which eventually percolate so as to afford a more global understanding of the subject. A good analogy would be putting together a jigsaw puzzle: initially only local regions seem to make sense but eventually one forms the necessary connections so that one recognizes the entire picture.

My presentation and choice of topics has been influenced by many books as well as by my own professors. I've reiterated extended some discussions from other texts, such as Barger and Olsson's treatment of the gravitational swing-by effect, and their discussion of rolling and skidding tops. The figures were, with very few exceptions, painstakingly made using Keynote and/or SM.

Originally these notes also included material on dynamical systems and on Hamiltonian mechanics. These sections have now been removed and placed within a separate set of notes on nonlinear dynamics (Physics 221A).

My only request, to those who would use these notes: please contact me if you find errors or typos, or if you have suggestions for additional material. My email address is `darovas@ucsd.edu`. I plan to update and extend these notes as my time and inclination permit.

Chapter 0

Reference Materials

Here I list several resources, arranged by topic. My personal favorites are marked with a diamond (◇).

0.1 Lagrangian Mechanics (mostly)

◇ L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Butterworth-Heinemann, 1976)

◇ A. L. Fetter and J. D. Walecka, *Nonlinear Mechanics* (Dover, 2006)

- O. D. Johns, *Analytical Mechanics for Relativity and Quantum Mechanics* (Oxford, 2005)

- D. T. Greenwood, *Classical Mechanics* (Dover, 1997)

- H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, 2001)

- V. Barger and M. Olsson, *Classical Mechanics : A Modern Perspective* (McGraw-Hill, 1994)

0.2 Hamiltonian Mechanics (mostly)

◇ J. V. José and E. J. Saletan, *Mathematical Methods of Classical Mechanics* (Springer, 1997)

- ◇ W. Dittrich and M. Reuter, *Classical and Quantum Dynamics* (Springer, 2001)
- • V. I. Arnold *Introduction to Dynamics* (Cambridge, 1982)
- • V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics* (Springer, 2006)
- • I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge, 1982)

0.3 Mathematics

- ◇ I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Dover, 1991)
- ◇ V. I. Arnold, *Ordinary Differential Equations* (MIT Press, 1973)
- • V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer, 1988)
- • R. Weinstock, *Calculus of Variations* (Dover, 1974)

Chapter 1

Introduction to Dynamics

1.1 Introduction and Review

Dynamics is the science of how things *move.* A complete solution to the motion of a system means that we know the coordinates of all its constituent particles as functions of time. For a single point particle moving in three-dimensional space, this means we want to know its position vector $\boldsymbol{r}(t)$ as a function of time. If there are many particles, the motion is described by a set of functions $\boldsymbol{r}_i(t)$, where i labels which particle we are talking about. So generally speaking, solving for the motion means being able to predict where a particle will be at any given instant of time. Of course, knowing the function $\boldsymbol{r}_i(t)$ means we can take its derivative and obtain the velocity $\boldsymbol{v}_i(t) = d\boldsymbol{r}_i/dt$ at any time as well.

The complete motion for a system is not given to us outright, but rather is encoded in a set of differential equations, called the *equations of motion.* An example of an equation of motion is

$$m\frac{d^2x}{dt^2} = -mg \tag{1.1}$$

with the solution

$$x(t) = x_0 + v_0 t - \tfrac{1}{2}gt^2 \tag{1.2}$$

where x_0 and v_0 are constants corresponding to the initial *boundary conditions* on the position and velocity: $x(0) = x_0$, $v(0) = v_0$. This particular solution describes the vertical motion of a particle of mass m moving near the earth's surface.

In this class, we shall discuss a general framework by which the equations of motion may be obtained, and methods for solving them. That "general framework" is Lagrangian Dynamics, which itself is really nothing more than an elegant restatement of Isaac Newton's Laws of Motion.

1.1.1 Newton's laws of motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens. To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

I. A body remains in uniform motion unless acted on by a force.
II. Force equals rate of change of momentum: $\boldsymbol{F} = d\boldsymbol{p}/dt$.
III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t$. Now $\boldsymbol{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\boldsymbol{r}(t)$ in a different coordinate system whose origin $\boldsymbol{R}$ moves according to the function $\boldsymbol{R}(t)$, then in this new "frame of reference" the position of our particle will be

$$\begin{aligned} \boldsymbol{r}'(t) &= \boldsymbol{r}(t) - \boldsymbol{R}(t) \\ &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t - \boldsymbol{R}(t) \ . \end{aligned} \tag{1.3}$$

If the acceleration $d^2\boldsymbol{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton's Laws hold. A transformation from one frame $\mathcal{K}$ to another frame $\mathcal{K}'$ which moves at constant velocity $\boldsymbol{V}$ relative to $\mathcal{K}$ is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth's surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum $\boldsymbol{p}$, defined as the product $\boldsymbol{p} = m\boldsymbol{v}$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force $\boldsymbol{F}$ depends on the coordinates (or possibly velocities) themselves. This is known as a *force law.* Examples of force laws include:

$$\begin{aligned}
\text{Constant force:} \qquad & \boldsymbol{F} = -m\boldsymbol{g} \\
\text{Hooke's Law:} \qquad & F = -kx \\
\text{Gravitation:} \qquad & \boldsymbol{F} = -GMm\,\hat{\boldsymbol{r}}/r^2 \\
\text{Lorentz force:} \qquad & \boldsymbol{F} = q\,\boldsymbol{E} + q\,\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \\
\text{Fluid friction } (v \text{ small}): \qquad & \boldsymbol{F} = -b\,\boldsymbol{v}\ .
\end{aligned}$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and hence the accelerations are parallel to each other, with

$$\frac{a_1}{a_2} = -\frac{m_2}{m_1}\ , \tag{1.4}$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is a constant, a result known as the *conservation of momentum.*

1.1.2 Aside : inertial *vs.* gravitational mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force $\boldsymbol{F}_{ij}$ exerted by a particle i by another particle j is

$$\boldsymbol{F}_{ij} = -Gm_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3}\ , \tag{1.5}$$

where G, the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11}\mathrm{N \cdot m^2/kg^2}\ . \tag{1.6}$$

Notice Newton's Third Law in action: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_{\rm e}$, with $\boldsymbol{r}_i - \boldsymbol{r}_j \simeq R_{\rm e}\hat{\boldsymbol{r}}$ and obtain

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} \equiv -m\boldsymbol{g} \tag{1.7}$$

where $\hat{\boldsymbol{r}}$ is a radial unit vector pointing from the earth's center and $g = GM_{\rm e}/R_{\rm e}^2 \simeq 9.8\,{\rm m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\boldsymbol{a} = -\boldsymbol{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$\boldsymbol{a} = -\frac{m'}{m}\,\boldsymbol{g} \tag{1.8}$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.2 Examples of Motion in One Dimension

To gain some experience with solving equations of motion in a physical setting, we consider some physically relevant examples of one-dimensional motion.

1.2.1 Uniform force

With $F = -mg$, appropriate for a particle falling under the influence of a uniform gravitational field, we have $m\,d^2x/dt^2 = -mg$, or $\ddot{x} = -g$. Notation:

$$\dot{x} \equiv \frac{dx}{dt}\ , \qquad \ddot{x} \equiv \frac{d^2x}{dt^2}\ , \qquad \overset{\vdots\vdots}{x} = \frac{d^7x}{dt^7}\ , \qquad etc. \tag{1.9}$$

With $v = \dot{x}$, we solve $dv/dt = -g$:

$$\int\limits_{v(0)}^{v(t)} dv = \int\limits_0^t ds\,(-g) \tag{1.10}$$

$$v(t) - v(0) = -gt\ . \tag{1.11}$$

Note that there is a constant of integration, $v(0)$, which enters our solution.

We are now in position to solve $dx/dt = v$:

$$\int\limits_{x(0)}^{x(t)} dx = \int\limits_0^t ds\, v(s) \tag{1.12}$$

$$x(t) = x(0) + \int\limits_0^t ds\, \big[v(0) - gs\big] \tag{1.13}$$

$$= x(0) + v(0)t - \tfrac{1}{2}gt^2 \ . \tag{1.14}$$

Note that a second constant of integration, $x(0)$, has appeared.

1.2.2 Uniform force with linear frictional damping

In this case,

$$m\frac{dv}{dt} = -mg - \gamma v \tag{1.15}$$

which may be rewritten

$$\frac{dv}{v + mg/\gamma} = -\frac{\gamma}{m} dt \tag{1.16}$$

$$d\ln(v + mg/\gamma) = -(\gamma/m)dt \ . \tag{1.17}$$

Integrating then gives

$$\ln\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) = -\gamma t/m \tag{1.18}$$

$$v(t) = -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma}\right) e^{-\gamma t/m} \ . \tag{1.19}$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$x(t) = x(0) + \frac{m}{\gamma}\left(v(0) + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}\, t \ . \tag{1.20}$$

The solution to the *second* order ODE $m\ddot{x} = -mg - \gamma\dot{x}$ thus entails *two* constants of integration: $v(0)$ and $x(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \ . \tag{1.21}$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity})$$

$$\begin{aligned} j &= n \cdot (-e) \cdot (-v_\infty) \\ &= \frac{ne^2\tau}{m}\, E \ . \end{aligned} \tag{1.22}$$

The ratio j/E is called the *conductivity* of the metal, σ. According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations ("phonons"). In high purity copper at low temperatures ($T \lesssim 4\,\text{K}$), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}\,\text{s}$).

1.2.3 Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_\text{f} = -cv^2\,\text{sgn}\,(v)$, where $\text{sgn}\,(v)$ is the *sign* of v: $\text{sgn}\,(v) = +1$ if $v > 0$ and $\text{sgn}\,(v) = -1$ if $v < 0$. (Note one can also write $\text{sgn}\,(v) = v/|v|$ where $|v|$ is the *absolute value*.) Why all this trouble with $\text{sgn}\,(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be *oppositely directed* with respect to the velocity v. We will assume that $v < 0$ always, hence $F_\text{f} = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot v = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \tag{1.23}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left[\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty}\right] \tag{1.24}$$

we obtain

$$\begin{aligned} \frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \\ &= \frac{1}{2v_\infty} d\ln\left(\frac{v_\infty - v}{v_\infty + v}\right) \\ &= \frac{g}{v_\infty^2}\, dt \ . \end{aligned} \tag{1.25}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\frac{1}{2v_\infty} \ln\left(\frac{v_\infty - v(t)}{v_\infty + v(t)}\right) = \frac{gt}{v_\infty^2} \tag{1.26}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \ . \tag{1.27}$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \ . \tag{1.28}$$

Again, as $t \to \infty$ one has $v(t) \to -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Advanced Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt, all the fluid molecules inside the volume $\Delta V = S \cdot v\,\Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\Delta u = 2(v - u)$. The total momentum change is obtained by multiplying Δu by the total mass $M = \varrho\,\Delta V$, where ϱ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v - u) \cdot \varrho\, S\, v\, \Delta t \tag{1.29}$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S\,\varrho\, v(v - u) \ . \tag{1.30}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u \rangle = 0$, we obtain the result $\langle F \rangle = 2S\varrho v^2$, where $\langle \cdots \rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_{\rm f} = -\langle F \rangle = -cv^2$, where $c = 2S\varrho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu\varrho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.2.4 Crossed electric and magnetic fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We'll throw in gravity for good measure. We take $\boldsymbol{E} = E\hat{\boldsymbol{x}}$,

$\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. The equation of motion is Newton's 2nd Law again:

$$m\,\ddot{\boldsymbol{r}} = m\boldsymbol{g} + q\boldsymbol{E} + \tfrac{q}{c}\,\dot{\boldsymbol{r}} \times \boldsymbol{B}\ . \tag{1.31}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$m\ddot{x} = qE + \frac{qB}{c}\,\dot{y} \tag{1.32}$$

$$m\ddot{y} = -\frac{qB}{c}\,\dot{x} \tag{1.33}$$

$$m\ddot{z} = -mg\ . \tag{1.34}$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0)\,t - \tfrac{1}{2}gt^2\ . \tag{1.35}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\dot{v}_x = \omega_\mathrm{c}(v_y + u_\mathrm{D}) \tag{1.36}$$

$$\dot{v}_y = -\omega_\mathrm{c}\,v_x\ , \tag{1.37}$$

where $\omega_\mathrm{c} = qB/mc$ is the *cyclotron frequency* and $u_\mathrm{D} = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$v_x(t) = v_x(0)\,\cos(\omega_\mathrm{c}t) + \big(v_y(0) + u_\mathrm{D}\big)\,\sin(\omega_\mathrm{c}t) \tag{1.38}$$

$$v_y(t) = -u_\mathrm{D} + \big(v_y(0) + u_\mathrm{D}\big)\cos(\omega_\mathrm{c}t) - v_x(0)\,\sin(\omega_\mathrm{c}t)\ . \tag{1.39}$$

Integrating again, the full motion is given by:

$$x(t) = x(0) + A\,\sin\delta + A\,\sin(\omega_\mathrm{c}t - \delta) \tag{1.40}$$

$$y(r) = y(0) - u_\mathrm{D}\,t - A\cos\delta + A\,\cos(\omega_\mathrm{c}t - \delta)\ , \tag{1.41}$$

where

$$A = \frac{1}{\omega_\mathrm{c}}\sqrt{\dot{x}^2(0) + \big(\dot{y}(0) + u_\mathrm{D}\big)^2} \quad , \quad \delta = \tan^{-1}\left(\frac{\dot{y}(0) + u_\mathrm{D}}{\dot{x}(0)}\right)\ . \tag{1.42}$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0)\ ,\ y(0)\ ,\ z(0)\ ,\ A\ ,\ \delta\ ,\ \dot{z}(0)\ . \tag{1.43}$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

1.3 Pause for Reflection

In mechanical systems, for each coordinate, or "degree of freedom," there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

Chapter 2

Systems of Particles

2.1 Work-Energy Theorem

Consider a system of many particles, with positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \ . \tag{2.1}$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \, \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i \ . \tag{2.2}$$

Here, we've used the relation

$$\frac{d}{dt}\big(\boldsymbol{A}^2\big) = 2\,\boldsymbol{A} \cdot \frac{d\boldsymbol{A}}{dt} \ . \tag{2.3}$$

We now invoke Newton's 2nd Law, $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, to write eqn. 2.2 as $\dot{T}_i = \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i$. We integrate this equation from time $t_{\rm A}$ to $t_{\rm B}$:

$$\begin{aligned} T_i^{\rm (B)} - T_i^{\rm (A)} &= \int\limits_{t_{\rm A}}^{t_{\rm B}} \! dt \, \frac{dT_i}{dt} \\ &= \int\limits_{t_{\rm A}}^{t_{\rm B}} \! dt \, \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \equiv \sum_i W_i^{\rm (A\to B)} \ , \end{aligned} \tag{2.4}$$

where $W_i^{\rm (A\to B)}$ is the total *work done* on particle i during its motion from state A to state B, Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{\rm (A\to B)} = \sum_i W_i^{\rm (A\to B)}$. Eqn. 2.4 is known as the *work-energy theorem*. It says that

In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done: $T^{\rm (B)} - T^{\rm (A)} = W^{\rm (A\to B)}$.

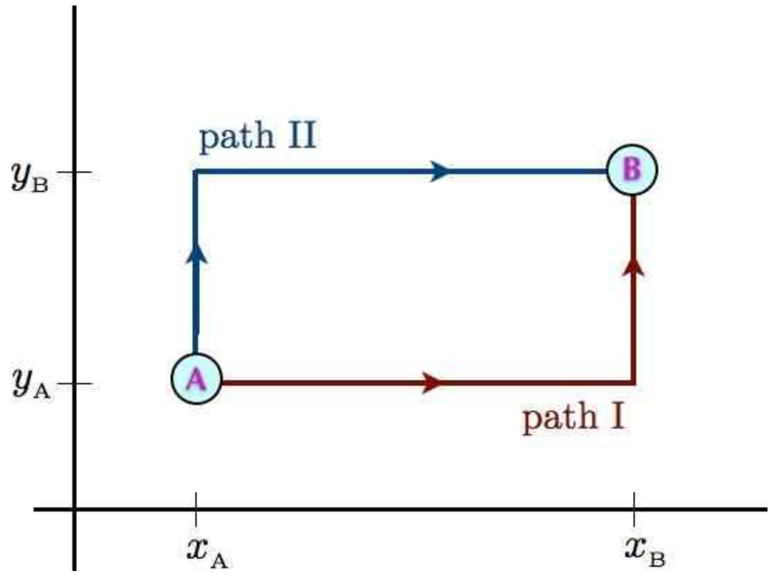


Figure 2.1: Two paths joining points A and B.

2.2 Conservative and Nonconservative Forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{(\mathrm{A}\to\mathrm{B})} = \int_{t_\mathrm{A}}^{t_\mathrm{B}} dt\, \boldsymbol{F}\cdot\boldsymbol{v} \ , \tag{2.5}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. This is the most general expression for the work done. If the force $\boldsymbol{F}$ depends only on the particle's position $\boldsymbol{r}$, we may write $d\boldsymbol{r} = \boldsymbol{v}\,dt$, and then

$$W^{(\mathrm{A}\to\mathrm{B})} = \int_{\boldsymbol{r}_\mathrm{A}}^{\boldsymbol{r}_\mathrm{B}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) \ . \tag{2.6}$$

Consider now the force

$$\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\,\hat{\boldsymbol{x}} + K_2\, x\,\hat{\boldsymbol{y}} \ , \tag{2.7}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 2.1:

$$W^{(\mathrm{I})} = K_1 \int_{x_\mathrm{A}}^{x_\mathrm{B}} dx\, y_\mathrm{A} + K_2 \int_{y_\mathrm{A}}^{y_\mathrm{B}} dy\, x_\mathrm{B} = K_1\, y_\mathrm{A}\,(x_\mathrm{B}-x_\mathrm{A}) + K_2\, x_\mathrm{B}\,(y_\mathrm{B}-y_\mathrm{A}) \tag{2.8}$$

$$W^{(\mathrm{II})} = K_1 \int_{x_\mathrm{A}}^{x_\mathrm{B}} dx\, y_\mathrm{B} + K_2 \int_{y_\mathrm{A}}^{y_\mathrm{B}} dy\, x_\mathrm{A} = K_1\, y_\mathrm{B}\,(x_\mathrm{B}-x_\mathrm{A}) + K_2\, x_\mathrm{A}\,(y_\mathrm{B}-y_\mathrm{A}) \ . \tag{2.9}$$

Note that in general $W^{(\mathrm{I})} \neq W^{(\mathrm{II})}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(\mathrm{I})} - W^{(\mathrm{II})} = (K_2 - K_1)\,(x_{\mathrm{B}} - x_{\mathrm{A}})\,(y_{\mathrm{B}} - y_{\mathrm{A}}) \ . \tag{2.10}$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 2.1. The reason for this is Stokes' theorem:

$$\oint\limits_{\partial\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F} = \int\limits_{\mathcal{C}} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F} \ . \tag{2.11}$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path[1], dS is the scalar differential area element, $\hat{\boldsymbol{n}}$ is the unit normal to that differential area element, and $\boldsymbol{\nabla} \times \boldsymbol{F}$ is the curl of $\boldsymbol{F}$:

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{F} &= \det \begin{pmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{pmatrix} \\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \ . \end{aligned} \tag{2.12}$$

For the force under consideration, $\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}}$, the curl is

$$\boldsymbol{\nabla} \times \boldsymbol{F} = (K_2 - K_1)\, \hat{\boldsymbol{z}} \ , \tag{2.13}$$

which is a constant. The RHS of eqn. 2.11 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 2.10, we evaluate the integral $\oint\limits_{\mathcal{C}} d\boldsymbol{\ell} \cdot \boldsymbol{F}$ along the path $\gamma_{\mathrm{II}}^{-1} \circ \gamma_{\mathrm{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the integral of $\hat{\boldsymbol{n}} \cdot \boldsymbol{\nabla} \times \boldsymbol{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 2.10.

When $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ everywhere in space, we can always write $\boldsymbol{F} = -\boldsymbol{\nabla} U$, where $U(\boldsymbol{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$\begin{aligned} W^{(\mathrm{A}\to\mathrm{B})} &= \int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r} \cdot \boldsymbol{F}(\boldsymbol{r}) \\ &= -\int\limits_{\boldsymbol{r}_{\mathrm{A}}}^{\boldsymbol{r}_{\mathrm{B}}} d\boldsymbol{r} \cdot \boldsymbol{\nabla} U \\ &= U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ . \end{aligned} \tag{2.14}$$

[1]If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

The work-energy theorem then gives

$$T^{(\mathrm{B})} - T^{(\mathrm{A})} = U(\boldsymbol{r}_{\mathrm{A}}) - U(\boldsymbol{r}_{\mathrm{B}}) \ , \tag{2.15}$$

which says

$$E^{(\mathrm{B})} = T^{(\mathrm{B})} + U(\boldsymbol{r}_{\mathrm{B}}) = T^{(\mathrm{A})} + U(\boldsymbol{r}_{\mathrm{A}}) = E^{(\mathrm{A})} \ . \tag{2.16}$$

Thus, the total energy $E = T + U$ is conserved.

2.2.1 Example : integrating $\boldsymbol{F} = -\boldsymbol{\nabla} U$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can compute $U(\boldsymbol{r})$ by integrating, *viz.*

$$U(\boldsymbol{r}) = U(\boldsymbol{0}) - \int\limits_{\boldsymbol{0}}^{\boldsymbol{r}} d\boldsymbol{r}' \cdot \boldsymbol{F}(\boldsymbol{r}') \ . \tag{2.17}$$

The integral does not depend on the path chosen connecting $\boldsymbol{0}$ and $\boldsymbol{r}$. For example, we can take

$$U(x,y,z) = U(0,0,0) \ - \int\limits_{(0,0,0)}^{(x,0,0)} \!\! dx' \, F_x(x',0,0) \ - \int\limits_{(x,0,0)}^{(x,y,0)} \!\! dy' \, F_y(x,y',0) \ - \int\limits_{(z,y,0)}^{(x,y,z)} \!\! dz' \, F_z(x,y,z') \ . \tag{2.18}$$

The constant $U(0,0,0)$ is arbitrary and impossible to determine from $\boldsymbol{F}$ alone.

As an example, consider the force

$$\boldsymbol{F}(\boldsymbol{r}) = -ky \, \hat{\boldsymbol{x}} - kx \, \hat{\boldsymbol{y}} - 4bz^3 \, \hat{\boldsymbol{z}} \ , \tag{2.19}$$

where k and b are constants. We have

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_x = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0 \tag{2.20}$$

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_y = \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0 \tag{2.21}$$

$$\big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_z = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \ , \tag{2.22}$$

so $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ and $\boldsymbol{F}$ must be expressible as $\boldsymbol{F} = -\boldsymbol{\nabla} U$. Integrating using eqn. 2.18, we have

$$U(x,y,z) = U(0,0,0) \ + \int\limits_{(0,0,0)}^{(x,0,0)} \!\! dx' \, k \cdot 0 \ + \int\limits_{(x,0,0)}^{(x,y,0)} \!\! dy' \, kxy' \ + \int\limits_{(z,y,0)}^{(x,y,z)} \!\! dz' \, 4bz'^3 \tag{2.23}$$

$$= U(0,0,0) + kxy + bz^4 \ . \tag{2.24}$$

Another approach is to integrate the partial differential equation $\boldsymbol{\nabla} U = -\boldsymbol{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\boldsymbol{x}}$-component,

$$\frac{\partial U}{\partial x} = ky \ . \tag{2.25}$$

Integrating, we obtain

$$U(x, y, z) = kxy + f(y, z) \ , \tag{2.26}$$

where $f(y, z)$ is at this point an *arbitrary function* of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \quad \Longrightarrow \quad U(x, y, z) = kxy + g(x, z) \ . \tag{2.27}$$

Finally, the z-component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \quad \Longrightarrow \quad U(x, y, z) = bz^4 + h(x, y) \ . \tag{2.28}$$

We now equate the first two expressions:

$$kxy + f(y, z) = kxy + g(x, z) \ . \tag{2.29}$$

Subtracting kxy from each side, we obtain the equation $f(y, z) = g(x, z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y, z) = g(x, z) = q(z) \ , \tag{2.30}$$

where $q(z)$ is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x, y) = kxy + q(z) \ . \tag{2.31}$$

The only possible solution is $h(x, y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x, y, z) = C + kxy + bz^4 \ . \tag{2.32}$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x, y, z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z. The fact that $f(y, z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y, z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y, z)$.

2.3 Conservative Forces in Many Particle Systems

$$T = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \tag{2.33}$$

$$U = \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.34}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r}-\boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \, \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\text{ext})} + \boldsymbol{F}_i^{(\text{int})} \ , \tag{2.35}$$

with

$$\boldsymbol{F}_i^{(\text{ext})} = -\frac{\partial V(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \tag{2.36}$$

$$\boldsymbol{F}_i^{(\text{int})} = -\sum_j \frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\boldsymbol{r}_i} \equiv \sum_j \boldsymbol{F}_{ij}^{(\text{int})} \ . \tag{2.37}$$

Here, $\boldsymbol{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j:

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\partial \boldsymbol{r}_i} = -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \, v'\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.38}$$

Note that $\boldsymbol{F}_{ij}^{(\text{int})} = -\boldsymbol{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$, in which case we may write the interparticle force as

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\hat{\boldsymbol{r}}_{ij} \, v'\big(r_{ij}\big) \ . \tag{2.39}$$

2.4 Linear and Angular Momentum

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$. Its rate of change is

$$\frac{d\boldsymbol{P}}{dt} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\sum_{i \neq j} \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{F}_{ij}^{(\text{int})} + \boldsymbol{F}_{ji}^{(\text{int})} = 0} = \boldsymbol{F}_{\text{tot}}^{(\text{ext})} \ , \tag{2.40}$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\boldsymbol{P} = \sum_i m_i \dot{\boldsymbol{r}}_i = M \dot{\boldsymbol{R}} \tag{2.41}$$

$$M = \sum_i m_i \quad \text{(total mass)} \tag{2.42}$$

$$\boldsymbol{R} = \frac{\sum_i m_i \, \boldsymbol{r}_i}{\sum_i m_i} \quad \text{(center-of-mass)} \ . \tag{2.43}$$

Next, consider the total angular momentum,

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \ . \tag{2.44}$$

The rate of change of $\boldsymbol{L}$ is then

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \sum_i \left\{ m_i \dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i + m_i \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i \right\} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \sum_{i \neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}^{(\text{int})} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\tfrac{1}{2} \sum_{i \neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij}^{(\text{int})} = 0} \\ &= \boldsymbol{N}_{\text{tot}}^{(\text{ext})} \ . \end{aligned} \tag{2.45}$$

Finally, it is useful to establish the result

$$T = \tfrac{1}{2} \sum_i m_i \, \dot{\boldsymbol{r}}_i^2 = \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \sum_i m_i \big(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{R}} \big)^2 \ , \tag{2.46}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$\begin{aligned} 0 &= \int\limits_{\text{initial}}^{\text{final}} dE = \int\limits_{\text{initial}}^{\text{final}} dT + \int\limits_{\text{initial}}^{\text{final}} dU \\ &= T^{(\text{B})} - T^{(\text{A})} - \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i \ , \end{aligned} \tag{2.47}$$

which is to say

$$\Delta T = T^{(\text{B})} - T^{(\text{A})} = \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i = -\Delta U \ . \tag{2.48}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \ . \tag{2.49}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i \, \phi(\boldsymbol{r}_i) \longrightarrow \int d^3\!r \, \rho(\boldsymbol{r}) \, \phi(\boldsymbol{r}) \ , \tag{2.50}$$

where $\rho(\boldsymbol{r})$ is the mass density, and $\phi(\boldsymbol{r})$ is any function.

2.5 Scaling of Solutions for Homogeneous Potentials

2.5.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U(\lambda\,\boldsymbol{r}_1,\ldots,\lambda\,\boldsymbol{r}_N) = \lambda^k\,U(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)\ . \tag{2.51}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N) = -G\sum_{i<j}\frac{m_i\,m_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|} \quad ; \quad k=-1\ , \tag{2.52}$$

and the harmonic oscillator,

$$U(q_1,\ldots,q_n) = \tfrac{1}{2}\sum_{\sigma,\sigma'} V_{\sigma\sigma'}\,q_\sigma\,q_{\sigma'} \quad ; \quad k=+2\ . \tag{2.53}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1,\ldots,x_n)$ is homogeneous:

$$H(\lambda\,x_1,\ldots,\lambda\,x_n) = \lambda^k\,H(x_1,\ldots,x_n)\ . \tag{2.54}$$

Then

$$\boxed{\frac{d}{d\lambda}\bigg|_{\lambda=1} H(\lambda\,x_1,\ldots,\lambda\,x_n) = \sum_{i=1}^n x_i\,\frac{\partial H}{\partial x_i} = k\,H} \tag{2.55}$$

2.5.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\boldsymbol{r}_i = \alpha\,\tilde{\boldsymbol{r}}_i \qquad , \qquad t = \beta\,\tilde{t}\ . \tag{2.56}$$

Then

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\alpha}{\beta}\,\frac{d\tilde{\boldsymbol{r}}_i}{d\tilde{t}} \qquad , \qquad \frac{d^2\boldsymbol{r}_i}{dt^2} = \frac{\alpha}{\beta^2}\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2}\ . \tag{2.57}$$

The force $\boldsymbol{F}_i$ is given by

$$\begin{aligned}\boldsymbol{F}_i &= -\frac{\partial}{\partial\boldsymbol{r}_i}\,U(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)\\ &= -\frac{\partial}{\partial(\alpha\tilde{\boldsymbol{r}}_i)}\,\alpha^k\,U(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N)\\ &= \alpha^{k-1}\,\tilde{\boldsymbol{F}}_i\ .\end{aligned} \tag{2.58}$$

Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2}\, m_i\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} = \alpha^{k-1}\, \tilde{\boldsymbol{F}}_i \ . \tag{2.59}$$

If we choose β such that

We now demand

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \ , \tag{2.60}$$

then the equation of motion is invariant under the rescaling transformation! This means that if $\boldsymbol{r}(t)$ is a solution to the equations of motion, then so is $\alpha\, \boldsymbol{r}\big(\alpha^{\frac{1}{2}k-1}\, t\big)$. This gives us an entire one-parameter family of solutions, for all real positive α.

If $\boldsymbol{r}(t)$ is periodic with period T, the $\boldsymbol{r}_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k}\, T$. Thus,

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \ . \tag{2.61}$$

Here, $\alpha = L'/L$ is the ratio of length scales. Velocities, energies and angular momenta scale accordingly:

$$\big[v\big] = \frac{L}{T} \qquad \Rightarrow \qquad \frac{v'}{v} = \frac{L'}{L}\bigg/\frac{T'}{T} = \alpha^{\frac{1}{2}k} \tag{2.62}$$

$$\big[E\big] = \frac{ML^2}{T^2} \qquad \Rightarrow \qquad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2\bigg/\left(\frac{T'}{T}\right)^2 = \alpha^k \tag{2.63}$$

$$\big[\boldsymbol{L}\big] = \frac{ML^2}{T} \qquad \Rightarrow \qquad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2\bigg/\frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \ . \tag{2.64}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\alpha) = \alpha\, q_\sigma(t) \ . \tag{2.65}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\alpha) = \alpha\, \boldsymbol{r}\big(\alpha^{-3/2}\, t\big) \ . \tag{2.66}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 , \tag{2.67}$$

also known as Kepler's Third Law.

2.6 Appendix I : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \ldots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \ldots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu\big(x_1, \ldots, x_d\big) \ . \tag{2.68}$$

In general these are nonlinear equations.

Let $\hat{\boldsymbol{e}}_i^0 = \hat{\boldsymbol{x}}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{\boldsymbol{e}}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\boldsymbol{s} = \sum_{i=1}^d \hat{\boldsymbol{e}}_i^0 \, dx_i = \sum_{\mu=1}^d \hat{\boldsymbol{e}}_\mu \, h_\mu(q)\, dq_\mu \ , \tag{2.69}$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν. Finding the coefficient of dq_μ then gives

$$h_\mu(q)\, \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \hat{\boldsymbol{e}}_i^0 \qquad \Rightarrow \qquad \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^d M_{\mu i} \, \hat{\boldsymbol{e}}_i^0 \ , \tag{2.70}$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \, \frac{\partial x_i}{\partial q_\mu} \ . \tag{2.71}$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{\boldsymbol{e}}_\mu \cdot \hat{\boldsymbol{e}}_\nu = \big(MM^{\rm t}\big)_{\mu\nu} = \frac{1}{h_\mu(q)\, h_\nu(q)} \sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} \ . \tag{2.72}$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^d \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} = h_\mu^2(q)\, \delta_{\mu\nu} \ . \tag{2.73}$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^d \bigg(\frac{\partial x_i}{\partial q_\mu}\bigg)^2} \ . \tag{2.74}$$

Note that

$$(d\boldsymbol{s})^2 = \sum_{\mu=1}^d h_\mu^2(q)\, (dq_\mu)^2 \ . \tag{2.75}$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\boldsymbol{s})^2 = \sum_{\mu,\nu=1}^d g_{\mu\nu}(q)\, dq_\mu \, dq_\nu \ , \tag{2.76}$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

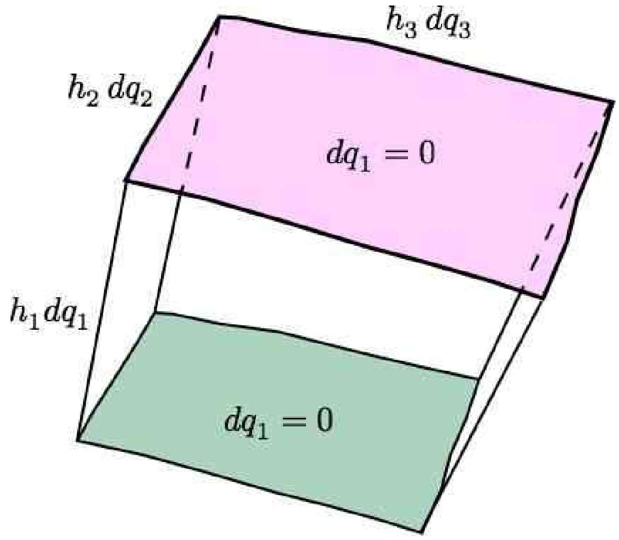


Figure 2.2: Volume element Ω for computing divergences.

2.6.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ):

$$x = \rho\, \sin\theta\, \cos\phi \quad , \quad y = \rho\, \sin\theta\, \sin\phi \quad , \quad z = \rho\, \cos\theta\ . \tag{2.77}$$

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2\, \sin^2\!\theta\ . \tag{2.78}$$

Thus,

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\theta}}\, d\theta + \rho\, \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi\ . \tag{2.79}$$

2.6.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient $\boldsymbol{\nabla} U$ of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1}\, dq_1 + \frac{\partial U}{\partial q_2}\, dq_2 + \frac{\partial U}{\partial q_3}\, dq_3 \\ &\equiv \boldsymbol{\nabla} U \cdot d\boldsymbol{s}\ . \end{aligned} \tag{2.80}$$

Thus,

$$\boldsymbol{\nabla} = \frac{\hat{\boldsymbol{e}}_1}{h_1(q)}\, \frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{e}}_2}{h_2(q)}\, \frac{\partial}{\partial q_2} + \frac{\hat{\boldsymbol{e}}_3}{h_3(q)}\, \frac{\partial}{\partial q_3}\ . \tag{2.81}$$

For the divergence, we use the divergence theorem, and we appeal to fig. 2.2:

$$\int\limits_{\Omega} \! dV\, \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int\limits_{\partial\Omega} \! dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A}\ , \tag{2.82}$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \boldsymbol{\nabla}\cdot\boldsymbol{A}\cdot(h_1\,dq_1)\,(h_2\,dq_2)\,(h_3\,dq_3)\ . \tag{2.83}$$

The RHS is

$$\begin{aligned}\text{RHS} &= A_1\,h_2\,h_3\Big|_{q_1}^{q_1+dq_1} dq_2\,dq_3 + A_2\,h_1\,h_3\Big|_{q_2}^{q_2+dq_2} dq_1\,dq_3 + A_3\,h_1\,h_2\Big|_{q_3}^{q_1+dq_3} dq_1\,dq_2\\ &= \bigg[\frac{\partial}{\partial q_1}\big(A_1\,h_2\,h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\,h_1\,h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\,h_1\,h_2\big)\bigg]\,dq_1\,dq_2\,dq_3\ .\end{aligned} \tag{2.84}$$

We therefore conclude

$$\boxed{\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{h_1\,h_2\,h_3}\bigg[\frac{\partial}{\partial q_1}\big(A_1\,h_2\,h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\,h_1\,h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\,h_1\,h_2\big)\bigg]}\ . \tag{2.85}$$

To obtain the curl $\boldsymbol{\nabla}\times\boldsymbol{A}$, we use Stokes' theorem again,

$$\int\limits_{\Sigma} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \oint\limits_{\partial\Sigma} d\boldsymbol{\ell}\cdot\boldsymbol{A}\ , \tag{2.86}$$

where Σ is a two-dimensional region of space and $\partial\Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2\,dq_2$ and $h_3\,dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\boldsymbol{e}}_1\cdot\boldsymbol{\nabla}\times\boldsymbol{A}\,(h_2\,dq_2)\,(h_3\,dq_3)\ . \tag{2.87}$$

The RHS is

$$\begin{aligned}\text{RHS} &= A_3\,h_3\Big|_{q_2}^{q_2+dq_2} dq_3 - A_2\,h_2\Big|_{q_3}^{q_3+dq_3} dq_2\\ &= \bigg[\frac{\partial}{\partial q_2}\big(A_3\,h_3\big) - \frac{\partial}{\partial q_3}\big(A_2\,h_2\big)\bigg]\,dq_2\,dq_3\ .\end{aligned} \tag{2.88}$$

Therefore

$$(\boldsymbol{\nabla}\times\boldsymbol{A})_1 = \frac{1}{h_2\,h_3}\bigg(\frac{\partial(h_3\,A_3)}{\partial q_2} - \frac{\partial(h_2\,A_2)}{\partial q_3}\bigg)\ . \tag{2.89}$$

This is one component of the full result

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{h_1\,h_2\,h_2}\,\det\begin{pmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3\\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3}\\ h_1\,A_1 & h_2\,A_2 & h_3\,A_3\end{pmatrix}\ . \tag{2.90}$$

The Laplacian of a scalar function U is given by

$$\begin{aligned}\nabla^2 U &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}U\\ &= \frac{1}{h_1\,h_2\,h_3}\bigg\{\frac{\partial}{\partial q_1}\bigg(\frac{h_2\,h_3}{h_1}\,\frac{\partial U}{\partial q_1}\bigg) + \frac{\partial}{\partial q_2}\bigg(\frac{h_1\,h_3}{h_2}\,\frac{\partial U}{\partial q_2}\bigg) + \frac{\partial}{\partial q_3}\bigg(\frac{h_1\,h_2}{h_3}\,\frac{\partial U}{\partial q_3}\bigg)\bigg\}\ .\end{aligned} \tag{2.91}$$

2.7 Common curvilinear orthogonal systems

2.7.1 Rectangular coordinates

In *rectangular* coordinates (x, y, z), we have

$$h_x = h_y = h_z = 1 \ . \tag{2.92}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz \tag{2.93}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \ . \tag{2.94}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{x}}\, \frac{\partial U}{\partial x} + \hat{\boldsymbol{y}}\, \frac{\partial U}{\partial y} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \ . \tag{2.95}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \ . \tag{2.96}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} \ . \tag{2.97}$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.98}$$

2.7.2 Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z), we have

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\, \cos\phi + \hat{\boldsymbol{y}}\, \sin\phi \qquad \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\, \cos\phi - \hat{\boldsymbol{\phi}}\, \sin\phi \qquad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\, d\phi \tag{2.99}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\, \sin\phi + \hat{\boldsymbol{y}}\, \cos\phi \qquad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\, \sin\phi + \hat{\boldsymbol{\phi}}\, \cos\phi \qquad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}\, d\phi \ . \tag{2.100}$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \ . \tag{2.101}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \hat{\boldsymbol{\phi}}\, \rho\, d\phi + \hat{\boldsymbol{z}}\, dz \tag{2.102}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \ . \tag{2.103}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{\rho}}\, \frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\, \frac{\partial U}{\partial \phi} + \hat{\boldsymbol{z}}\, \frac{\partial U}{\partial z} \ . \tag{2.104}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{\rho}\, \frac{\partial(\rho\, A_\rho)}{\partial \rho} + \frac{1}{\rho}\, \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \ . \tag{2.105}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{1}{\rho}\, \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho}\, \frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\, \frac{\partial A_\rho}{\partial \phi} \right) \hat{\boldsymbol{z}} \ . \tag{2.106}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho}\, \frac{\partial}{\partial \rho} \left(\rho\, \frac{\partial U}{\partial \rho} \right) + \frac{1}{\rho^2}\, \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} \ . \tag{2.107}$$

2.7.3 Spherical coordinates

In *spherical* coordinates (r, θ, ϕ), we have

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \sin\theta \tag{2.108}$$

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}} \cos\theta \cos\phi + \hat{\boldsymbol{y}} \cos\theta \sin\phi - \hat{\boldsymbol{z}} \cos\theta \tag{2.109}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi \ , \tag{2.110}$$

for which

$$\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}} \quad , \quad \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \ . \tag{2.111}$$

The inverse is

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi \tag{2.112}$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi \tag{2.113}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta \ . \tag{2.114}$$

The differential relations are

$$d\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}\, d\theta + \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \tag{2.115}$$

$$d\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{r}}\, d\theta + \cos\theta\, \hat{\boldsymbol{\phi}}\, d\phi \tag{2.116}$$

$$d\hat{\boldsymbol{\phi}} = -\big(\sin\theta\, \hat{\boldsymbol{r}} + \cos\theta\, \hat{\boldsymbol{\theta}} \big)\, d\phi \tag{2.117}$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r\, \sin\theta \ . \tag{2.118}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{r}}\, dr + \hat{\boldsymbol{\theta}}\, r\, d\theta + \hat{\boldsymbol{\phi}}\, r \sin\theta\, d\phi \tag{2.119}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2 \ . \tag{2.120}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{r}}\,\frac{\partial U}{\partial\rho} + \frac{\hat{\boldsymbol{\theta}}}{r}\,\frac{\partial U}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\,\frac{\partial U}{\partial\phi} \ . \tag{2.121}$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\,\frac{\partial(r^2 A_r)}{r} + \frac{1}{r\sin\theta}\,\frac{\partial(\sin\theta\,A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial\phi} \ . \tag{2.122}$$

The curl is

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}&\left(\frac{\partial(\sin\theta\,A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \ . \end{aligned} \tag{2.123}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\!\theta}\,\frac{\partial^2 U}{\partial\phi^2} \ . \tag{2.124}$$

2.7.4 Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \tfrac{1}{2}m\left(\frac{d\boldsymbol{s}}{dt}\right)^2 &= \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) && \text{(3D Cartesian)} && (2.125)\\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2\big) && \text{(2D polar)} && (2.126)\\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\big) && \text{(3D cylindrical)} && (2.127)\\ &= \tfrac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\big) && \text{(3D polar)}\ . && (2.128) \end{aligned}$$

Chapter 3

One-Dimensional Conservative Systems

3.1 Description as a Dynamical System

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \ . \tag{3.1}$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x) \ , \tag{3.2}$$

is then conserved. This may be verified explicitly:

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt}\Big[\tfrac{1}{2}m\dot{x}^2 + U(x)\Big] \\ &= \Big[m\ddot{x} + U'(x)\Big]\,\dot{x} = 0 \ . \end{aligned} \tag{3.3}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\bigg(E - U(x)\bigg)} \ . \tag{3.4}$$

Note that the constant E is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \ , \tag{3.5}$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E, the motion of the system is bounded by the turning points, and confined to the region(s)

$U(x) \leq E$. We can integrate eqn. 3.4 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}} \int\limits_{x_0}^{x} \frac{dx'}{\sqrt{E - U(x')}} \ . \tag{3.6}$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0. Since

$$E = E_0 = \tfrac{1}{2} m v_0^2 + U(x_0) \ , \tag{3.7}$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m} \int\limits_{x_-(E)}^{x_+(E)} \frac{dx'}{\sqrt{E - U(x')}} \ . \tag{3.8}$$

3.1.1 Example : harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \frac{1}{2}kx^2$, hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2E - kx^2}} \ . \tag{3.9}$$

The turning points are $x \pm (E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}} \, \sin\theta \ . \tag{3.10}$$

We then find

$$dt = \sqrt{\frac{m}{k}} \, d\theta \ , \tag{3.11}$$

with solution

$$\theta(t) = \theta_0 + \omega t \ , \tag{3.12}$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the complete motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \, \sin(\omega t + \theta_0) \ . \tag{3.13}$$

Note the two constants of integration, E and θ_0.

3.2 One-Dimensional Mechanics as a Dynamical System

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\frac{dx}{dt} = v \tag{3.14}$$

$$\frac{dv}{dt} = \frac{1}{m}\,F(x)\ . \tag{3.15}$$

This may be written in matrix-vector form, as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}\,F(x) \end{pmatrix}\ . \tag{3.16}$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi})\ , \tag{3.17}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ is an N-dimensional vector in *phase space*. For the model of eqn. 3.16, we evidently have $N = 2$. The object $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field*. It is itself a vector, existing at every point in phase space, R^N. Each of the components of $\boldsymbol{V}(\boldsymbol{\varphi})$ is a function (in general) of *all* the components of $\boldsymbol{\varphi}$:

$$V_j = V_j(\varphi_1, \ldots, \varphi_N) \qquad (j = 1, \ldots, N)\ . \tag{3.18}$$

Solutions to the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ are called *integral curves*. Each such integral curve $\boldsymbol{\varphi}(t)$ is uniquely determined by N constants of integration, which may be taken to be the initial value $\boldsymbol{\varphi}(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$. The phase curves are thus curves of constant energy.

3.2.1 Sketching phase curves

To plot the phase curves,

(i) Sketch the potential $U(x)$.

(ii) Below this plot, sketch $v(x;E) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$.

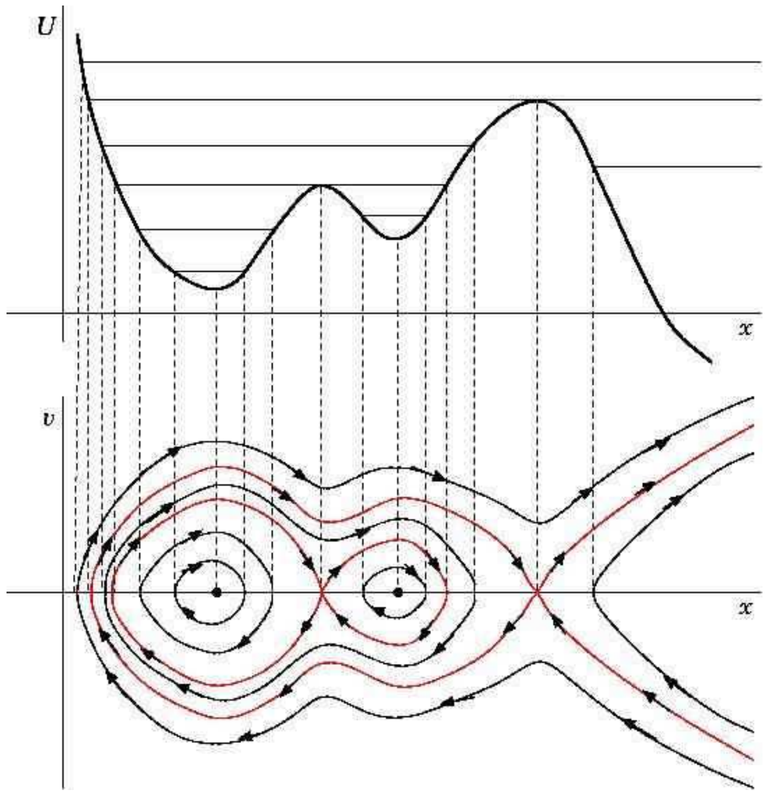


Figure 3.1: A potential $U(x)$ and the corresponding phase portraits. Separatrices are shown in red.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

 (a) For E slightly above E_{min}, the phase curves are ellipses.
 (b) For E slightly below E_{max}, the phase curves are (locally) hyperbolae.
 (c) For $E = E_{\text{max}}$ the phase curve is called a *separatrix*.

(iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint\limits_E v\,dx = \sqrt{\tfrac{8}{m}} \int\limits_{x_-(E)}^{x_+(E)} dx'\,\sqrt{E - U(x')}\ . \tag{3.19}$$

Thus, the period is proportional to the rate of change of $A(E)$ with E:

$$T = m\,\frac{\partial \mathcal{A}}{\partial E}\ . \tag{3.20}$$

3.3 Fixed Points and their Vicinity

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*)\,(x - x^*) + \tfrac{1}{2}U''(x^*)\,(x - x^*)^2 + \tfrac{1}{6}U'''(x^*)\,(x - x^*)^3 + \ldots \; . \tag{3.21}$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \tfrac{1}{2}k(\delta x)^2 \;, \tag{3.22}$$

where $U_0 = U(x^*)$ and $k = U''(x^*) > 0$. The solutions to the motion in this potential are:

$$U''(x^*) > 0 \;:\; \delta x(t) = \delta x_0 \,\cos(\omega t) + \frac{\delta v_0}{\omega}\,\sin(\omega t) \tag{3.23}$$

$$U''(x^*) < 0 \;:\; \delta x(t) = \delta x_0 \,\cosh(\gamma t) + \frac{\delta v_0}{\gamma}\,\sinh(\gamma t) \;, \tag{3.24}$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \tfrac{1}{2}m\,(\delta v_0)^2 + \tfrac{1}{2}k\,(\delta x_0)^2 \;. \tag{3.25}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm\gamma\,\delta x_0$. Let's take $\delta v_0 = -\gamma\,\delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0 \,\exp(-\gamma t) \;. \tag{3.26}$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \ln\left(\frac{\delta x_0}{\delta x}\right) \;. \tag{3.27}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, *the period of motion along a separatrix is infinite*.

3.3.1 Linearized dynamics in the vicinity of a fixed point

Linearizing in the vicinity of such a fixed point, we write $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{m}\,U''(x^*) & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} + \ldots \,, \tag{3.28}$$

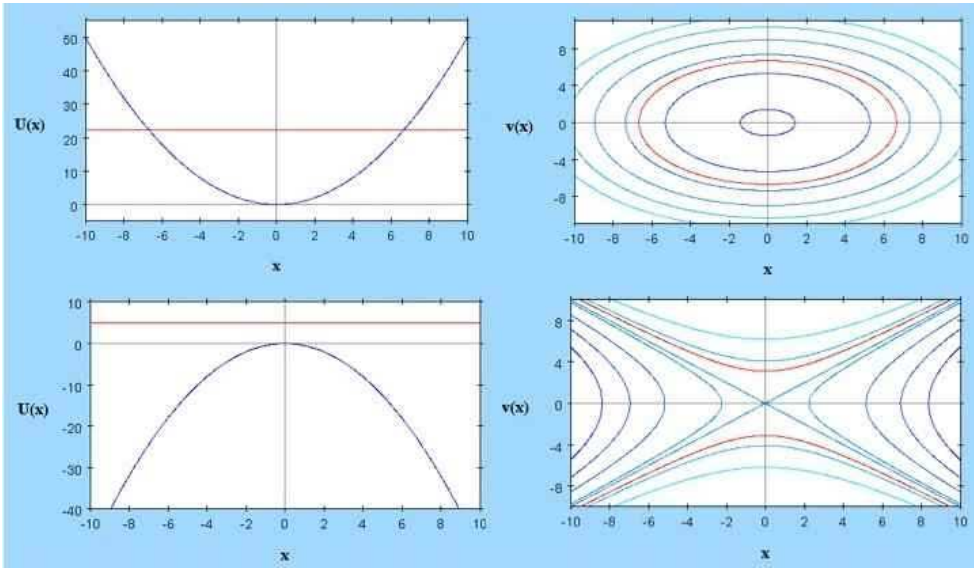


Figure 3.2: Phase curves in the vicinity of centers and saddles.

This is a *linear* equation, which we can solve completely.

Consider the general linear equation $\dot{\boldsymbol{\varphi}} = A\,\boldsymbol{\varphi}$, where A is a fixed real matrix. Now whenever we have a problem involving matrices, we should start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$A\,\boldsymbol{\psi}_\alpha = \lambda_\alpha\,\boldsymbol{\psi}_\alpha\ . \tag{3.29}$$

Here $\boldsymbol{\psi}_\alpha$ is the α^{th} *right eigenvector*[1] of A. The eigenvalues are roots of the characteristic equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - A)$. Let's expand $\boldsymbol{\varphi}(t)$ in terms of the right eigenvectors of A:

$$\boldsymbol{\varphi}(t) = \sum_\alpha C_\alpha(t)\,\boldsymbol{\psi}_\alpha\ . \tag{3.30}$$

Assuming, for the purposes of this discussion, that A is nondegenerate, and its eigenvectors span R^N, the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha\,C_\alpha\ , \tag{3.31}$$

with solutions

$$C_\alpha(t) = C_\alpha(0)\,\exp(\lambda_\alpha t)\ . \tag{3.32}$$

If $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $\mathsf{Im}\,(\lambda_\alpha)$.

[1]If A is symmetric, the right and left eigenvectors are the same. If A is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \ , \tag{3.33}$$

where $T = \mathrm{Tr}(A)$ and $D = \det(A)$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4D} \ . \tag{3.34}$$

We'll study the general case in Physics 110B. For now, we focus on our conservative mechanical system of eqn. 3.28. The trace and determinant of the above matrix are $T = 0$ and $D = \frac{1}{m} U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in Fig. 3.1.

3.4 Examples of Conservative One-Dimensional Systems

3.4.1 Harmonic oscillator

Recall again the harmonic oscillator, discussed in lecture 3. The potential energy is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m \, \frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \ , \tag{3.35}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 \, x \end{pmatrix} \ , \tag{3.36}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$x(t) = x_0 \, \cos(\omega t) + \frac{v_0}{\omega} \, \sin(\omega t) \tag{3.37}$$

$$v(t) = v_0 \, \cos(\omega t) - \omega \, x_0 \, \sin(\omega t) \ . \tag{3.38}$$

The phase curves are ellipses:

$$\omega_0 \, x^2(t) + \omega_0^{-1} \, v^2(t) = C \ , \tag{3.39}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in Fig. 3.3. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \ . \tag{3.40}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\mathrm{max}} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\mathrm{max}} = \sqrt{\frac{2E}{m}} \ . \tag{3.41}$$

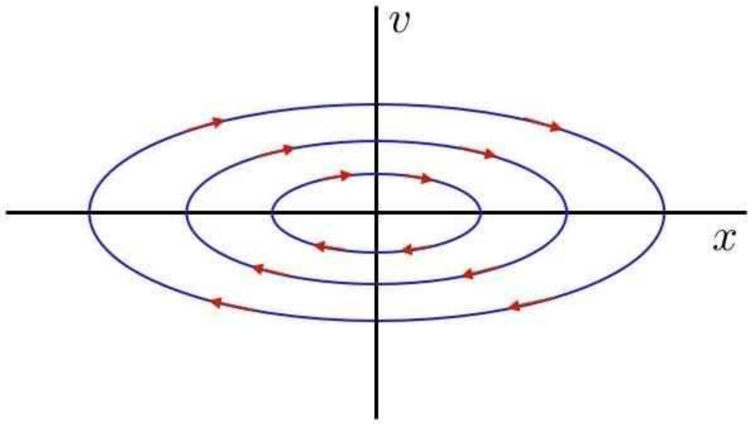


Figure 3.3: Phase curves for the harmonic oscillator.

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi \, x_{\text{max}} \, v_{\text{max}} = \frac{2\pi E}{\sqrt{mk}} \ . \tag{3.42}$$

The period of motion is therefore

$$T(E) = m \, \frac{\partial \mathcal{A}}{\partial E} = 2\pi \sqrt{\frac{m}{k}} \ , \tag{3.43}$$

which is independent of E.

3.4.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell \cos\theta$, hence

$$m\ell^2 \, \ddot{\theta} = -\frac{dU}{d\theta} = -mg\ell \sin\theta \ . \tag{3.44}$$

This is equivalent to

$$\frac{d}{dt} \begin{pmatrix} \theta \\ \omega \end{pmatrix} = \begin{pmatrix} \omega \\ -\omega_0^2 \, \sin\theta \end{pmatrix} \ , \tag{3.45}$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \tfrac{1}{2} \, m\ell^2 \, \dot{\theta}^2 + U(\theta) \ . \tag{3.46}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot{\theta}^2 - 2\omega_0^2 \, \cos\theta = -2\omega_0^2 \, \cos\theta_0 \ . \tag{3.47}$$

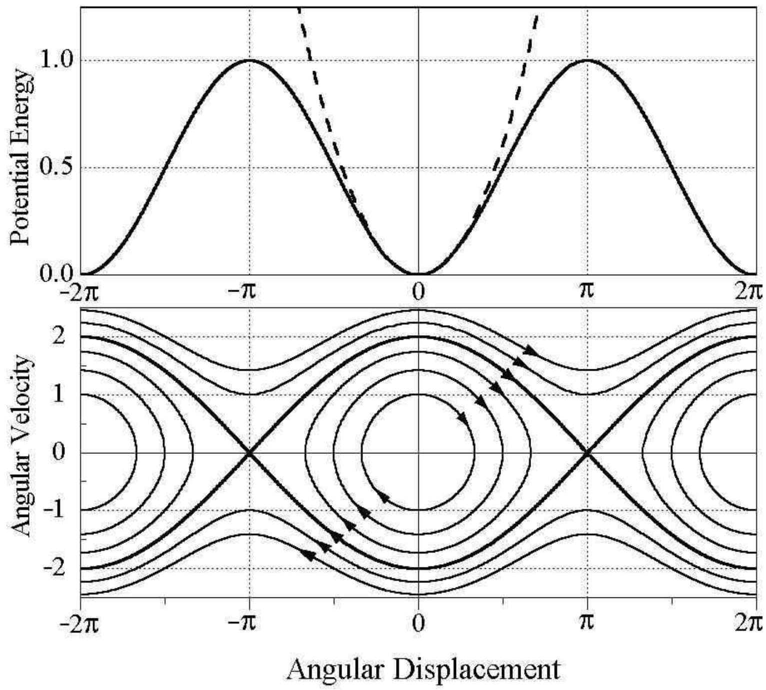


Figure 3.4: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

The period for motion of amplitude θ_0 is then

$$T(\theta_0) = \frac{\sqrt{8}}{\omega_0} \int\limits_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0} \, \mathrm{K}\big(\sin^2 \tfrac{1}{2}\theta_0\big) \ , \tag{3.48}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind. Expanding $K(z)$, we have

$$T(\theta_0) = \frac{2\pi}{\omega_0} \bigg\{ 1 + \tfrac{1}{4} \sin^2\left(\tfrac{1}{2}\theta_0\right) + \tfrac{9}{64} \sin^4\left(\tfrac{1}{2}\theta_0\right) + \ldots \bigg\} \ . \tag{3.49}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\,\theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in Fig. 3.4. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

3.4.3 Other potentials

Using the phase plotter application written by Ben Schmidel, available on the Physics 110A course web page, it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \ , \tag{3.50}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses $m_{1,2}$, $\mu = m_1 m_2/(m_1 + m_2)$ is the 'reduced mass', and $k = G m_1 m_2$, where G is the Cavendish constant. We can then write

$$U_{\text{eff}}(r) = U_0 \bigg\{ -\frac{1}{x} + \frac{1}{2x^2} \bigg\} \ , \tag{3.51}$$

where $r_0 = \ell^2/\mu k$ has the dimensions of length, and $x \equiv r/r_0$, and where $U_0 = k/r_0 = \mu k^2/\ell^2$. Thus, if distances are measured in units of r_0 and the potential in units of U_0, the potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \, \text{sech}^2(x/a) \ , \tag{3.52}$$

which, in dimensionless form, is $\mathcal{U}(x) = -\text{sech}^2(x)$, after measuring distances in units of a and potential in units of U_0.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} . \tag{3.53}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.

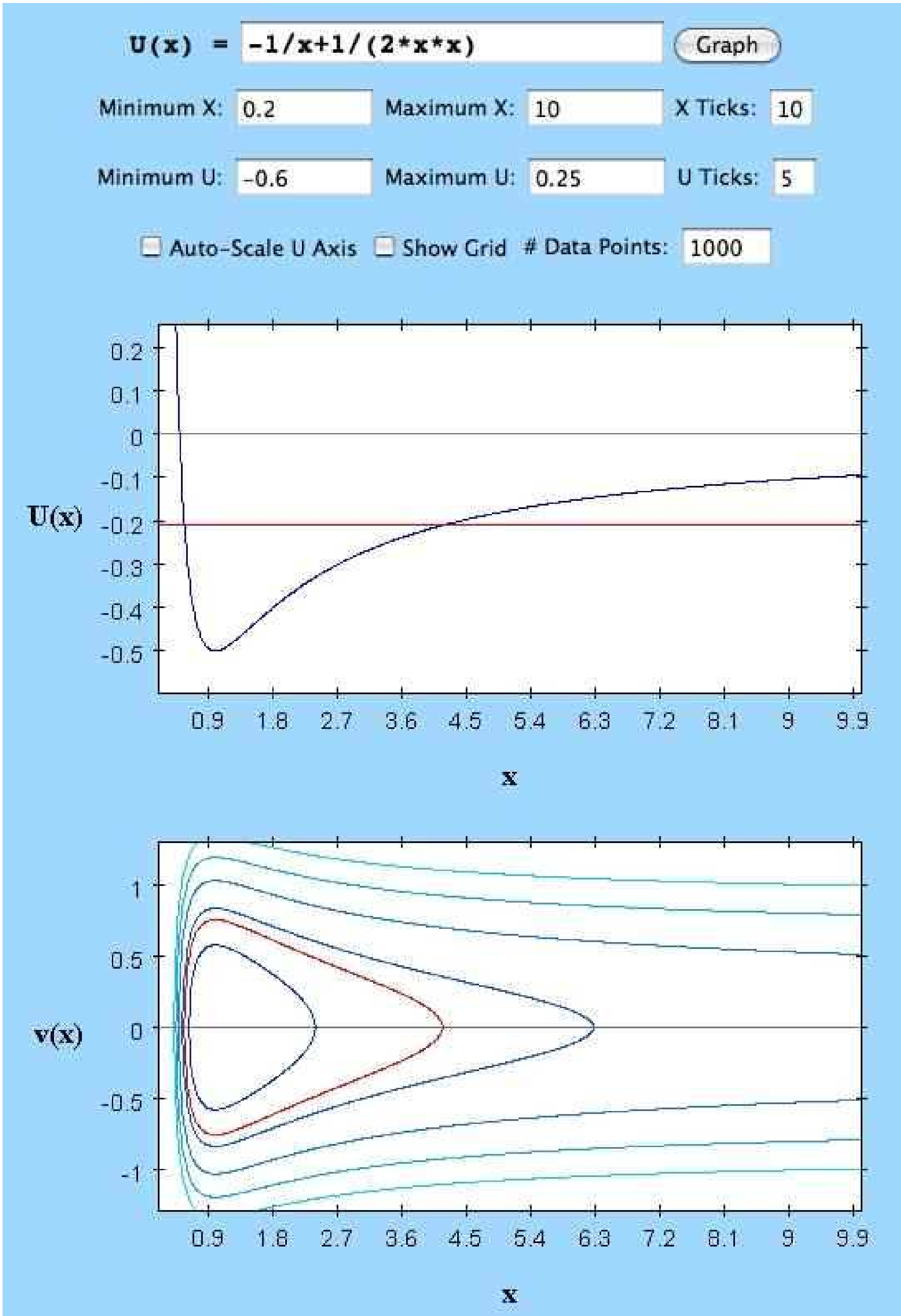


Figure 3.5: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

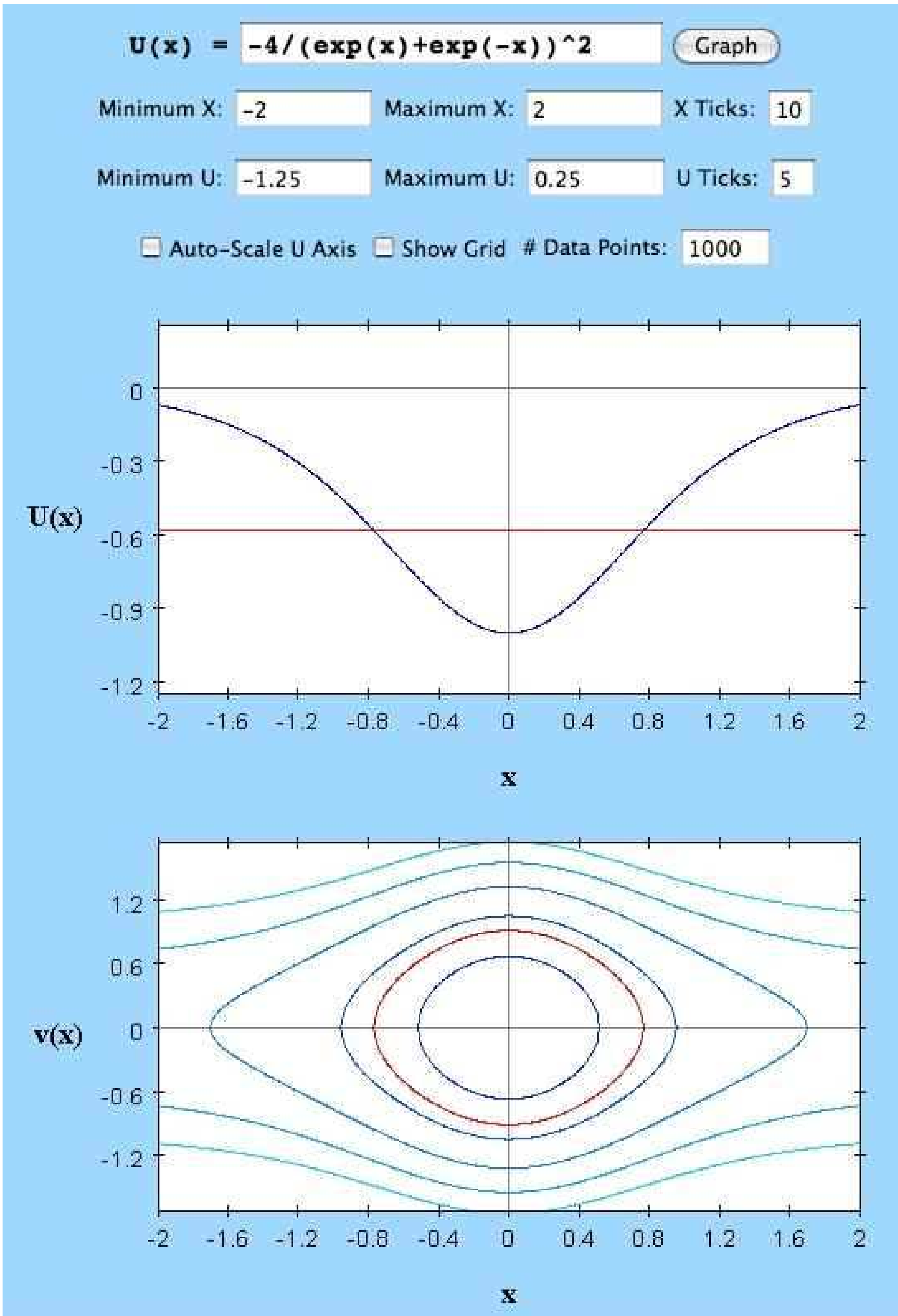


Figure 3.6: Phase curves for the potential $U(x) = -\text{sech}^2(x)$.

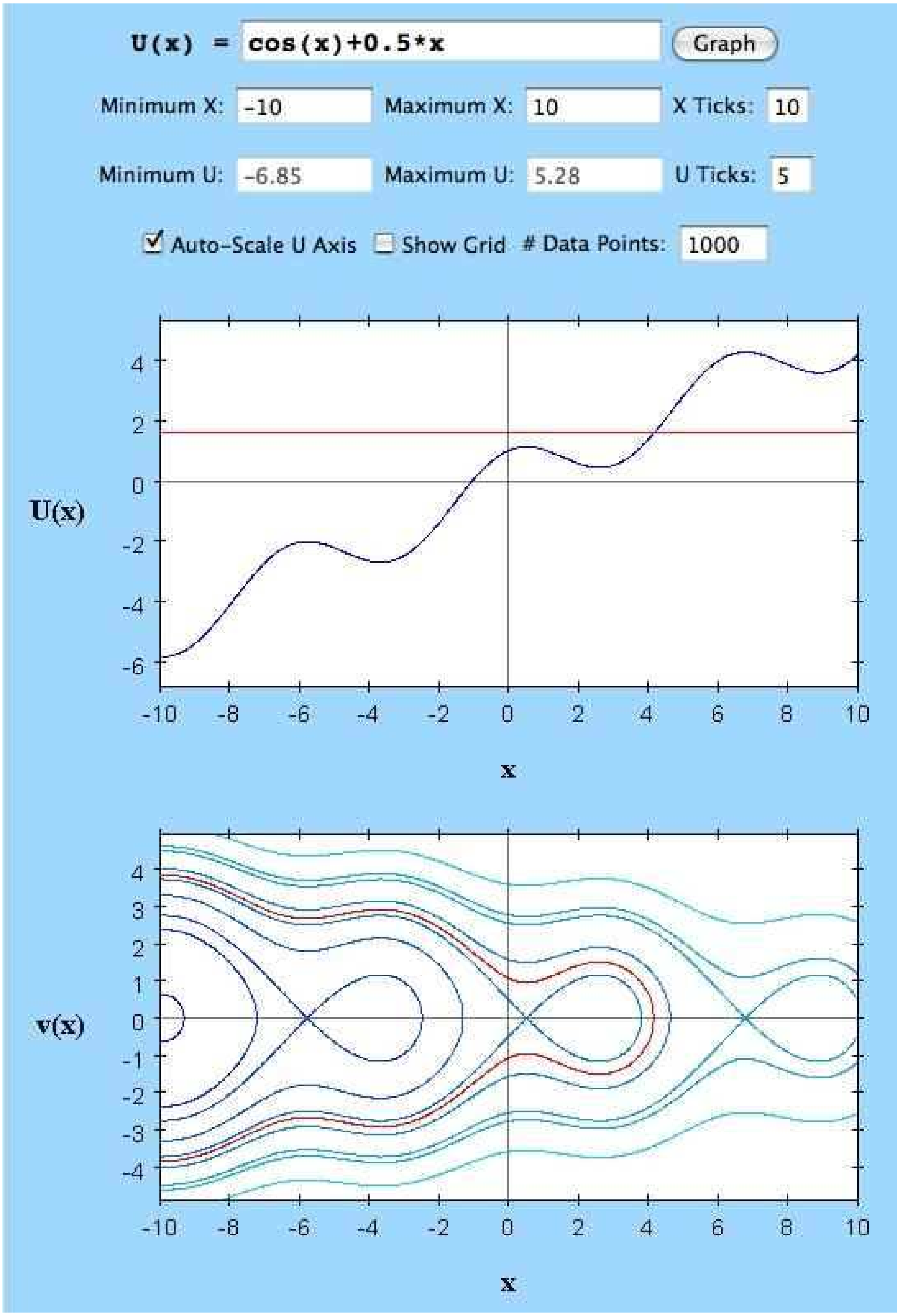


Figure 3.7: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.

Chapter 4

Linear Oscillations

Harmonic motion is ubiquitous in Physics. The reason is that any potential energy function, when expanded in a Taylor series in the vicinity of a local minimum, is a harmonic function:

$$U(\vec{q}\,) = U(\vec{q}^{\,*}) + \sum_{j=1}^{N} \overbrace{\frac{\partial U}{\partial q_j}\bigg|_{\vec{q}=\vec{q}^{\,*}}}^{\boldsymbol{\nabla} U(\vec{q}^{\,*})=0} (q_j - q_j^*) + \tfrac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 U}{\partial q_j\,\partial q_k}\bigg|_{\vec{q}=\vec{q}^{\,*}} (q_j - q_j^*)\,(q_k - q_k^*) + \ldots\ , \tag{4.1}$$

where the $\{q_j\}$ are *generalized coordinates* – more on this when we discuss Lagrangians. In one dimension, we have simply

$$U(x) = U(x^*) + \tfrac{1}{2}\,U''(x^*)\,(x - x^*)^2 + \ldots\ . \tag{4.2}$$

Provided the deviation $\eta = x - x^*$ is small enough in magnitude, the remaining terms in the Taylor expansion may be ignored. Newton's Second Law then gives

$$m\,\ddot{\eta} = -U''(x^*)\,\eta + \mathcal{O}(\eta^2)\ . \tag{4.3}$$

This, to lowest order, is the equation of motion for a harmonic oscillator. If $U''(x^*) > 0$, the equilibrium point $x = x^*$ is *stable*, since for small deviations from equilibrium the restoring force pushes the system back toward the equilibrium point. When $U''(x^*) < 0$, the equilibrium is *unstable*, and the forces push one further away from equilibrium.

4.1 Damped Harmonic Oscillator

In the real world, there are frictional forces, which we here will approximate by $F = -\gamma v$. We begin with the homogeneous equation for a damped harmonic oscillator,

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = 0\ , \tag{4.4}$$

where $\gamma = 2\beta m$. To solve, write $x(t) = \sum_i C_i \, e^{-i\omega_i t}$. This renders the differential equation 4.4 an *algebraic* equation for the two eigenfrequencies ω_i, each of which must satisfy

$$\omega^2 + 2i\beta\omega - \omega_0^2 = 0 \; , \tag{4.5}$$

hence

$$\omega_\pm = -i\beta \pm (\omega_0^2 - \beta^2)^{1/2} \; . \tag{4.6}$$

The most general solution to eqn. 4.4 is then

$$x(t) = C_+ \, e^{-i\omega_+ t} + C_- \, e^{-i\omega_- t} \tag{4.7}$$

where $C_\pm$ are arbitrary constants. Notice that the eigenfrequencies are in general complex, with a negative imaginary part (so long as the damping coefficient β is positive). Thus $e^{-i\omega_\pm t}$ decays to zero as $t \to \infty$.

4.1.1 Classes of damped harmonic motion

We identify three classes of motion:

$$\begin{aligned} &\text{(i) Underdamped } (\omega_0^2 > \beta^2) \\ &\text{(ii) Overdamped } (\omega_0^2 < \beta^2) \\ &\text{(iii) Critically Damped } (\omega_0^2 = \beta^2) \; . \end{aligned}$$

Underdamped motion

The solution for underdamped motion is

$$\begin{aligned} x(t) &= A\cos(\nu t + \phi)\, e^{-\beta t} \\ \dot x(t) &= -\omega_0 A \cos\big(\nu t + \phi + \sin^{-1}(\beta/\omega_0)\big)\, e^{-\beta t} \; , \end{aligned} \tag{4.8}$$

where $\nu = \sqrt{\omega_0^2 - \beta^2}$, and where A and ϕ are constants determined by initial conditions. From $x_0 = A\cos\phi$ and $\dot x_0 = -\beta A\cos\phi - \nu A \sin\phi$, we have $\dot x_0 + \beta x_0 = -\nu A \sin\phi$, and

$$A = \sqrt{x_0^2 + \left(\frac{\dot x_0 + \beta\, x_0}{\nu}\right)^2} \qquad , \qquad \phi = -\tan^{-1}\left(\frac{\dot x_0 + \beta\, x_0}{\nu\, x_0}\right) \; . \tag{4.9}$$

Overdamped motion

The solution in the case of overdamped motion is

$$\begin{aligned} x(t) &= C\, e^{-(\beta-\lambda)t} + D\, e^{-(\beta+\lambda)t} \\ \dot x(t) &= -(\beta-\lambda)\, C\, e^{-(\beta-\lambda)t} - (\beta+\lambda)\, D\, e^{-(\beta+\lambda)t} \; , \end{aligned} \tag{4.10}$$

where $\lambda = \sqrt{\beta^2 - \omega_0^2}$ and where C and D are constants determined by the initial conditions:

$$\begin{pmatrix} 1 & 1 \\ -(\beta - \lambda) & -(\beta + \lambda) \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} x_0 \\ \dot{x}_0 \end{pmatrix} . \tag{4.11}$$

Inverting the above matrix, we have the solution

$$C = \frac{(\beta + \lambda)\, x_0}{2\lambda} + \frac{\dot{x}_0}{2\lambda} \qquad , \qquad D = -\frac{(\beta - \lambda)\, x_0}{2\lambda} - \frac{\dot{x}_0}{2\lambda} . \tag{4.12}$$

Critically damped motion

The solution in the case of critically damped motion is

$$\begin{aligned} x(t) &= E\, e^{-\beta t} + F t\, e^{-\beta t} \\ \dot{x}(t) &= -\Big(\beta E + (\beta t - 1) F\Big) e^{-\beta t} . \end{aligned} \tag{4.13}$$

Thus, $x_0 = E$ and $\dot{x}_0 = F - \beta E$, so

$$E = x_0 \qquad , \qquad F = \dot{x}_0 + \beta x_0 . \tag{4.14}$$

The screen door analogy

The three types of behavior are depicted in fig. 4.1. To concretize these cases in one's mind, it is helpful to think of the case of a screen door or a shock absorber. If the hinges on the door are underdamped, the door will swing back and forth (assuming it doesn't have a rim which smacks into the door frame) several times before coming to a stop. If the hinges are overdamped, the door may take a very long time to close. To see this, note that for $\beta \gg \omega_0$ we have

$$\begin{aligned} \sqrt{\beta^2 - \omega_0^2} &= \beta \left(1 - \frac{\omega_0^2}{\beta^2}\right)^{-1/2} \\ &= \beta \left(1 - \frac{\omega_0^2}{2\beta^2} - \frac{\omega_0^4}{8\beta^4} + \ldots \right) , \end{aligned} \tag{4.15}$$

which leads to

$$\begin{aligned} \beta - \sqrt{\beta^2 - \omega_0^2} &= \frac{\omega_0^2}{2\beta} + \frac{\omega_0^4}{8\beta^3} + \ldots \\ \beta + \sqrt{\beta^2 - \omega_0^2} &= 2\beta - \frac{\omega_0^2}{2\beta} - + \ldots . \end{aligned} \tag{4.16}$$

Thus, we can write

$$x(t) = C\, e^{-t/\tau_1} + D\, e^{-t/\tau_2} , \tag{4.17}$$

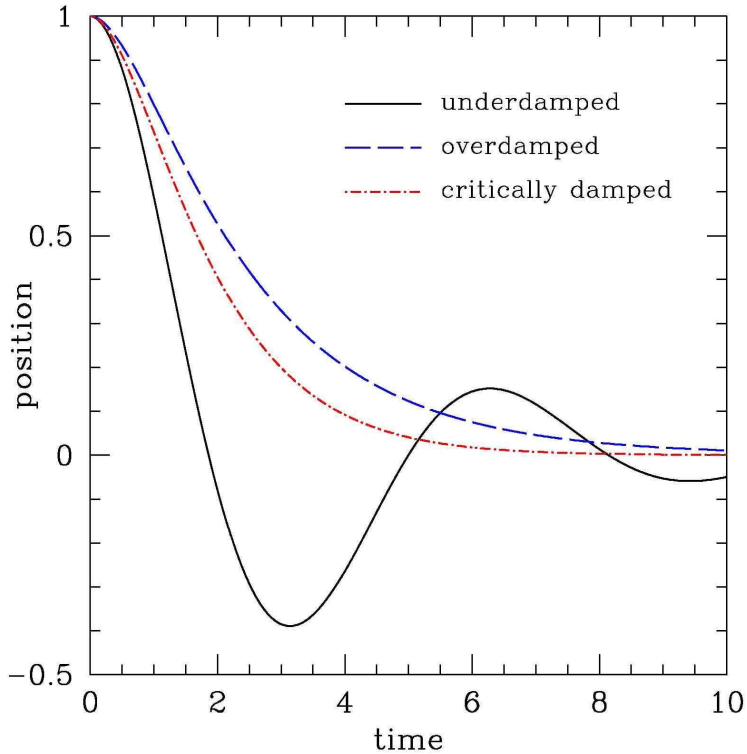


Figure 4.1: Three classifications of damped harmonic motion. The initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$.

with

$$\tau_1 = \frac{1}{\beta - \sqrt{\beta^2 - \omega_0^2}} \approx \frac{2\beta}{\omega_0^2} \tag{4.18}$$

$$\tau_2 = \frac{1}{\beta + \sqrt{\beta^2 - \omega_0^2}} \approx \frac{1}{2\beta} \ . \tag{4.19}$$

Thus $x(t)$ is a sum of exponentials, with decay times $\tau_{1,2}$. For $\beta \gg \omega_0$, we have that τ_1 is much larger than τ_2 – the ratio is $\tau_1/\tau_2 \approx 4\beta^2/\omega_0^2 \gg 1$. Thus, on time scales on the order of τ_1, the second term has completely damped away. The decay time τ_1, though, is very long, since β is so large. So a highly overdamped oscillator will take a very long time to come to equilbrium.

4.1.2 Remarks on the case of critical damping

Define the first order differential operator

$$\mathcal{D}_t = \frac{d}{dt} + \beta \ . \tag{4.20}$$

The solution to $\mathcal{D}_t\, x(t) = 0$ is $\tilde{x}(t) = A\, e^{-\beta t}$, where A is a constant. Note that the *commutator* of $\mathcal{D}_t$ and t is unity:

$$\big[\mathcal{D}_t\,, t\big] = 1 \ , \tag{4.21}$$

where $[A, B] \equiv AB - BA$. The simplest way to verify eqn. 4.21 is to compute its action upon an arbitrary function $f(t)$:

$$\begin{aligned}\big[\mathcal{D}_t\,,\, t\big]\, f(t) &= \left(\frac{d}{dt} + \beta\right) t\, f(t) - t \left(\frac{d}{dt} + \beta\right) f(t) \\ &= \frac{d}{dt}\big(t\, f(t)\big) - t\, \frac{d}{dt}\, f(t) = f(t) \ .\end{aligned} \tag{4.22}$$

We know that $x(t) = \tilde{x}(t) = A\, e^{-\beta t}$ satisfies $\mathcal{D}_t\, x(t) = 0$. Therefore

$$\begin{aligned}0 &= \mathcal{D}_t\, \big[\mathcal{D}_t\,, t\big]\, \tilde{x}(t) \\ &= \mathcal{D}_t^2 \Big(t\, \tilde{x}(t)\Big) - \mathcal{D}_t\, t\, \overbrace{\mathcal{D}_t\, \tilde{x}(t)}^{0} \\ &= \mathcal{D}_t^2 \Big(t\, \tilde{x}(t)\Big) \ .\end{aligned} \tag{4.23}$$

We already know that $\mathcal{D}_t^2\, \tilde{x}(t) = \mathcal{D}_t\, \mathcal{D}_t\, \tilde{x}(t) = 0$. The above equation establishes that the second independent solution to the second order ODE $\mathcal{D}_t^2\, x(t) = 0$ is $x(t) = t\, \tilde{x}(t)$. Indeed, we can keep going, and show that

$$\mathcal{D}_t^n \left(t^{n-1}\, \tilde{x}(t)\right) = 0 \ . \tag{4.24}$$

Thus, the n independent solutions to the n^{th} order ODE

$$\left(\frac{d}{dt} + \beta\right)^n x(t) = 0 \tag{4.25}$$

are

$$x_k(t) = A\, t^k\, e^{-\beta t} \quad , \quad k = 0, 1, \ldots, n-1 \ . \tag{4.26}$$

4.1.3 Phase portraits for the damped harmonic oscillator

Expressed as a dynamical system, the equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ is written as two coupled first order ODEs, *viz.*

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\omega_0^2\, x - 2\beta v \ .\end{aligned} \tag{4.27}$$

In the theory of dynamical systems, a *nullcline* is a curve along which one component of the phase space velocity $\dot{\boldsymbol{\varphi}}$ vanishes. In our case, there are two nullclines: $\dot{x} = 0$ and $\dot{v} = 0$. The equation of the first nullcline, $\dot{x} = 0$, is simply $v = 0$, *i.e.* the first nullcline is the x-axis. The equation of the second nullcline, $\dot{v} = 0$, is $v = -(\omega_0^2/2\beta)\, x$. This is a line which runs through the origin and has negative slope. Everywhere along the first nullcline $\dot{x} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the v-axis. Similarly, everywhere along the second nullcline $\dot{v} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the x-axis. The situation is depicted in fig. 4.2.

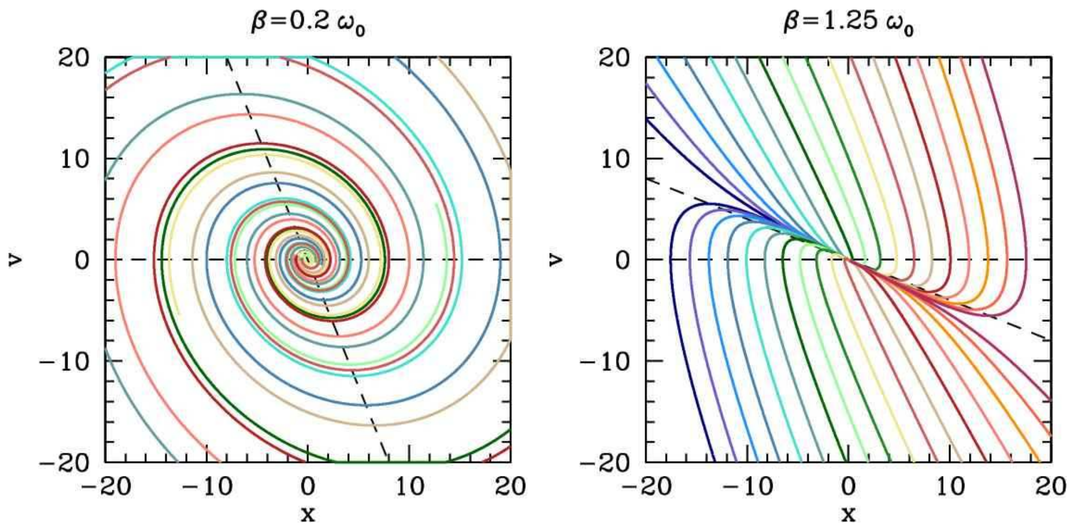


Figure 4.2: Phase curves for the damped harmonic oscillator. Left panel: underdamped motion. Right panel: overdamped motion. Note the *nullclines* along $x = 0$ and $v = -(\omega_0^2/2\beta)x$, which are shown as dashed lines.

4.2 Damped Harmonic Oscillator with Forcing

When forced, the equation for the damped oscillator becomes

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = f(t)\ , \tag{4.28}$$

where $f(t) = F(t)/m$. Since this equation is linear in $x(t)$, we can, without loss of generality, restrict out attention to harmonic forcing terms of the form

$$f(t) = f_0\cos(\Omega t + \varphi_0) = \mathrm{Re}\Big[f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\Big] \tag{4.29}$$

where Re stands for "real part". Here, Ω is the forcing frequency.

Consider first the complex equation

$$\frac{d^2z}{dt^2} + 2\beta\,\frac{dz}{dt} + \omega_0^2\,z = f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\ . \tag{4.30}$$

We try a solution $z(t) = z_0\,e^{-i\Omega t}$. Plugging in, we obtain the algebraic equation

$$z_0 = \frac{f_0\,e^{-i\varphi_0}}{\omega_0^2 - 2i\beta\Omega - \Omega^2} \equiv A(\Omega)\,e^{i\delta(\Omega)}\,f_0 e^{-i\varphi_0}\ . \tag{4.31}$$

The amplitude $A(\Omega)$ and phase shift $\delta(\Omega)$ are given by the equation

$$A(\Omega)\,e^{i\delta(\Omega)} = \frac{1}{\omega_0^2 - 2i\beta\Omega - \Omega^2}\ . \tag{4.32}$$

A basic fact of complex numbers:

$$\frac{1}{a-ib} = \frac{a+ib}{a^2+b^2} = \frac{e^{i\tan^{-1}(b/a)}}{\sqrt{a^2+b^2}} \ . \tag{4.33}$$

Thus,

$$A(\Omega) = \Big((\omega_0^2-\Omega^2)^2 + 4\beta^2\Omega^2\Big)^{-1/2} \tag{4.34}$$

$$\delta(\Omega) = \tan^{-1}\!\left(\frac{2\beta\Omega}{\omega_0^2-\Omega^2}\right) \ . \tag{4.35}$$

Now since the coefficients β and ω_0^2 are real, we can take the complex conjugate of eqn. 4.30, and write

$$\ddot{z} + 2\beta\,\dot{z} + \omega_0^2\,z = f_0\,e^{-i\varphi_0}\,e^{-i\Omega t} \tag{4.36}$$

$$\ddot{\bar{z}} + 2\beta\,\dot{\bar{z}} + \omega_0^2\,\bar{z} = f_0\,e^{+i\varphi_0}\,e^{+i\Omega t} \ , \tag{4.37}$$

where $\bar{z}$ is the complex conjugate of z. We now add these two equations and divide by two to arrive at

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f_0\,\cos(\Omega t + \varphi_0) \ . \tag{4.38}$$

Therefore, the real, physical solution we seek is

$$\begin{aligned} x_{\rm inh}(t) &= \mathrm{Re}\Big[A(\Omega)\,e^{i\delta(\Omega)}\cdot f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\Big] \\ &= A(\Omega)\,f_0\,\cos\big(\Omega t + \varphi_0 - \delta(\Omega)\big) \ . \end{aligned} \tag{4.39}$$

The quantity $A(\Omega)$ is the *amplitude* of the response (in units of f_0), while $\delta(\Omega)$ is the (dimensionless) *phase lag* (typically expressed in radians).

The maximum of the amplitude $A(\Omega)$ occurs when $A'(\Omega) = 0$. From

$$\frac{dA}{d\Omega} = -\frac{2\Omega}{\big[A(\Omega)\big]^3}\,\big(\Omega^2 - \omega_0^2 + 2\beta^2\big) \ , \tag{4.40}$$

we conclude that $A'(\Omega) = 0$ for $\Omega = 0$ and for $\Omega = \Omega_{\rm R}$, where

$$\Omega_{\rm R} = \sqrt{\omega_0^2 - 2\beta^2} \ . \tag{4.41}$$

The solution at $\Omega = \Omega_{\rm R}$ pertains only if $\omega_0^2 > 2\beta^2$, of course, in which case $\Omega = 0$ is a local minimum and $\Omega = \Omega_{\rm R}$ a local maximum. If $\omega_0^2 < 2\beta^2$ there is only a local maximum, at $\Omega = 0$. See Fig. 4.3.

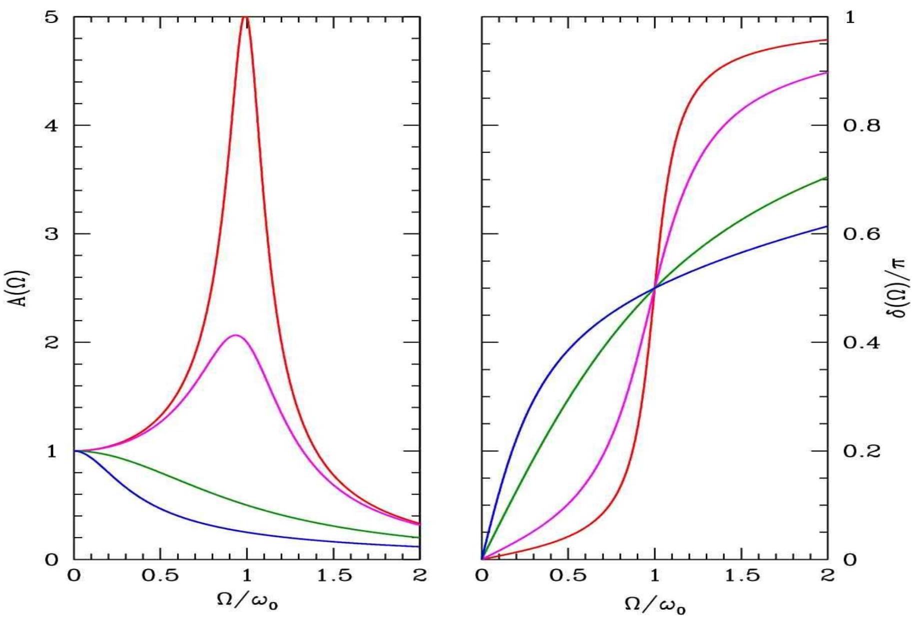


Figure 4.3: Amplitude and phase shift *versus* oscillator frequency (units of ω_0) for β/ω_0 values of 0.1 (red), 0.25 (magenta), 1.0 (green), and 2.0 (blue).

Since equation 4.28 is linear, we can add a solution to the homogeneous equation to $x_{\rm inh}(t)$ and we will still have a solution. Thus, the most general solution to eqn. 4.28 is

$$\begin{aligned}
x(t) &= x_{\rm inh}(t) + x_{\rm hom}(t) \\
&= {\rm Re}\Big[A(\Omega)\, e^{i\delta(\Omega)} \cdot f_0\, e^{-i\varphi_0}\, e^{-i\Omega t}\Big] + C_+\, e^{-i\omega_+ t} + C_-\, e^{-i\omega_- t} \\
&= \overbrace{A(\Omega)\, f_0 \cos\big(\Omega t + \varphi_0 - \delta(\Omega)\big)}^{x_{\rm inh}(t)} + \overbrace{C\, e^{-\beta t} \cos(\nu t) + D\, e^{-\beta t} \sin(\nu t)}^{x_{\rm hom}(t)} \ ,
\end{aligned} \tag{4.42}$$

where $\nu = \sqrt{\omega_0^2 - \beta^2}$ as before.

The last two terms in eqn. 4.42 are the solution to the homogeneous equation, *i.e.* with $f(t) = 0$. They are necessary to include because they carry with them the two constants of integration which always arise in the solution of a second order ODE. That is, C and D are adjusted so as to satisfy $x(0) = x_0$ and $\dot{x}_0 = v_0$. However, due to their $e^{-\beta t}$ prefactor, these terms decay to zero once t reaches a relatively low multiple of β^{-1}. They are called *transients*, and may be set to zero if we are only interested in the long time behavior of the system. This means, incidentally, that the initial conditions are effectively forgotten over a time scale on the order of β^{-1}.

For $\Omega_{\rm R} > 0$, one defines the *quality factor*, Q, of the oscillator by $Q = \Omega_{\rm R}/2\beta$. Q is a rough measure of how many periods the unforced oscillator executes before its initial amplitude is damped down to a small value. For a forced oscillator driven near resonance, and for weak damping, Q is also related to the ratio of average energy in the oscillator to the energy lost per cycle by the external source. To see this, let us compute the energy lost per cycle,

$$\begin{aligned}\Delta E &= m \int_0^{2\pi/\Omega} dt\, \dot{x}\, f(t) \\ &= -m \int_0^{2\pi/\Omega} dt\, \Omega\, A\, f_0^2\, \sin(\Omega t + \varphi_0 - \delta)\, \cos(\Omega t + \varphi_0) \\ &= \pi A\, f_0^2\, m\, \sin\delta \\ &= 2\pi\beta\, m\, \Omega\, A^2(\Omega)\, f_0^2 \ , \end{aligned} \tag{4.43}$$

since $\sin\delta(\Omega) = 2\beta\Omega\, A(\Omega)$. The oscillator energy, averaged over the cycle, is

$$\begin{aligned}\langle E\rangle &= \frac{\Omega}{2\pi} \int_0^{2\pi/\Omega} dt\, \tfrac{1}{2} m \big(\dot{x}^2 + \omega_0^2\, x^2\big) \\ &= \tfrac{1}{4} m\, (\Omega^2 + \omega_0^2)\, A^2(\Omega)\, f_0^2 \ . \end{aligned} \tag{4.44}$$

Thus, we have

$$\frac{2\pi\langle E\rangle}{\Delta E} = \frac{\Omega^2 + \omega_0^2}{4\beta\Omega} \ . \tag{4.45}$$

Thus, for $\Omega \approx \Omega_{\mathrm{R}}$ and $\beta^2 \ll \omega_0^2$, we have

$$Q \approx \frac{2\pi\langle E\rangle}{\Delta E} \approx \frac{\omega_0}{2\beta} \ . \tag{4.46}$$

4.2.1 Resonant forcing

When the damping β vanishes, the response diverges at resonance. The solution to the resonantly forced oscillator

$$\ddot{x} + \omega_0^2\, x = f_0 \cos(\omega_0\, t + \varphi_0) \tag{4.47}$$

is given by

$$x(t) = \frac{f_0}{2\omega_0}\, t\, \sin(\omega_0\, t + \varphi_0) + \overbrace{A\, \cos(\omega_0\, t) + B\, \sin(\omega_0\, t)}^{x_{\mathrm{hom}}(t)} \ . \tag{4.48}$$

The amplitude of this solution grows linearly due to the energy pumped into the oscillator by the resonant external forcing. In the real world, nonlinearities can mitigate this unphysical, unbounded response.

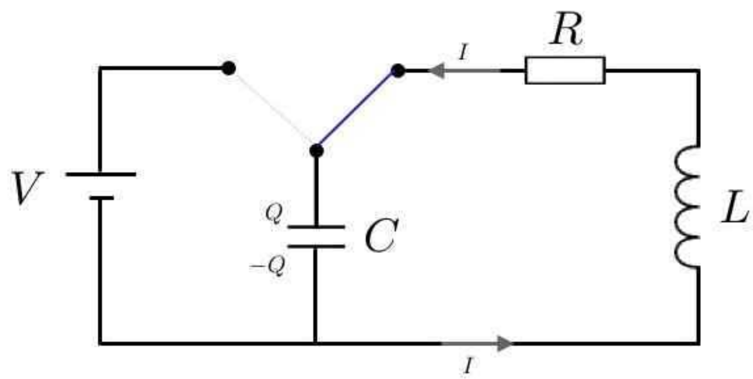


Figure 4.4: An *R*-*L*-*C* circuit which behaves as a damped harmonic oscillator.

4.2.2 *R*-*L*-*C* circuits

Consider the *R*-*L*-*C* circuit of Fig. 4.4. When the switch is to the left, the capacitor is charged, eventually to a steady state value $Q = CV$. At $t = 0$ the switch is thrown to the right, completing the *R*-*L*-*C* circuit. Recall that the sum of the voltage drops across the three elements must be zero:

$$L\,\frac{dI}{dt} + IR + \frac{Q}{C} = 0 \ . \tag{4.49}$$

We also have $\dot{Q} = I$, hence

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\,\frac{dQ}{dt} + \frac{1}{LC}\,Q = 0 \ , \tag{4.50}$$

which is the equation for a damped harmonic oscillator, with $\omega_0 = (LC)^{-1/2}$ and $\beta = R/2L$.

The boundary conditions at $t = 0$ are $Q(0) = CV$ and $\dot{Q}(0) = 0$. Under these conditions, the full solution at all times is

$$Q(t) = CV\,e^{-\beta t}\Big(\cos\nu t + \frac{\beta}{\nu}\,\sin\nu t\Big) \tag{4.51}$$

$$I(t) = -CV\,\frac{\omega_0^2}{\nu}\,e^{-\beta t}\,\sin\nu t \ , \tag{4.52}$$

again with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

If we put a time-dependent voltage source in series with the resistor, capacitor, and inductor, we would have

$$L\,\frac{dI}{dt} + IR + \frac{Q}{C} = V(t) \ , \tag{4.53}$$

which is the equation of a *forced* damped harmonic oscillator.

4.2.3 Examples

Third order linear ODE with forcing

The problem is to solve the equation

$$\mathcal{L}_t\, x \equiv \dddot{x} + (a+b+c)\,\ddot{x} + (ab+ac+bc)\,\dot{x} + abc\,x = f_0 \cos(\Omega t)\ . \tag{4.54}$$

The key to solving this is to note that the differential operator $\mathcal{L}_t$ factorizes:

$$\begin{aligned} \mathcal{L}_t &= \frac{d^3}{dt^3} + (a+b+c)\,\frac{d^2}{dt^2} + (ab+ac+bc)\,\frac{d}{dt} + abc \\ &= \Big(\frac{d}{dt} + a\Big)\Big(\frac{d}{dt} + b\Big)\Big(\frac{d}{dt} + c\Big)\ , \end{aligned} \tag{4.55}$$

which says that the third order differential operator appearing in the ODE is in fact a product of first order differential operators. Since

$$\frac{dx}{dt} + \alpha x = 0 \quad \Longrightarrow \quad x(t) = A\, e^{-\alpha x}\ , \tag{4.56}$$

we see that the homogeneous solution takes the form

$$x_{\rm h}(t) = A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct}\ , \tag{4.57}$$

where A, B, and C are constants.

To find the inhomogeneous solution, we solve $L_t\, x = f_0\, e^{-i\Omega t}$ and take the real part. Writing $x(t) = x_0\, e^{-i\Omega t}$, we have

$$\mathcal{L}_t\, x_0\, e^{-i\Omega t} = (a - i\Omega)\,(b - i\Omega)\,(c - i\Omega)\, x_0\, e^{-i\Omega t} \tag{4.58}$$

and thus

$$x_0 = \frac{f_0\, e^{-i\Omega t}}{(a - i\Omega)(b - i\Omega)(c - i\Omega)} \equiv A(\Omega)\, e^{i\delta(\Omega)}\, f_0\, e^{-i\Omega t}\ ,$$

where

$$A(\Omega) = \Big[(a^2 + \Omega^2)\,(b^2 + \Omega^2)\,(c^2 + \Omega^2)\Big]^{-1/2} \tag{4.59}$$

$$\delta(\Omega) = \tan^{-1}\Big(\frac{\Omega}{a}\Big) + \tan^{-1}\Big(\frac{\Omega}{b}\Big) + \tan^{-1}\Big(\frac{\Omega}{c}\Big)\ . \tag{4.60}$$

Thus, the most general solution to $L_t\, x(t) = f_0 \cos(\Omega t)$ is

$$x(t) = A(\Omega)\, f_0\, \cos\big(\Omega t - \delta(\Omega)\big) + A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct}\ . \tag{4.61}$$

Note that the phase shift increases monotonically from $\delta(0) = 0$ to $\delta(\infty) = \frac{3}{2}\pi$.

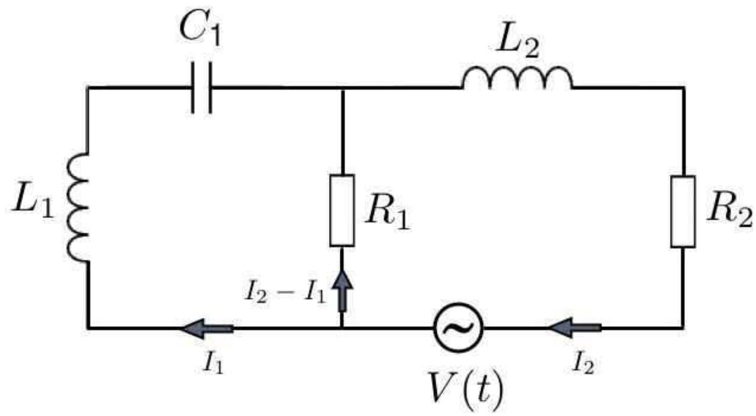


Figure 4.5: A driven L-C-R circuit, with $V(t) = V_0 \cos(\omega t)$.

Mechanical analog of RLC circuit

Consider the electrical circuit in fig. 4.5. Our task is to construct its mechanical analog. To do so, we invoke Kirchoff's laws around the left and right loops:

$$L_1 \dot{I}_1 + \frac{Q_1}{C_1} + R_1 (I_1 - I_2) = 0 \tag{4.62}$$

$$L_2 \dot{I}_2 + R_2 I_2 + R_1 (I_2 - I_1) = V(t) \ . \tag{4.63}$$

Let $Q_1(t)$ be the charge on the left plate of capacitor C_1, and define

$$Q_2(t) = \int_0^t dt' \, I_2(t') \ . \tag{4.64}$$

Then Kirchoff's laws may be written

$$\ddot{Q}_1 + \frac{R_1}{L_1} (\dot{Q}_1 - \dot{Q}_2) + \frac{1}{L_1 C_1} Q_1 = 0 \tag{4.65}$$

$$\ddot{Q}_2 + \frac{R_2}{L_2} \dot{Q}_2 + \frac{R_1}{L_2} (\dot{Q}_2 - \dot{Q}_1) = \frac{V(t)}{L_2} \ . \tag{4.66}$$

Now consider the mechanical system in Fig. 4.6. The blocks have masses M_1 and M_2. The friction coefficient between blocks 1 and 2 is b_1, and the friction coefficient between block 2 and the floor is b_2. Here we assume a velocity-dependent frictional force $F_{\rm f} = -b\dot{x}$, rather than the more conventional constant $F_{\rm f} = -\mu W$, where W is the weight of an object. Velocity-dependent friction is applicable when the relative velocity of an object and a surface is sufficiently large. There is a spring of spring constant k_1 which connects block 1 to the wall. Finally, block 2 is driven by a periodic acceleration $f_0 \cos(\omega t)$. We now identify

$$X_1 \leftrightarrow Q_1 \quad , \quad X_2 \leftrightarrow Q_2 \quad , \quad b_1 \leftrightarrow \frac{R_1}{L_1} \quad , \quad b_2 \leftrightarrow \frac{R_2}{L_2} \quad , \quad k_1 \leftrightarrow \frac{1}{L_1 C_1} \ , \tag{4.67}$$

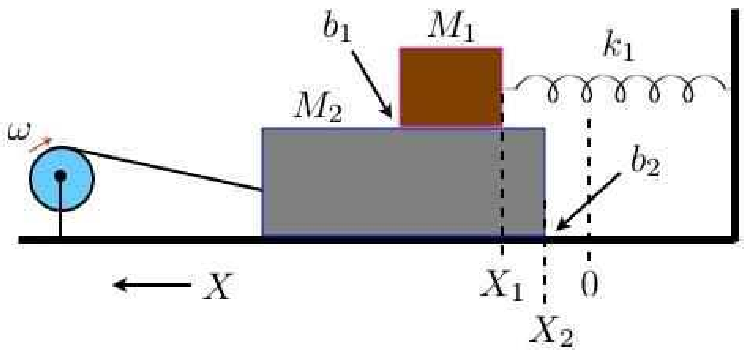


Figure 4.6: The equivalent mechanical circuit for fig. 4.5.

as well as $f(t) \leftrightarrow V(t)/L_2$.

The solution again proceeds by Fourier transform. We write

$$V(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{V}(\omega)\, e^{-i\omega t} \tag{4.68}$$

and

$$\begin{Bmatrix} Q_1(t) \\ \hat{I}_2(t) \end{Bmatrix} = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \begin{Bmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{Bmatrix} e^{-i\omega t} \tag{4.69}$$

The frequency space version of Kirchoff's laws for this problem is

$$\overbrace{\begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1\, C_1 & R_1/L_1 \\ i\omega\, R_1/L_2 & -i\omega + (R_1 + R_2)/L_2 \end{pmatrix}}^{\hat{G}(\omega)} \begin{pmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{V}(\omega)/L_2 \end{pmatrix} \tag{4.70}$$

The homogeneous equation has eigenfrequencies given by the solution to $\det \hat{G}(\omega) = 0$, which is a cubic equation. Correspondingly, there are three initial conditions to account for: $Q_1(0)$, $I_1(0)$, and $I_2(0)$. As in the case of the single damped harmonic oscillator, these transients are damped, and for large times may be ignored. The solution then is

$$\begin{pmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1\, C_1 & R_1/L_1 \\ i\omega\, R_1/L_2 & -i\omega + (R_1 + R_2)/L_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \hat{V}(\omega)/L_2 \end{pmatrix} . \tag{4.71}$$

To obtain the time-dependent $Q_1(t)$ and $I_2(t)$, we must compute the Fourier transform back to the time domain.

4.3 General solution by Green's function method

For a general forcing function $f(t)$, we solve by Fourier transform. Recall that a function $F(t)$ in the time domain has a Fourier transform $\hat{F}(\omega)$ in the frequency domain. The relation between the two is:[1]

$$F(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{F}(\omega) \quad \Longleftrightarrow \quad \hat{F}(\omega) = \int_{-\infty}^{\infty} dt\, e^{+i\omega t}\, F(t)\ . \tag{4.72}$$

We can convert the differential equation 4.3 to an algebraic equation in the frequency domain, $\hat{x}(\omega) = \hat{G}(\omega)\, \hat{f}(\omega)$, where

$$\hat{G}(\omega) = \frac{1}{\omega_0^2 - 2i\beta\omega - \omega^2} \tag{4.73}$$

is the *Green's function* in the frequency domain. The general solution is written

$$x(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{G}(\omega)\, \hat{f}(\omega) + x_{\rm h}(t)\ , \tag{4.74}$$

where $x_{\rm h}(t) = \sum_i C_i\, e^{-i\omega_i t}$ is a solution to the homogeneous equation. We may also write the above integral over the time domain:

$$x(t) = \int_{-\infty}^{\infty} dt'\, G(t-t')\, f(t') + x_{\rm h}(t) \tag{4.75}$$

$$\begin{aligned} G(s) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega s}\, \hat{G}(\omega) \\ &= \nu^{-1}\, \exp(-\beta s)\, \sin(\nu s)\, \Theta(s) \end{aligned} \tag{4.76}$$

where $\Theta(s)$ is the *step function*,

$$\Theta(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases} \tag{4.77}$$

where once again $\nu \equiv \sqrt{\omega_0^2 - \beta^2}$.

Example: force pulse

Consider a pulse force

$$f(t) = f_0\, \Theta(t)\, \Theta(T-t) = \begin{cases} f_0 & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases} \tag{4.78}$$

[1]Different texts often use different conventions for Fourier and inverse Fourier transforms. Sometimes the factor of $(2\pi)^{-1}$ is associated with the time integral, and sometimes a factor of $(2\pi)^{-1/2}$ is assigned to both frequency and time integrals. The convention I use is obviously the best.

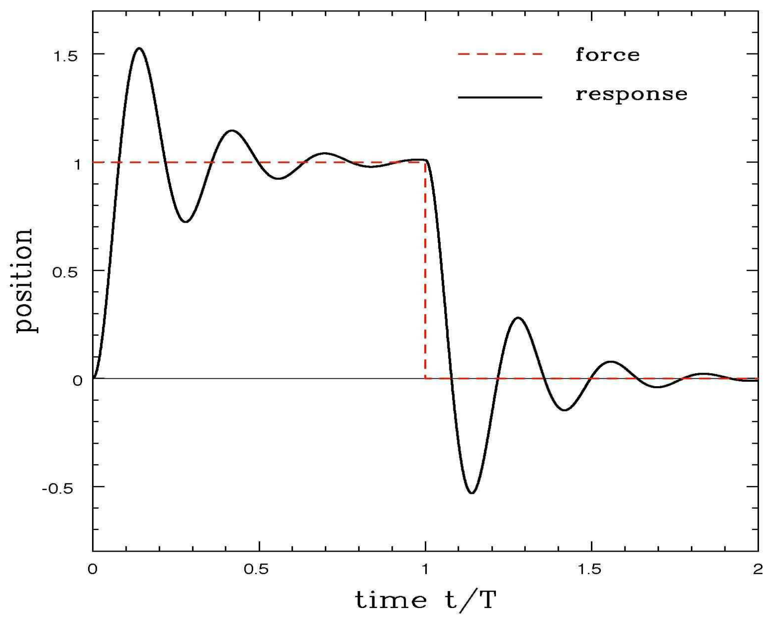


Figure 4.7: Response of an underdamped oscillator to a pulse force.

In the underdamped regime, for example, we find the solution

$$x(t) = \frac{f_0}{\omega_0^2} \left\{ 1 - e^{-\beta t} \cos \nu t - \frac{\beta}{\nu} e^{-\beta t} \sin \nu t \right\} \tag{4.79}$$

if $0 \leq t \leq T$ and

$$\begin{aligned} x(t) &= \frac{f_0}{\omega_0^2} \left\{ \left(e^{-\beta(t-T)} \cos \nu(t-T) - e^{-\beta t} \cos \nu t \right) \right. \\ &\qquad \left. + \frac{\beta}{\nu} \left(e^{-\beta(t-T)} \sin \nu(t-T) - e^{-\beta t} \sin \nu t \right) \right\} \end{aligned} \tag{4.80}$$

if $t > T$.

4.4 General Linear Autonomous Inhomogeneous ODEs

This method immediately generalizes to the case of general autonomous linear inhomogeneous ODEs of the form

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 \, x = f(t) \ . \tag{4.81}$$

We can write this as

$$\mathcal{L}_t \, x(t) = f(t) \ , \tag{4.82}$$

where $\mathcal{L}_t$ is the $n^{\rm th}$ order differential operator

$$\mathcal{L}_t = \frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\,\frac{d}{dt} + a_0 \ . \tag{4.83}$$

The general solution to the inhomogeneous equation is given by

$$x(t) = x_{\rm h}(t) + \int\limits_{-\infty}^{\infty}\!dt'\,G(t,t')\,f(t') \ , \tag{4.84}$$

where $G(t,t')$ is the Green's function. Note that $\mathcal{L}_t\,x_{\rm h}(t) = 0$. Thus, in order for eqns. 4.82 and 4.84 to be true, we must have

$$\mathcal{L}_t\,x(t) = \overbrace{\mathcal{L}_t\,x_{\rm h}(t)}^{\rm this\ vanishes} + \int\limits_{-\infty}^{\infty}\!dt'\,\mathcal{L}_t\,G(t,t')\,f(t') = f(t) \ , \tag{4.85}$$

which means that

$$\mathcal{L}_t\,G(t,t') = \delta(t-t') \ , \tag{4.86}$$

where $\delta(t-t')$ is the Dirac δ-function. Some properties of $\delta(x)$:

$$\int\limits_a^b\!dx\,f(x)\,\delta(x-y) = \begin{cases} f(y) & \text{if } a < y < b \\ \\ 0 & \text{if } y < a \text{ or } y > b \ . \end{cases} \tag{4.87}$$

$$\delta\big(g(x)\big) = \sum_{\substack{x_i\,{\rm with} \\ g(x_i)=0}} \frac{\delta(x-x_i)}{\big|g'(x_i)\big|} \ , \tag{4.88}$$

valid for any functions $f(x)$ and $g(x)$. The sum in the second equation is over the zeros x_i of $g(x)$.

Incidentally, the Dirac δ-function enters into the relation between a function and its Fourier transform, in the following sense. We have

$$f(t) = \int\limits_{-\infty}^{\infty}\!\frac{d\omega}{2\pi}\,e^{-i\omega t}\,\hat f(\omega) \tag{4.89}$$

$$\hat f(\omega) = \int\limits_{-\infty}^{\infty}\!dt\,e^{+i\omega t}\,f(t) \ . \tag{4.90}$$

Substituting the second equation into the first, we have

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \int_{-\infty}^{\infty} dt'\, e^{i\omega t'}\, f(t') \\ &= \int_{-\infty}^{\infty} dt' \left\{ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t'-t)} \right\} f(t') \ , \end{aligned} \tag{4.91}$$

which is indeed correct because the term in brackets is a representation of $\delta(t-t')$:

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega s} = \delta(s) \ . \tag{4.92}$$

If the differential equation $\mathcal{L}_t\, x(t) = f(t)$ is defined over some finite t interval with prescribed boundary conditions on $x(t)$ at the endpoints, then $G(t,t')$ will depend on t and t' separately. For the case we are considering, the interval is the entire real line $t \in (-\infty, \infty)$, and $G(t,t') = G(t-t')$ is a function of the single variable $t-t'$.

Note that $\mathcal{L}_t = \mathcal{L}\big(\frac{d}{dt}\big)$ may be considered a function of the differential operator $\frac{d}{dt}$. If we now Fourier transform the equation $\mathcal{L}_t\, x(t) = f(t)$, we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, f(t) &= \int_{-\infty}^{\infty} dt\, e^{i\omega t} \left\{ \frac{d^n}{dt^n} + a_{n-1}\, \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\, \frac{d}{dt} + a_0 \right\} x(t) \\ &= \int_{-\infty}^{\infty} dt\, e^{i\omega t} \left\{ (-i\omega)^n + a_{n-1}\, (-i\omega)^{n-1} + \ldots + a_1\, (-i\omega) + a_0 \right\} x(t) \ , \end{aligned} \tag{4.93}$$

where we integrate by parts on t, assuming the boundary terms at $t = \pm\infty$ vanish, *i.e.* $x(\pm\infty) = 0$, so that, inside the t integral,

$$e^{i\omega t} \left(\frac{d}{dt} \right)^k x(t) \to \left[\left(-\frac{d}{dt} \right)^k e^{i\omega t} \right] x(t) = (-i\omega)^k\, e^{i\omega t}\, x(t) \ . \tag{4.94}$$

Thus, if we define

$$\hat{\mathcal{L}}(\omega) = \sum_{k=0}^{n} a_k\, (-i\omega)^k \ , \tag{4.95}$$

then we have

$$\hat{\mathcal{L}}(\omega)\, \hat{\boldsymbol{x}}(\omega) = \hat{f}(\omega) \ , \tag{4.96}$$

where $a_n \equiv 1$. According to the Fundamental Theorem of Algebra, the n^{th} degree polynomial $\hat{\mathcal{L}}(\omega)$ may be uniquely factored over the complex ω plane into a product over n roots:

$$\hat{\mathcal{L}}(\omega) = (-i)^n\, (\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n) \ . \tag{4.97}$$

If the $\{a_k\}$ are all real, then $\big[\hat{\mathcal{L}}(\omega)\big]^* = \hat{\mathcal{L}}(-\omega^*)$, hence if Ω is a root then so is $-\Omega^*$. Thus, the roots appear in pairs which are symmetric about the imaginary axis. *I.e.* if $\Omega = a + ib$ is a root, then so is $-\Omega^* = -a + ib$.

The general solution to the homogeneous equation is

$$x_{\rm h}(t) = \sum_{i=1}^{n} A_i \, e^{-i\omega_i t} \ , \tag{4.98}$$

which involves n arbitrary complex constants A_i. The susceptibility, or Green's function in Fourier space, $\hat{G}(\omega)$ is then

$$\hat{G}(\omega) = \frac{1}{\hat{\mathcal{L}}(\omega)} = \frac{i^n}{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)} \ , \tag{4.99}$$

and the general solution to the inhomogeneous equation is again given by

$$x(t) = x_{\rm h}(t) + \int\limits_{-\infty}^{\infty} \! dt' \, G(t - t') \, f(t') \ , \tag{4.100}$$

where $x_{\rm h}(t)$ is the solution to the homogeneous equation, *i.e.* with zero forcing, and where

$$\begin{aligned} G(s) &= \int\limits_{-\infty}^{\infty} \! \frac{d\omega}{2\pi} \, e^{-i\omega s} \, \hat{G}(\omega) \\ &= i^n \int\limits_{-\infty}^{\infty} \! \frac{d\omega}{2\pi} \, \frac{e^{-i\omega s}}{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)} \\ &= \sum_{j=1}^{n} \frac{e^{-i\omega_j s}}{i \, \mathcal{L}'(\omega_j)} \, \Theta(s) \ , \end{aligned} \tag{4.101}$$

where we assume that $\text{Im}\, \omega_j < 0$ for all j. The integral above was done using Cauchy's theorem and the calculus of residues – a beautiful result from the theory of complex functions.

As an example, consider the familiar case

$$\begin{aligned} \hat{\mathcal{L}}(\omega) &= \omega_0^2 - 2i\beta\omega - \omega^2 \\ &= -(\omega - \omega_+) \, (\omega - \omega_-) \ , \end{aligned} \tag{4.102}$$

with $\omega_\pm = -i\beta \pm \nu$, and $\nu = (\omega_0^2 - \beta^2)^{1/2}$. This yields

$$\mathcal{L}'(\omega_\pm) = \mp(\omega_+ - \omega_-) = \mp 2\nu \ . \tag{4.103}$$

Then according to equation 4.101,

$$\begin{aligned} G(s) &= \left\{ \frac{e^{-i\omega_+ s}}{i\mathcal{L}'(\omega_+)} + \frac{e^{-i\omega_- s}}{i\mathcal{L}'(\omega_-)} \right\} \Theta(s) \\ &= \left\{ \frac{e^{-\beta s}\, e^{-i\nu s}}{-2i\nu} + \frac{e^{-\beta s}\, e^{i\nu s}}{2i\nu} \right\} \Theta(s) \\ &= \nu^{-1}\, e^{-\beta s}\, \sin(\nu s)\, \Theta(s) \ , \end{aligned} \tag{4.104}$$

exactly as before.

4.5 Kramers-Krönig Relations (advanced material)

Suppose $\hat{\chi}(\omega) \equiv \hat{G}(\omega)$ is analytic in the UHP[2]. Then for all ν, we must have

$$\int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{\hat{\chi}(\nu)}{\nu - \omega + i\epsilon} = 0 \ , \tag{4.105}$$

where ϵ is a positive infinitesimal. The reason is simple: just close the contour in the UHP, assuming $\hat{\chi}(\omega)$ vanishes sufficiently rapidly that Jordan's lemma can be applied. Clearly this is an extremely weak restriction on $\hat{\chi}(\omega)$, given the fact that the denominator already causes the integrand to vanish as $|\omega|^{-1}$.

Let us examine the function

$$\frac{1}{\nu - \omega + i\epsilon} = \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} - \frac{i\epsilon}{(\nu - \omega)^2 + \epsilon^2} \ . \tag{4.106}$$

which we have separated into real and imaginary parts. Under an integral sign, the first term, in the limit $\epsilon \to 0$, is equivalent to taking a *principal part* of the integral. That is, for any function $F(\nu)$ which is regular at $\nu = \omega$,

$$\lim_{\epsilon \to 0} \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} \, F(\nu) \equiv \mathcal{P} \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{F(\nu)}{\nu - \omega} \ . \tag{4.107}$$

The principal part symbol $\mathcal{P}$ means that the singularity at $\nu = \omega$ is elided, either by smoothing out the function $1/(\nu - \epsilon)$ as above, or by simply cutting out a region of integration of width ϵ on either side of $\nu = \omega$.

The imaginary part is more interesting. Let us write

$$h(u) \equiv \frac{\epsilon}{u^2 + \epsilon^2} \ . \tag{4.108}$$

[2] In this section, we use the notation $\hat{\chi}(\omega)$ for the susceptibility, rather than $\hat{G}(\omega)$

For $|u| \gg \epsilon$, $h(u) \simeq \epsilon/u^2$, which vanishes as $\epsilon \to 0$. For $u = 0$, $h(0) = 1/\epsilon$ which diverges as $\epsilon \to 0$. Thus, $h(u)$ has a huge peak at $u = 0$ and rapidly decays to 0 as one moves off the peak in either direction a distance greater that ϵ. Finally, note that

$$\int_{-\infty}^{\infty} du\, h(u) = \pi \ , \tag{4.109}$$

a result which itself is easy to show using contour integration. Putting it all together, this tells us that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{u^2 + \epsilon^2} = \pi\delta(u) \ . \tag{4.110}$$

Thus, for positive infinitesimal ϵ,

$$\frac{1}{u \pm i\epsilon} = \mathcal{P}\,\frac{1}{u} \mp i\pi\delta(u) \ , \tag{4.111}$$

a most useful result.

We now return to our initial result 4.105, and we separate $\hat{\chi}(\omega)$ into real and imaginary parts:

$$\hat{\chi}(\omega) = \hat{\chi}'(\omega) + i\hat{\chi}''(\omega) \ . \tag{4.112}$$

(In this equation, the primes do not indicate differentiation with respect to argument.) We therefore have, for every real value of ω,

$$0 = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \Big[\chi'(\nu) + i\chi''(\nu)\Big] \Big[\mathcal{P}\,\frac{1}{\nu - \omega} - i\pi\delta(\nu - \omega)\Big] \ . \tag{4.113}$$

Taking the real and imaginary parts of this equation, we derive the *Kramers-Krönig relations*:

$$\chi'(\omega) = +\mathcal{P}\int_{-\infty}^{\infty} \frac{d\nu}{\pi}\, \frac{\hat{\chi}''(\nu)}{\nu - \omega} \tag{4.114}$$

$$\chi''(\omega) = -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\nu}{\pi}\, \frac{\hat{\chi}'(\nu)}{\nu - \omega} \ . \tag{4.115}$$

Chapter 5

Calculus of Variations

5.1 Snell's Law

Warm-up problem: You are standing at point (x_1, y_1) on the beach and you want to get to a point (x_2, y_2) in the water, a few meters offshore. The interface between the beach and the water lies at $x = 0$. What path results in the shortest travel time? It is not a straight line! This is because your speed v_1 on the sand is greater than your speed v_2 in the water. The optimal path actually consists of two line segments, as shown in Fig. 5.1. Let the path pass through the point $(0, y)$ on the interface. Then the time T is a function of y:

$$T(y) = \frac{1}{v_1}\sqrt{x_1^2 + (y - y_1)^2} + \frac{1}{v_2}\sqrt{x_2^2 + (y_2 - y)^2}\ . \tag{5.1}$$

To find the minimum time, we set

$$\begin{aligned} \frac{dT}{dy} = 0 &= \frac{1}{v_1}\frac{y - y_1}{\sqrt{x_1^2 + (y - y_1)^2}} - \frac{1}{v_2}\frac{y_2 - y}{\sqrt{x_2^2 + (y_2 - y)^2}} \\ &= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}\ . \end{aligned} \tag{5.2}$$

Thus, the optimal path satisfies

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}\ , \tag{5.3}$$

which is known as *Snell's Law*.

Snell's Law is familiar from optics, where the speed of light in a polarizable medium is written $v = c/n$, where n is the index of refraction. In terms of n,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2\ . \tag{5.4}$$

If there are several interfaces, Snell's law holds at each one, so that

$$n_i \sin\theta_i = n_{i+1} \sin\theta_{i+1}\ , \tag{5.5}$$

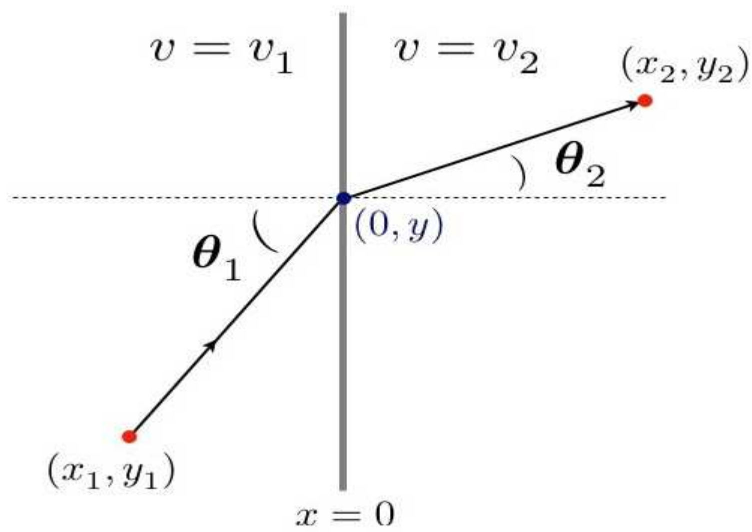


Figure 5.1: The shortest path between (x_1, y_1) and (x_2, y_2) is not a straight line, but rather two successive line segments of different slope.

at the interface between media i and $i + 1$.

In the limit where the number of slabs goes to infinity but their thickness is infinitesimal, we can regard n and θ as functions of a continuous variable x. One then has

$$\frac{\sin\theta(x)}{v(x)} = \frac{y'}{v\sqrt{1+y'^2}} = P \ , \tag{5.6}$$

where P is a constant. Here wve have used the result $\sin\theta = y'/\sqrt{1+y'^2}$, which follows from drawing a right triangle with side lengths dx, dy, and $\sqrt{dx^2+dy^2}$. If we differentiate the above equation with respect to x, we eliminate the constant and obtain the second order ODE

$$\frac{1}{1+y'^2}\,\frac{y''}{y'} = \frac{v'}{v} \ . \tag{5.7}$$

This is a differential equation that $y(x)$ must satisfy if the *functional*

$$T\big[y(x)\big] = \int \frac{ds}{v} = \int\limits_{x_1}^{x_2} \! dx \, \frac{\sqrt{1+y'^2}}{v(x)} \tag{5.8}$$

is to be minimized.

5.2 Functions and Functionals

A *function* is a mathematical object which takes a real (or complex) variable, or several such variables, and returns a real (or complex) number. A *functional* is a mathematical

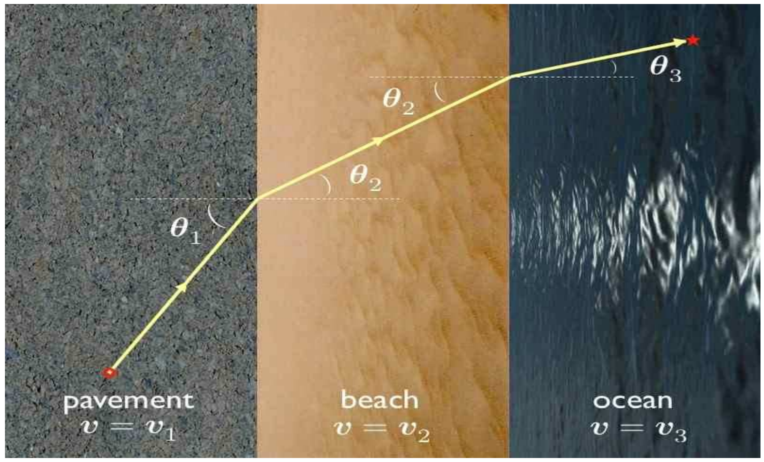


Figure 5.2: The path of shortest length is composed of three line segments. The relation between the angles at each interface is governed by Snell's Law.

object which takes an entire function and returns a number. In the case at hand, we have

$$T\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\, L(y, y', x) \ , \tag{5.9}$$

where the function $L(y, y', x)$ is given by

$$L(y, y', x) = \frac{1}{v(x)} \sqrt{1 + y'^2} \ . \tag{5.10}$$

Here $n(x)$ is a given function characterizing the medium, and $y(x)$ is the path whose time is to be evaluated.

In ordinary calculus, we extremize a function $f(x)$ by demanding that f not change to lowest order when we change $x \to x + dx$:

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2}\, f''(x)\, (dx)^2 + \ldots \ . \tag{5.11}$$

We say that $x = x^*$ is an extremum when $f'(x^*) = 0$.

For a functional, the first *functional variation* is obtained by sending $y(x) \to y(x) + \delta y(x)$,

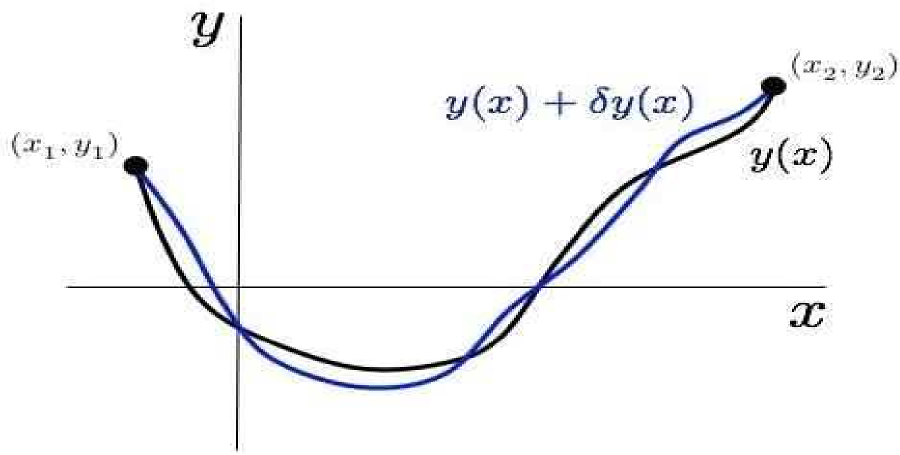


Figure 5.3: A path $y(x)$ and its variation $y(x) + \delta y(x)$.

and extracting the variation in the functional to order δy. Thus, we compute

$$\begin{aligned}
T\big[y(x) + \delta y(x)\big] &= \int\limits_{x_1}^{x_2} dx\, L(y + \delta y, y' + \delta y', x) \\
&= \int\limits_{x_1}^{x_2} dx \left\{ L + \frac{\partial L}{\partial y}\, \delta y + \frac{\partial L}{\partial y'}\, \delta y' + \mathcal{O}\big((\delta y)^2\big) \right\} \\
&= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left\{ \frac{\partial L}{\partial y}\, \delta y + \frac{\partial L}{\partial y'}\, \frac{d}{dx}\, \delta y \right\} \\
&= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right] \delta y + \frac{\partial L}{\partial y'}\, \delta y \,\Bigg|_{x_1}^{x_2} . \qquad (5.12)
\end{aligned}$$

Now one very important thing about the variation $\delta y(x)$ is that it must vanish at the endpoints: $\delta y(x_1) = \delta y(x_2) = 0$. This is because the space of functions under consideration satisfy fixed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Thus, the last term in the above equation vanishes, and we have

$$\delta T = \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right] \delta y \ . \qquad (5.13)$$

We say that the first functional derivative of T with respect to $y(x)$ is

$$\frac{\delta T}{\delta y(x)} = \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right]_x , \qquad (5.14)$$

where the subscript indicates that the expression inside the square brackets is to be evaluated at x. The functional $T\big[y(x)\big]$ is *extremized* when its first functional derivative vanishes, which results in a differential equation for $y(x)$,

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 \ , \tag{5.15}$$

known as the *Euler-Lagrange* equation.

$L(y, y', x)$ independent of y

Suppose $L(y, y', x)$ is independent of y. Then from the Euler-Lagrange equations we have that

$$P \equiv \frac{\partial L}{\partial y'} \tag{5.16}$$

is a constant. In classical mechanics, this will turn out to be a *generalized momentum*. For $L = \frac{1}{v}\sqrt{1+y'^2}$, we have

$$P = \frac{y'}{v\sqrt{1+y'^2}} \ . \tag{5.17}$$

Setting $dP/dx = 0$, we recover the second order ODE of eqn. 5.7. Solving for y',

$$\frac{dy}{dx} = \pm\frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \ , \tag{5.18}$$

where $v_0 = 1/P$.

$L(y, y', x)$ independent of x

When $L(y, y', x)$ is independent of x, we can again integrate the equation of motion. Consider the quantity

$$H = y' \frac{\partial L}{\partial y'} - L \ . \tag{5.19}$$

Then

$$\begin{aligned} \frac{dH}{dx} = \frac{d}{dx}\left[y' \frac{\partial L}{\partial y'} - L\right] &= y'' \frac{\partial L}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y'} \, y'' - \frac{\partial L}{\partial y} y' - \frac{\partial L}{\partial x} \\ &= y'\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] - \frac{\partial L}{\partial x} \ , \end{aligned} \tag{5.20}$$

where we have used the Euler-Lagrange equations to write $\frac{d}{dx}\big(\frac{\partial L}{\partial y'}\big) = \frac{\partial L}{\partial y}$. So if $\partial L/\partial x = 0$, we have $dH/dx = 0$, *i.e.* H is a constant.

5.2.1 Functional Taylor series

In general, we may expand a functional $F[y + \delta y]$ in a *functional Taylor series*,

$$\begin{aligned} F[y+\delta y] = F[y] + \int\! dx_1 \, K_1(x_1)\, \delta y(x_1) + \tfrac{1}{2!} \int\! dx_1 \int\! dx_2 \, K_2(x_1,x_2)\, \delta y(x_1)\, \delta y(x_2) \\ + \tfrac{1}{3!} \int\! dx_1 \int\! dx_2 \int\! dx_3 \, K_3(x_1,x_2,x_3)\, \delta y(x_1)\, \delta y(x_2)\, \delta y(x_3) + \ldots \end{aligned} \tag{5.21}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta y(x_1)\cdots\delta y(x_n)} \tag{5.22}$$

for the $n^{\rm th}$ functional derivative.

5.3 Examples from the Calculus of Variations

Here we present three useful examples of variational calculus as applied to problems in mathematics and physics.

5.3.1 Example 1 : minimal surface of revolution

Consider a surface formed by rotating the function $y(x)$ about the x-axis. The area is then

$$A\big[y(x)\big] = \int\limits_{x_1}^{x_2} \! dx\, 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \ , \tag{5.23}$$

and is a functional of the curve $y(x)$. Thus we can define $L(y,y') = 2\pi y\sqrt{1+y'^2}$ and make the identification $y(x) \leftrightarrow q(t)$. Since $L(y,y',x)$ is independent of x, we have

$$H = y'\,\frac{\partial L}{\partial y'} - L \quad \Rightarrow \quad \frac{dH}{dx} = -\frac{\partial L}{\partial x} \ , \tag{5.24}$$

and when L has no explicit x-dependence, H is conserved. One finds

$$H = 2\pi y \cdot \frac{y'^2}{\sqrt{1+y'^2}} - 2\pi y\,\sqrt{1+y'^2} = -\frac{2\pi y}{\sqrt{1+y'^2}} \ . \tag{5.25}$$

Solving for y',

$$\frac{dy}{dx} = \pm\sqrt{\left(\frac{2\pi y}{H}\right)^2 - 1} \ , \tag{5.26}$$

which may be integrated with the substitution $y = \frac{H}{2\pi}\cosh u$, yielding

$$y(x) = b\,\cosh\left(\frac{x-a}{b}\right) \ , \tag{5.27}$$

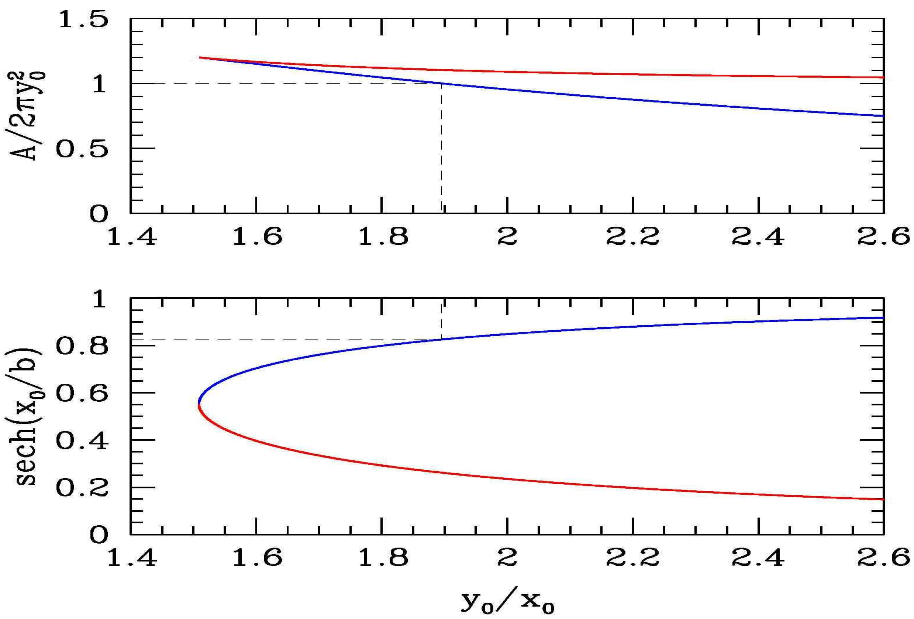


Figure 5.4: Minimal surface solution, with $y(x) = b\cosh(x/b)$ and $y(x_0) = y_0$. Top panel: $A/2\pi y_0^2$ *vs.* y_0/x_0. Bottom panel: $\text{sech}(x_0/b)$ *vs.* y_0/x_0. The blue curve corresponds to a global minimum of $A[y(x)]$, and the red curve to a local minimum or saddle point.

where a and $b = \frac{H}{2\pi}$ are constants of integration. Note there are two such constants, as the original equation was second order. This shape is called a *catenary.* As we shall later find, it is also the shape of a uniformly dense rope hanging between two supports, under the influence of gravity. To fix the constants a and b, we invoke the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Consider the case where $-x_1 = x_2 \equiv x_0$ and $y_1 = y_2 \equiv y_0$. Then clearly $a = 0$, and we have

$$y_0 = b\,\cosh\left(\frac{x_0}{b}\right) \qquad \Rightarrow \qquad \gamma = \kappa^{-1}\,\cosh\kappa\ , \tag{5.28}$$

with $\gamma \equiv y_0/x_0$ and $\kappa \equiv x_0/b$. One finds that for any $\gamma > 1.5089$ there are two solutions, one of which is a global minimum and one of which is a local minimum or saddle of $A[y(x)]$. The solution with the smaller value of κ (*i.e.* the larger value of $\text{sech}\,\kappa$) yields the smaller value of A, as shown in Fig. 5.4. Note that

$$\frac{y}{y_0} = \frac{\cosh(x/b)}{\cosh(x_0/b)}\ , \tag{5.29}$$

so $y(x = 0) = y_0\,\text{sech}(x_0/b)$.

When extremizing functions that are defined over a finite or semi-infinite interval, one must take care to evaluate the function at the boundary, for it may be that the boundary yields a global extremum even though the derivative may not vanish there. Similarly, when extremizing functionals, one must investigate the functions at the boundary of function space. In this case, such a function would be the discontinuous solution, with

$$y(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ 0 & \text{if } x_1 < x < x_2 \\ y_2 & \text{if } x = x_2 \ . \end{cases} \tag{5.30}$$

This solution corresponds to a surface consisting of two discs of radii y_1 and y_2, joined by an infinitesimally thin thread. The area functional evaluated for this particular $y(x)$ is clearly $A = \pi(y_1^2 + y_2^2)$. In Fig. 5.4, we plot $A/2\pi y_0^2$ *versus* the parameter $\gamma = y_0/x_0$. For $\gamma > \gamma_c \approx 1.564$, one of the catenary solutions is the global minimum. For $\gamma < \gamma_c$, the minimum area is achieved by the discontinuous solution.

Note that the functional derivative,

$$K_1(x) = \frac{\delta A}{\delta y(x)} = \left\{ \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) \right\} = \frac{2\pi\big(1 + y'^2 - yy''\big)}{(1 + y'^2)^{3/2}} \ , \tag{5.31}$$

indeed vanishes for the catenary solutions, but does not vanish for the discontinuous solution, where $K_1(x) = 2\pi$ throughout the interval $(-x_0, x_0)$. Since $y = 0$ on this interval, y cannot be decreased. The fact that $K_1(x) > 0$ means that increasing y will result in an increase in A, so the boundary value for A, which is $2\pi y_0^2$, is indeed a local minimum.

We furthermore see in Fig. 5.4 that for $\gamma < \gamma_* \approx 1.5089$ the local minimum and saddle are no longer present. This is the familiar saddle-node bifurcation, here in function space. Thus, for $\gamma \in [0, \gamma_*)$ there are no extrema of $A[y(x)]$, and the minimum area occurs for the discontinuous $y(x)$ lying at the boundary of function space. For $\gamma \in (\gamma_*, \gamma_c)$, two extrema exist, one of which is a local minimum and the other a saddle point. Still, the area is minimized for the discontinuous solution. For $\gamma \in (\gamma_c, \infty)$, the local minimum is the global minimum, and has smaller area than for the discontinuous solution.

5.3.2 Example 2 : geodesic on a surface of revolution

We use cylindrical coordinates (ρ, ϕ, z) on the surface $z = z(\rho)$. Thus,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2\, d\phi^2 + dx^2 \\ &= \Big\{ 1 + \big[z'(\rho)\big]^2 \Big\}\, d\rho + \rho^2\, d\phi^2 \ , \end{aligned} \tag{5.32}$$

and the distance functional $D\big[\phi(\rho)\big]$ is

$$D\big[\phi(\rho)\big] = \int\limits_{\rho_1}^{\rho_2} d\rho\, L(\phi, \phi', \rho) \ , \tag{5.33}$$

where

$$L(\phi,\phi',\rho)=\sqrt{1+z'^2(\rho)+\rho^2\,\phi'^2(\rho)}\ . \tag{5.34}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial\phi}-\frac{d}{d\rho}\bigg(\frac{\partial L}{\partial\phi'}\bigg)=0 \quad\Rightarrow\quad \frac{\partial L}{\partial\phi'}=\text{const.} \tag{5.35}$$

Thus,

$$\frac{\partial L}{\partial\phi'}=\frac{\rho^2\,\phi'}{\sqrt{1+z'^2+\rho^2\,\phi'^2}}=a\ , \tag{5.36}$$

where a is a constant. Solving for ϕ', we obtain

$$d\phi=\frac{a\,\sqrt{1+\big[z'(\rho)\big]^2}}{\rho\,\sqrt{\rho^2-a^2}}\,d\rho\ , \tag{5.37}$$

which we must integrate to find $\phi(\rho)$, subject to boundary conditions $\phi(\rho_i)=\phi_i$, with $i=1,2$.

On a cone, $z(\rho)=\lambda\rho$, and we have

$$d\phi=a\,\sqrt{1+\lambda^2}\,\frac{d\rho}{\rho\,\sqrt{\rho^2-a^2}}=\sqrt{1+\lambda^2}\,d\tan^{-1}\sqrt{\frac{\rho^2}{a^2}-1}\ , \tag{5.38}$$

which yields

$$\phi(\rho)=\beta+\sqrt{1+\lambda^2}\ \tan^{-1}\sqrt{\frac{\rho^2}{a^2}-1}\ , \tag{5.39}$$

which is equivalent to

$$\rho\,\cos\bigg(\frac{\phi-\beta}{\sqrt{1+\lambda^2}}\bigg)=a\ . \tag{5.40}$$

The constants β and a are determined from $\phi(\rho_i)=\phi_i$.

5.3.3 Example 3 : brachistochrone

Problem: find the path between (x_1,y_1) and (x_2,y_2) which a particle sliding frictionlessly and under constant gravitational acceleration will traverse in the shortest time. To solve this we first must invoke some elementary mechanics. Assuming the particle is released from (x_1,y_1) at rest, energy conservation says

$$\tfrac{1}{2}mv^2+mgy=mgy_1\ . \tag{5.41}$$

Then the time, which is a functional of the curve $y(x)$, is

$$\begin{aligned}T\big[y(x)\big]&=\int\limits_{x_1}^{x_2}\frac{ds}{v}=\frac{1}{\sqrt{2g}}\int\limits_{x_1}^{x_2}\!dx\,\sqrt{\frac{1+y'^2}{y_1-y}}\\&\equiv\int\limits_{x_1}^{x_2}\!dx\,L(y,y',x)\ ,\end{aligned} \tag{5.42}$$

with

$$L(y,y',x)=\sqrt{\frac{1+y'^2}{2g(y_1-y)}}\ . \tag{5.43}$$

Since L is independent of x, eqn. 5.20, we have that

$$H=y'\,\frac{\partial L}{\partial y'}-L=-\Big[2g\,(y_1-y)\,\big(1+y'^2\big)\Big]^{-1/2} \tag{5.44}$$

is conserved. This yields

$$dx=-\sqrt{\frac{y_1-y}{2a-y_1+y}}\,dy\ , \tag{5.45}$$

with $a=(4gH^2)^{-1}$. This may be integrated parametrically, writing

$$y_1-y=2a\sin^2(\tfrac{1}{2}\theta)\quad\Rightarrow\quad dx=2a\sin^2(\tfrac{1}{2}\theta)\,d\theta\ , \tag{5.46}$$

which results in the parametric equations

$$x-x_1=a\,\big(\theta-\sin\theta\big) \tag{5.47}$$

$$y-y_1=-a\,(1-\cos\theta)\ . \tag{5.48}$$

This curve is known as a *cycloid.*

5.3.4 Ocean waves

Surface waves in fluids propagate with a definite relation between their angular frequency ω and their wavevector $k=2\pi/\lambda$, where λ is the wavelength. The *dispersion relation* is a function $\omega=\omega(k)$. The *group velocity* of the waves is then $v(k)=d\omega/dk$.

In a fluid with a flat bottom at depth h, the dispersion relation turns out to be

$$\omega(k)=\sqrt{gk\,\tanh kh}\approx\begin{cases}\sqrt{gh}\,k & \text{shallow } (kh\ll 1)\\ \\ \sqrt{gk} & \text{deep } (kh\gg 1)\ .\end{cases} \tag{5.49}$$

Suppose we are in the shallow case, where the wavelength λ is significantly greater than the depth h of the fluid. This is the case for ocean waves which break at the shore. The phase velocity and group velocity are then identical, and equal to $v(h)=\sqrt{gh}$. The waves propagate more slowly as they approach the shore.

Let us choose the following coordinate system: x represents the distance parallel to the shoreline, y the distance perpendicular to the shore (which lies at $y=0$), and $h(y)$ is the depth profile of the bottom. We assume $h(y)$ to be a slowly varying function of y which satisfies $h(0)=0$. Suppose a disturbance in the ocean at position (x_2,y_2) propagates until it reaches the shore at $(x_1,y_1=0)$. The time of propagation is

$$T\big[y(x)\big]=\int\frac{ds}{v}=\int\limits_{x_1}^{x_2}\!dx\,\sqrt{\frac{1+y'^2}{g\,h(y)}}\ . \tag{5.50}$$

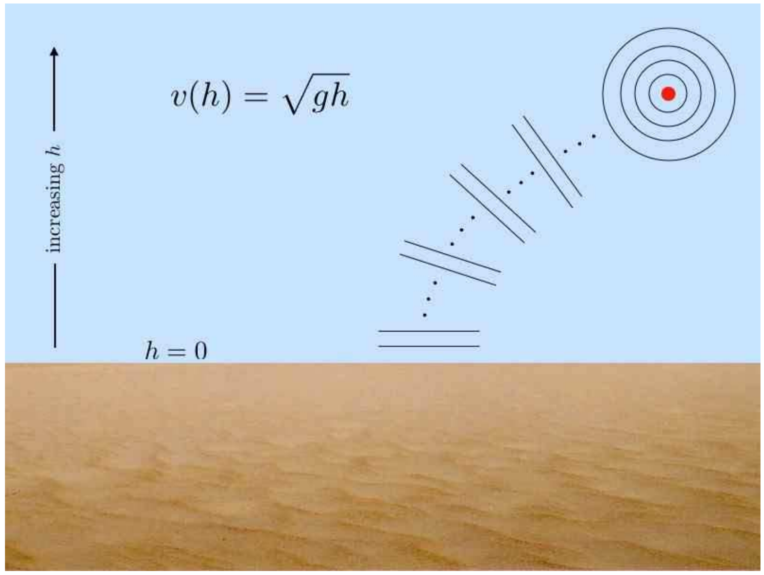


Figure 5.5: For shallow water waves, $v = \sqrt{gh}$. To minimize the propagation time from a source to the shore, the waves break parallel to the shoreline.

We thus identify the integrand

$$L(y, y', x) = \sqrt{\frac{1 + y'^2}{g\, h(y)}} \; . \tag{5.51}$$

As with the brachistochrone problem, to which this bears an obvious resemblance, L is cyclic in the independent variable x, hence

$$H = y' \, \frac{\partial L}{\partial y'} - L = -\Big[g\, h(y)\, \big(1 + y'^2\big)\Big]^{-1/2} \tag{5.52}$$

is constant. Solving for $y'(x)$, we have

$$\tan\theta = \frac{dy}{dx} = \sqrt{\frac{a}{h(y)} - 1} \; , \tag{5.53}$$

where $a = (gH)^{-1}$ is a constant, and where θ is the local slope of the function $y(x)$. Thus, we conclude that near $y = 0$, where $h(y) \to 0$, the waves come in *parallel to the shoreline.* If $h(y) = \alpha y$ has a linear profile, the solution is again a cycloid, with

$$x(\theta) = b\,(\theta - \sin\theta) \tag{5.54}$$

$$y(\theta) = b\,(1 - \cos\theta) \; , \tag{5.55}$$

where $b = 2a/\alpha$ and where the shore lies at $\theta = 0$. Expanding in a Taylor series in θ for small θ, we may eliminate θ and obtain $y(x)$ as

$$y(x) = \big(\tfrac{9}{2}\big)^{1/3}\, b^{1/3}\, x^{2/3} + \ldots \; . \tag{5.56}$$

A *tsunami* is a shallow water wave that manages propagates in deep water. This requires $\lambda > h$, as we've seen, which means the disturbance must have a very long spatial extent out in the open ocean, where $h \sim 10\,\text{km}$. An undersea earthquake is the only possible source; the characteristic length of earthquake fault lines can be hundreds of kilometers. If we take $h = 10\,\text{km}$, we obtain $v = \sqrt{gh} \approx 310\,\text{m/s}$ or $1100\,\text{km/hr}$. At these speeds, a tsunami can cross the Pacific Ocean in less than a day.

As the wave approaches the shore, it must slow down, since $v = \sqrt{gh}$ is diminishing. But energy is conserved, which means that the amplitude must concomitantly rise. In extreme cases, the water level rise at shore may be 20 meters or more.

5.4 Appendix : More on Functionals

We remarked in section 5.2 that a function f is an animal which gets fed a real number x and excretes a real number $f(x)$. We say f maps the reals to the reals, or

$$f\colon \mathbf{R} \to \mathbf{R} \tag{5.57}$$

Of course we also have functions $g\colon \mathbf{C} \to \mathbf{C}$ which eat and excrete complex numbers, multivariable functions $h\colon \mathbf{R}^N \to \mathbf{R}$ which eat N-tuples of numbers and excrete a single number, *etc.*

A *functional* $F[f(x)]$ eats entire functions (!) and excretes numbers. That is,

$$F\colon \Big\{ f(x) \;\big|\; x \in \mathbf{R} \Big\} \to \mathbf{R} \tag{5.58}$$

This says that F operates on the set of real-valued functions of a single real variable, yielding a real number. Some examples:

$$F[f(x)] = \tfrac{1}{2}\int\limits_{-\infty}^{\infty} dx\, \big[f(x)\big]^2 \tag{5.59}$$

$$F[f(x)] = \tfrac{1}{2}\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dx'\, K(x,x')\, f(x)\, f(x') \tag{5.60}$$

$$F[f(x)] = \int\limits_{-\infty}^{\infty} dx \left\{ \tfrac{1}{2}A\, f^2(x) + \tfrac{1}{2}B\left(\frac{df}{dx}\right)^2 \right\} . \tag{5.61}$$

In classical mechanics, the action S is a functional of the path $q(t)$:

$$S[q(t)] = \int\limits_{t_{\rm a}}^{t_{\rm b}} dt \left\{ \tfrac{1}{2}m\dot{q}^2 - U(q) \right\} . \tag{5.62}$$

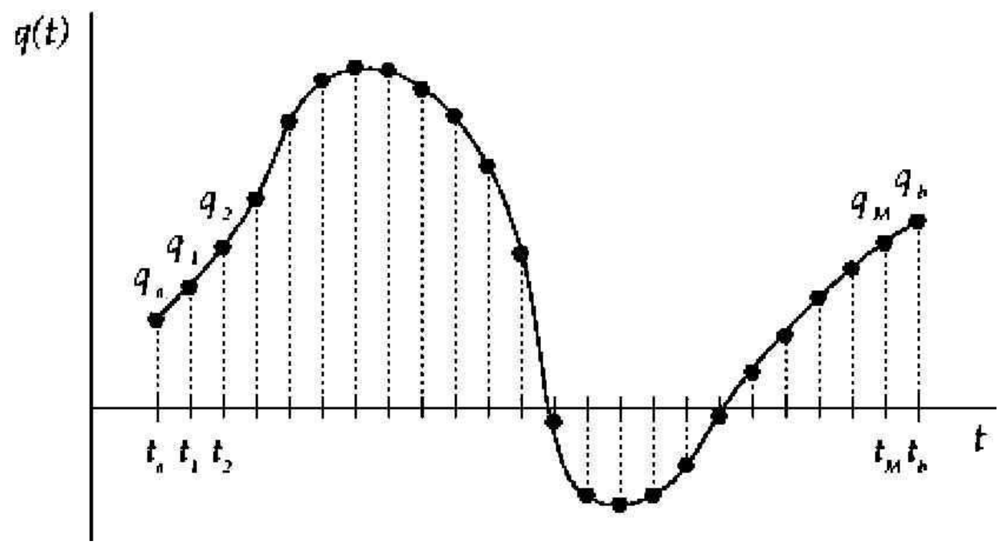


Figure 5.6: A functional $S[q(t)]$ is the continuum limit of a function of a large number of variables, $S(q_1, \ldots, q_M)$.

We can also have functionals which feed on functions of more than one independent variable, such as

$$S[y(x,t)] = \int_{t_a}^{t_b} \!dt \int_{x_a}^{x_b} \!dx \left\{ \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 \right\} , \tag{5.63}$$

which happens to be the functional for a string of mass density μ under uniform tension τ. Another example comes from electrodynamics:

$$S[A^\mu(\boldsymbol{x},t)] = -\int \! d^3\!x \int \! dt \left\{ \frac{1}{16\pi} F_{\mu\nu}\, F^{\mu\nu} + \frac{1}{c} j_\mu \, A^\mu \right\} , \tag{5.64}$$

which is a functional of the four fields $\{A^0, A^1, A^2, A^3\}$, where $A^0 = c\phi$. These are the components of the 4-potential, each of which is itself a function of four independent variables (x^0, x^1, x^2, x^3), with $x^0 = ct$. The field strength tensor is written in terms of derivatives of the A^μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where we use a metric $g_{\mu\nu} = \text{diag}(+,-,-,-)$ to raise and lower indices. The 4-potential couples linearly to the source term J_μ, which is the electric 4-current $(c\rho, \boldsymbol{J})$.

We extremize functions by sending the independent variable x to $x + dx$ and demanding that the variation $df = 0$ to first order in dx. That is,

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)(dx)^2 + \ldots \ , \tag{5.65}$$

whence $df = f'(x)\, dx + \mathcal{O}\big((dx)^2\big)$ and thus

$$f'(x^*) = 0 \quad \Longleftrightarrow \quad x^* \text{ an extremum.} \tag{5.66}$$

We extremize *functionals* by sending

$$f(x) \to f(x) + \delta f(x) \tag{5.67}$$

and demanding that the variation δF in the functional $F[f(x)]$ vanish to first order in $\delta f(x)$. The variation $\delta f(x)$ must sometimes satisfy certain boundary conditions. For example, if $F[f(x)]$ only operates on functions which vanish at a pair of endpoints, *i.e.* $f(x_a) = f(x_b) = 0$, then when we extremize the functional F we must do so *within the space of allowed functions.* Thus, we would in this case require $\delta f(x_a) = \delta f(x_b) = 0$. We may expand the functional $F[f + \delta f]$ in a *functional Taylor series*,

$$\begin{aligned} F[f+\delta f] = F[f] &+ \int\! dx_1\, K_1(x_1)\, \delta f(x_1) + \tfrac{1}{2!} \int\! dx_1 \int\! dx_2\, K_2(x_1,x_2)\, \delta f(x_1)\, \delta f(x_2) \\ &+ \tfrac{1}{3!} \int\! dx_1 \int\! dx_2 \int\! dx_3\, K_3(x_1,x_2,x_3)\, \delta f(x_1)\, \delta f(x_2)\, \delta f(x_3) + \ldots \end{aligned} \tag{5.68}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta f(x_1) \cdots \delta f(x_n)} \ . \tag{5.69}$$

In a more general case, $F = F[\{f_i(\boldsymbol{x})\}$ is a functional of several functions, each of which is a function of several independent variables.[1] We then write

$$\begin{aligned} F[\{f_i + \delta f_i\}] = F[\{f_i\}] &+ \int\! d\boldsymbol{x}_1\, K_1^i(\boldsymbol{x}_1)\, \delta f_i(\boldsymbol{x}_1) \\ &+ \tfrac{1}{2!} \int\! d\boldsymbol{x}_1 \int\! d\boldsymbol{x}_2\, K_2^{ij}(\boldsymbol{x}_1,\boldsymbol{x}_2)\, \delta f_i(\boldsymbol{x}_1)\, \delta f_j(\boldsymbol{x}_2) \\ &+ \tfrac{1}{3!} \int\! d\boldsymbol{x}_1 \int\! d\boldsymbol{x}_2 \int\! dx_3\ K_3^{ijk}(\boldsymbol{x}_1,\boldsymbol{x}_2,x_3)\, \delta f_i(\boldsymbol{x}_1)\, \delta f_j(\boldsymbol{x}_2)\, \delta f_k(\boldsymbol{x}_3) + \ldots\ , \end{aligned} \tag{5.70}$$

with

$$K_n^{i_1 i_2 \cdots i_n}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \frac{\delta^n F}{\delta f_{i_1}(\boldsymbol{x}_1)\, \delta f_{i_2}(\boldsymbol{x}_2)\, \delta f_{i_n}(\boldsymbol{x}_n)} \ . \tag{5.71}$$

Another way to compute functional derivatives is to send

$$f(x) \to f(x) + \epsilon_1\, \delta(x - x_1) + \ldots + \epsilon_n\, \delta(x - x_n) \tag{5.72}$$

and then differentiate n times with respect to ϵ_1 through ϵ_n. That is,

$$\frac{\delta^n F}{\delta f(x_1) \cdots \delta f(x_n)} = \frac{\partial^n}{\partial \epsilon_1 \cdots \partial \epsilon_n} \bigg|_{\epsilon_1 = \epsilon_2 = \cdots \epsilon_n = 0} F\big[f(x) + \epsilon_1\, \delta(x - x_1) + \ldots + \epsilon_n\, \delta(x - x_n)\big] \ . \tag{5.73}$$

Let's see how this works. As an example, we'll take the action functional from classical mechanics,

$$S[q(t)] = \int_{t_a}^{t_b} \! dt\, \Big\{ \tfrac{1}{2} m \dot{q}^2 - U(q) \Big\} \ . \tag{5.74}$$

[1] It may be also be that different functions depend on a different number of independent variables. *E.g.* $F = F[f(x),\, g(x,y),\, h(x,y,z)]$.

To compute the first functional derivative, we replace the function $q(t)$ with $q(t)+\epsilon\,\delta(t-t_1)$, and expand in powers of ϵ:

$$\begin{aligned} S\big[q(t)+\epsilon\delta(t-t_1)\big] &= S[q(t)] + \epsilon \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}} dt\, \Big\{ m\,\dot q\,\delta'(t-t_1) - U'(q)\,\delta(t-t_1)\Big\} \\ &= -\epsilon \Big\{ m\,\ddot q(t_1) + U'\big(q(t_1)\big)\Big\} \ , \end{aligned} \tag{5.75}$$

hence

$$\frac{\delta S}{\delta q(t)} = -\Big\{ m\,\ddot q(t) + U'\big(q(t)\big)\Big\} \tag{5.76}$$

and setting the first functional derivative to zero yields Newton's Second Law, $m\ddot q = -U'(q)$, for all $t\in[t_\mathrm{a},t_\mathrm{b}]$. Note that we have used the result

$$\int\limits_{-\infty}^{\infty} dt\,\delta'(t-t_1)\,h(t) = -h'(t_1) \ , \tag{5.77}$$

which is easily established upon integration by parts.

To compute the second functional derivative, we replace

$$q(t) \longrightarrow q(t) + \epsilon_1\,\delta(t-t_1) + \epsilon_2\,\delta(t-t_2) \tag{5.78}$$

and extract the term of order $\epsilon_1\,\epsilon_2$ in the double Taylor expansion. One finds this term to be

$$\epsilon_1\,\epsilon_2 \int\limits_{t_\mathrm{a}}^{t_\mathrm{b}} dt\, \Big\{ m\,\delta'(t-t_1)\,\delta'(t-t_2) - U''(q)\,\delta(t-t_1)\,\delta(t-t_2)\Big\} \ . \tag{5.79}$$

Note that we needn't bother with terms proportional to ϵ_1^2 or ϵ_2^2 since the recipe is to differentiate once with respect to each of ϵ_1 and ϵ_2 and then to set $\epsilon_1 = \epsilon_2 = 0$. This procedure uniquely selects the term proportional to $\epsilon_1\,\epsilon_2$, and yields

$$\frac{\delta^2 S}{\delta q(t_1)\,\delta q(t_2)} = -\Big\{ m\,\delta''(t_1-t_2) + U''\big(q(t_1)\big)\,\delta(t_1-t_2)\Big\} \ . \tag{5.80}$$

In multivariable calculus, the stability of an extremum is assessed by computing the matrix of second derivatives at the extremal point, known as the Hessian matrix. One has

$$\frac{\partial f}{\partial x_i}\bigg|_{\boldsymbol{x}^*} = 0 \quad \forall\ i \qquad ; \qquad H_{ij} = \frac{\partial^2 f}{\partial x_i\,\partial x_j}\bigg|_{\boldsymbol{x}^*} \ . \tag{5.81}$$

The eigenvalues of the Hessian H_{ij} determine the stability of the extremum. Since H_{ij} is a symmetric matrix, its eigenvectors η^α may be chosen to be orthogonal. The associated eigenvalues λ_α, defined by the equation

$$H_{ij}\,\eta^\alpha_j = \lambda_\alpha\,\eta^\alpha_i \ , \tag{5.82}$$

are the respective curvatures in the directions η^α, where $\alpha \in \{1, \ldots, n\}$ where n is the number of variables. The extremum is a local minimum if all the eigenvalues λ_α are positive, a maximum if all are negative, and otherwise is a saddle point. Near a saddle point, there are some directions in which the function increases and some in which it decreases.

In the case of functionals, the second functional derivative $K_2(x_1, x_2)$ defines an eigenvalue problem for $\delta f(x)$:

$$\int\limits_{x_a}^{x_b} dx_2\, K_2(x_1, x_2)\, \delta f(x_2) = \lambda\, \delta f(x_1) \ . \tag{5.83}$$

In general there are an infinite number of solutions to this equation which form a basis in function space, subject to appropriate boundary conditions at $x_{\rm a}$ and $x_{\rm b}$. For example, in the case of the action functional from classical mechanics, the above eigenvalue equation becomes a differential equation,

$$-\bigg\{ m\, \frac{d^2}{dt^2} + U''\big(q^*(t)\big) \bigg\}\, \delta q(t) = \lambda\, \delta q(t) \ , \tag{5.84}$$

where $q^*(t)$ is the solution to the Euler-Lagrange equations. As with the case of ordinary multivariable functions, the functional extremum is a local minimum (in function space) if every eigenvalue λ_α is positive, a local maximum if every eigenvalue is negative, and a saddle point otherwise.

Consider the simple harmonic oscillator, for which $U(q) = \frac{1}{2}\, m\omega_0^2\, q^2$. Then $U''\big(q^*(t)\big) = m\, \omega_0^2$; note that we don't even need to know the solution $q^*(t)$ to obtain the second functional derivative in this special case. The eigenvectors obey $m(\delta\ddot{q} + \omega_0^2\, \delta q) = -\lambda\, \delta q$, hence

$$\delta q(t) = A \cos\Big(\sqrt{\omega_0^2 + (\lambda/m)}\; t + \varphi \Big) \ , \tag{5.85}$$

where A and φ are constants. Demanding $\delta q(t_{\rm a}) = \delta q(t_{\rm b}) = 0$ requires

$$\sqrt{\omega_0^2 + (\lambda/m)}\; (t_{\rm b} - t_{\rm a}) = n\pi \ , \tag{5.86}$$

where n is an integer. Thus, the eigenfunctions are

$$\delta q_n(t) = A \sin\left(n\pi \cdot \frac{t - t_{\rm a}}{t_{\rm b} - t_{\rm a}} \right) \ , \tag{5.87}$$

and the eigenvalues are

$$\lambda_n = m \Big(\frac{n\pi}{T} \Big)^2 - m\omega_0^2 \ , \tag{5.88}$$

where $T = t_{\rm b} - t_{\rm a}$. Thus, so long as $T > \pi/\omega_0$, there is at least one negative eigenvalue. Indeed, for $\frac{n\pi}{\omega_0} < T < \frac{(n+1)\pi}{\omega_0}$ there will be n negative eigenvalues. This means the action is generally not a minimum, but rather lies at a *saddle point* in the (infinite-dimensional) function space.

To test this explicitly, consider a harmonic oscillator with the boundary conditions $q(0) = 0$ and $q(T) = Q$. The equations of motion, $\ddot{q} + \omega_0^2\, q = 0$, along with the boundary conditions, determine the motion,

$$q^*(t) = \frac{Q\,\sin(\omega_0 t)}{\sin(\omega_0 T)}\ . \tag{5.89}$$

The action for this path is then

$$\begin{aligned} S[q^*(t)] &= \int_0^T dt\,\Big\{ \tfrac{1}{2} m\,\dot{q}^{*2} - \tfrac{1}{2} m\omega_0^2\,q^{*2} \Big\} \\ &= \frac{m\,\omega_0^2\,Q^2}{2\sin^2\!\omega_0 T} \int_0^T dt\,\Big\{ \cos^2\!\omega_0 t - \sin^2\!\omega_0 t \Big\} \\ &= \tfrac{1}{2} m\omega_0\,Q^2\ \mathrm{ctn}\,(\omega_0 T)\ . \end{aligned} \tag{5.90}$$

Next consider the path $q(t) = Q\,t/T$ which satisfies the boundary conditions but does not satisfy the equations of motion (it proceeds with constant velocity). One finds the action for this path is

$$S[q(t)] = \tfrac{1}{2} m\omega_0\,Q^2 \left(\frac{1}{\omega_0 T} - \tfrac{1}{3}\omega_0 T \right)\ . \tag{5.91}$$

Thus, provided $\omega_0 T \neq n\pi$, in the limit $T \to \infty$ we find that the constant velocity path has lower action.

Finally, consider the general mechanical action,

$$S\big[q(t)\big] = \int_{t_a}^{t_b} dt\, L(q,\dot{q},t)\ . \tag{5.92}$$

We now evaluate the first few terms in the functional Taylor series:

$$\begin{aligned} S\big[q^*(t) + \delta q(t)\big] = \int_{t_a}^{t_b} dt\,\bigg\{ & L(q^*,\dot{q}^*,t) + \frac{\partial L}{\partial q_i}\bigg|_{q^*} \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{q^*} \delta \dot{q}_i \\ & + \frac{1}{2}\,\frac{\partial^2 L}{\partial q_i\,\partial q_j}\bigg|_{q^*} \delta q_i\,\delta q_j + \frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j}\bigg|_{q^*} \delta q_i\,\delta \dot{q}_j + \frac{1}{2}\,\frac{\partial^2 L}{\partial \dot{q}_i\,\partial \dot{q}_j}\bigg|_{q^*} \delta \dot{q}_i\,\delta \dot{q}_j + \ldots \bigg\}\ . \end{aligned} \tag{5.93}$$

To identify the functional derivatives, we integrate by parts. Let $\Phi_{\ldots}(t)$ be an arbitrary function of time. Then

$$\int_{t_a}^{t_b} dt\, \Phi_i(t)\, \delta\dot{q}_i(t) = -\int_{t_a}^{t_b} dt\, \dot{\Phi}_i(t)\, \delta q_i(t) \tag{5.94}$$

$$\begin{aligned}\int_{t_a}^{t_b} dt\, \Phi_{ij}(t)\, \delta q_i(t)\, \delta\dot{q}_j(t) &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t))\, \delta'(t-t')\, \delta q_i(t)\, \delta q_j(t')\end{aligned} \tag{5.95}$$

$$\begin{aligned}\int_{t_a}^{t_b} dt\, \Phi_{ij}(t)\, d\dot{q}_i(t)\, \delta\dot{q}_j(t) &= \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt'\, \Phi_{ij}(t)\, \delta(t-t')\, \frac{d}{dt}\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= -\int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \left[\dot{\Phi}_{ij}(t)\, \delta'(t-t') + \Phi_{ij}(t)\, \delta''(t-t') \right] \delta q_i(t)\, \delta q_j(t')\ .\end{aligned} \tag{5.96}$$

Thus,

$$\frac{\delta S}{\delta q_i(t)} = \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) \right]_{q^*(t)} \tag{5.97}$$

$$\begin{aligned}\frac{\delta^2 S}{\delta q_i(t)\, \delta q_j(t')} = \Bigg\{ &\frac{\partial^2 L}{\partial q_i\, \partial q_j}\bigg|_{q^*(t)} \delta(t-t') - \frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\bigg|_{q^*(t)} \delta''(t-t') \\ &+ \left[2\, \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j} \right) \right]_{q^*(t)} \delta'(t-t') \Bigg\}\ .\end{aligned} \tag{5.98}$$

Chapter 6

Lagrangian Mechanics

6.1 Generalized Coordinates

A set of *generalized coordinates* $q_1, \ldots, q_n$ completely describes the positions of all particles in a mechanical system. In a system with $d_{\rm f}$ degrees of freedom and k constraints, $n = d_{\rm f} - k$ independent generalized coordinates are needed to completely specify all the positions. A constraint is a relation among coordinates, such as $x^2 + y^2 + z^2 = a^2$ for a particle moving on a sphere of radius a. In this case, $d_{\rm f} = 3$ and $k = 1$. In this case, we could eliminate z in favor of x and y, *i.e.* by writing $z = \pm\sqrt{a^2 - x^2 - y^2}$, or we could choose as coordinates the polar and azimuthal angles θ and ϕ.

For the moment we will assume that $n = d_{\rm f} - k$, and that the generalized coordinates are independent, satisfying no additional constraints among them. Later on we will learn how to deal with any remaining constraints among the $\{q_1, \ldots, q_n\}$.

The generalized coordinates may have units of length, or angle, or perhaps something totally different. In the theory of small oscillations, the normal coordinates are conventionally chosen to have units of $(\text{mass})^{1/2} \times (\text{length})$. However, once a choice of generalized coordinate is made, with a concomitant set of units, the units of the conjugate momentum and force are determined:

$$\big[p_\sigma\big] = \frac{ML^2}{T} \cdot \frac{1}{\big[q_\sigma\big]} \qquad , \qquad \big[F_\sigma\big] = \frac{ML^2}{T^2} \cdot \frac{1}{\big[q_\sigma\big]} \ , \tag{6.1}$$

where $\big[A\big]$ means 'the units of A', and where M, L, and T stand for mass, length, and time, respectively. Thus, if q_σ has dimensions of length, then p_σ has dimensions of momentum and F_σ has dimensions of force. If q_σ is dimensionless, as is the case for an angle, p_σ has dimensions of angular momentum (ML^2/T) and F_σ has dimensions of torque (ML^2/T^2).

6.2 Hamilton's Principle

The equations of motion of classical mechanics are embodied in a variational principle, called *Hamilton's principle.* Hamilton's principle states that the motion of a system is such that the *action functional*

$$S\big[q(t)\big] = \int\limits_{t_1}^{t_2} dt\, L(q, \dot{q}, t) \tag{6.2}$$

is an extremum, *i.e.* $\delta S = 0$. Here, $q = \{q_1, \ldots, q_n\}$ is a complete set of *generalized coordinates* for our mechanical system, and

$$L = T - U \tag{6.3}$$

is the *Lagrangian*, where T is the kinetic energy and U is the potential energy. Setting the first variation of the action to zero gives the Euler-Lagrange equations,

$$\frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{momentum } p_\sigma} = \overbrace{\frac{\partial L}{\partial q_\sigma}}^{\text{force } F_\sigma} \quad . \tag{6.4}$$

Thus, we have the familiar $\dot{p}_\sigma = F_\sigma$, also known as Newton's second law. Note, however, that the $\{q_\sigma\}$ are *generalized coordinates*, so p_σ may not have dimensions of momentum, nor F_σ of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

6.2.1 Invariance of the equations of motion

Suppose

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}\, G(q, t) \ . \tag{6.5}$$

Then

$$\tilde{S}[q(t)] = S[q(t)] + G(q_b, t_b) - G(q_a, t_a) \ . \tag{6.6}$$

Since the difference $\tilde{S} - S$ is a function only of the endpoint values $\{q_a, q_b\}$, their variations are identical: $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Thus, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time.

6.2.2 Remarks on the order of the equations of motion

The equations of motion are second order in time. This follows from the fact that $L = L(q, \dot{q}, t)$. Using the chain rule,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial \dot{q}_{\sigma'}}\, \ddot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial q_{\sigma'}}\, \dot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial t} \ . \tag{6.7}$$

That the equations are second order in time can be regarded as an empirical fact. It follows, as we have just seen, from the fact that L depends on q and on $\dot{q}$, but on no higher time derivative terms. Suppose the Lagrangian did depend on the generalized accelerations $\ddot{q}$ as well. What would the equations of motion look like?

Taking the variation of S,

$$\delta\int_{t_a}^{t_b} dt\, L(q,\dot{q},\ddot{q},t) = \left[\frac{\partial L}{\partial \dot{q}_\sigma}\,\delta q_\sigma + \frac{\partial L}{\partial \ddot{q}_\sigma}\,\delta\dot{q}_\sigma - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\delta q_\sigma\right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_\sigma}\right)\right\}\delta q_\sigma \ . \tag{6.8}$$

The boundary term vanishes if we require $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = \delta\dot{q}_\sigma(t_a) = \delta\dot{q}_\sigma(t_b) = 0 \ \forall\ \sigma$. The equations of motion would then be *fourth order* in time.

6.2.3 Lagrangian for a free particle

For a free particle, we can use Cartesian coordinates for each particle as our system of generalized coordinates. For a single particle, the Lagrangian $L(\boldsymbol{x},\boldsymbol{v},t)$ must be a function solely of $\boldsymbol{v}^2$. This is because homogeneity with respect to space and time preclude any dependence of L on $\boldsymbol{x}$ or on t, and isotropy of space means L must depend on $\boldsymbol{v}^2$. We next invoke Galilean relativity, which says that the equations of motion are invariant under transformation to a reference frame moving with constant velocity. Let $\boldsymbol{V}$ be the velocity of the new reference frame $\mathcal{K}'$ relative to our initial reference frame $\mathcal{K}$. Then $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V}t$, and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$. In order that the equations of motion be invariant under the change in reference frame, we demand

$$L'(\boldsymbol{v}') = L(\boldsymbol{v}) + \frac{d}{dt}\,G(\boldsymbol{x},t) \ . \tag{6.9}$$

The only possibility is $L = \frac{1}{2}m\boldsymbol{v}^2$, where the constant m is the mass of the particle. Note:

$$L' = \tfrac{1}{2}m(\boldsymbol{v}-\boldsymbol{V})^2 = \tfrac{1}{2}m\boldsymbol{v}^2 + \frac{d}{dt}\Big(\tfrac{1}{2}m\boldsymbol{V}^2\,t - m\boldsymbol{V}\cdot\boldsymbol{x}\Big) = L + \frac{dG}{dt} \ . \tag{6.10}$$

For N interacting particles,

$$L = \tfrac{1}{2}\sum_{a=1}^{N} m_a \left(\frac{d\boldsymbol{x}_a}{dt}\right)^2 - U\big(\{\boldsymbol{x}_a\},\{\dot{\boldsymbol{x}}_a\}\big) \ . \tag{6.11}$$

Here, U is the *potential energy*. Generally, U is of the form

$$U = \sum_a U_1(\boldsymbol{x}_a) + \sum_{a<a'} v(\boldsymbol{x}_a - \boldsymbol{x}_{a'}) \ , \tag{6.12}$$

however, as we shall see, velocity-dependent potentials appear in the case of charged particles interacting with electromagnetic fields. In general, though,

$$L = T - U \ , \tag{6.13}$$

where T is the kinetic energy, and U is the potential energy.

6.3 Conserved Quantities

A conserved quantity $\Lambda(q, \dot{q}, t)$ is one which does not vary throughout the motion of the system. This means

$$\frac{d\Lambda}{dt}\bigg|_{q=q(t)} = 0 \ . \tag{6.14}$$

We shall discuss conserved quantities in detail in the chapter on Noether's Theorem, which follows.

6.3.1 Momentum conservation

The simplest case of a conserved quantity occurs when the Lagrangian does not explicitly depend on one or more of the generalized coordinates, *i.e.* when

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = 0 \ . \tag{6.15}$$

We then say that L is *cyclic* in the coordinate q_σ. In this case, the Euler-Lagrange equations $\dot{p}_\sigma = F_\sigma$ say that the conjugate momentum p_σ is conserved. Consider, for example, the motion of a particle of mass m near the surface of the earth. Let (x, y) be coordinates parallel to the surface and z the height. We then have

$$T = \tfrac{1}{2} m \big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) \tag{6.16}$$

$$U = mgz \tag{6.17}$$

$$L = T - U = \tfrac{1}{2} m \big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) - mgz \ . \tag{6.18}$$

Since

$$F_x = \frac{\partial L}{\partial x} = 0 \quad \text{and} \quad F_y = \frac{\partial L}{\partial y} = 0 \ , \tag{6.19}$$

we have that p_x and p_y are conserved, with

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \ . \tag{6.20}$$

These first order equations can be integrated to yield

$$x(t) = x(0) + \frac{p_x}{m}\, t \quad , \quad y(t) = y(0) + \frac{p_y}{m}\, t \ . \tag{6.21}$$

The z equation is of course

$$\dot{p}_z = m\ddot{z} = -mg = F_z \ , \tag{6.22}$$

with solution

$$z(t) = z(0) + \dot{z}(0)\, t - \tfrac{1}{2} g t^2 \ . \tag{6.23}$$

As another example, consider a particle moving in the (x, y) plane under the influence of a potential $U(x, y) = U\big(\sqrt{x^2 + y^2}\big)$ which depends only on the particle's distance from the origin $\rho = \sqrt{x^2 + y^2}$. The Lagrangian, expressed in two-dimensional polar coordinates (ρ, ϕ), is

$$L = \tfrac{1}{2} m\big(\dot{\rho}^2 + \rho^2 \dot{\phi}^2\big) - U(\rho) \ . \tag{6.24}$$

We see that L is cyclic in the angle ϕ, hence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} \tag{6.25}$$

is conserved. p_ϕ is the angular momentum of the particle about the $\hat{\boldsymbol{z}}$ axis. In the language of the calculus of variations, momentum conservation is what follows when the integrand of a functional is independent of the *independent variable*.

6.3.2 Energy conservation

When the integrand of a functional is independent of the *dependent* variable, another conservation law follows. For Lagrangian mechanics, consider the expression

$$H(q, \dot{q}, t) = \sum_{\sigma=1}^{n} p_\sigma \, \dot{q}_\sigma - L \ . \tag{6.26}$$

Now we take the total time derivative of H:

$$\frac{dH}{dt} = \sum_{\sigma=1}^{n} \bigg\{ p_\sigma \, \ddot{q}_\sigma + \dot{p}_\sigma \, \dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma} \, \dot{q}_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} \, \ddot{q}_\sigma \bigg\} - \frac{\partial L}{\partial t} \ . \tag{6.27}$$

We evaluate $\dot{H}$ along the motion of the system, which entails that the terms in the curly brackets above cancel for each σ:

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad \dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma} \ . \tag{6.28}$$

Thus, we find

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{6.29}$$

which means that H is conserved *whenever the Lagrangian contains no explicit time dependence*. For a Lagrangian of the form

$$L = \sum_a \tfrac{1}{2} m_a \dot{\boldsymbol{r}}_a^2 - U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \ , \tag{6.30}$$

we have that $\boldsymbol{p}_a = m_a \, \dot{\boldsymbol{r}}_a$, and

$$H = T + U = \sum_a \tfrac{1}{2} \, m_a \dot{\boldsymbol{r}}_a^2 + U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \ . \tag{6.31}$$

However, it is not always the case that $H = T + U$ is the total energy, as we shall see in the next chapter.

6.4 Choosing Generalized Coordinates

Any choice of generalized coordinates will yield an equivalent set of equations of motion. However, some choices result in an apparently simpler set than others. This is often true with respect to the form of the potential energy. Additionally, certain constraints that may be present are more amenable to treatment using a particular set of generalized coordinates.

The kinetic energy T is always simple to write in Cartesian coordinates, and it is good practice, at least when one is first learning the method, to write T in Cartesian coordinates and then convert to generalized coordinates. In Cartesian coordinates, the kinetic energy of a single particle of mass m is

$$T = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2 \right) \ . \tag{6.32}$$

If the motion is two-dimensional, and confined to the plane $z =$ const., one of course has $T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$.

Two other commonly used coordinate systems are the cylindrical and spherical systems. In cylindrical coordinates (ρ, ϕ, z), ρ is the radial coordinate in the (x, y) plane and ϕ is the azimuthal angle:

$$x = \rho \, \cos\phi \qquad\qquad \dot{x} = \cos\phi \, \dot{\rho} - \rho \sin\phi \, \dot{\phi} \tag{6.33}$$

$$y = \rho \, \sin\phi \qquad\qquad \dot{y} = \sin\phi \, \dot{\rho} + \rho \cos\phi \, \dot{\phi} \ , \tag{6.34}$$

and the third, orthogonal coordinate is of course z. The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2 \right) \\ &= \tfrac{1}{2} m \left(\dot{\rho}^2 + \rho^2 \, \dot{\phi}^2 + \dot{z}^2 \right) \ . \end{aligned} \tag{6.35}$$

When the motion is confined to a plane with $z =$ const., this coordinate system is often referred to as 'two-dimensional polar' coordinates.

In spherical coordinates (r, θ, ϕ), r is the radius, θ is the polar angle, and ϕ is the azimuthal angle. On the globe, θ would be the 'colatitude', which is $\theta = \frac{\pi}{2} - \lambda$, where λ is the latitude. *I.e.* $\theta = 0$ at the north pole. In spherical polar coordinates,

$$x = r \, \sin\theta \, \cos\phi \qquad \dot{x} = \sin\theta \, \cos\phi \, \dot{r} + r \, \cos\theta \, \cos\phi \, \dot{\theta} - r \, \sin\theta \sin\phi \, \dot{\phi} \tag{6.36}$$

$$y = r \, \sin\theta \, \sin\phi \qquad \dot{y} = \sin\theta \, \sin\phi \, \dot{r} + r \, \cos\theta \, \sin\phi \, \dot{\theta} + r \, \sin\theta \cos\phi \, \dot{\phi} \tag{6.37}$$

$$z = r \, \cos\theta \qquad \dot{z} = \cos\theta \, \dot{r} - r \, \sin\theta \, \dot{\theta} \ . \tag{6.38}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \\ &= \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\!\theta\,\dot{\phi}^2\right) \ . \end{aligned} \tag{6.39}$$

6.5 How to Solve Mechanics Problems

Here are some simple steps you can follow toward obtaining the equations of motion:

1. Choose a set of generalized coordinates $\{q_1, \ldots, q_n\}$.
2. Find the kinetic energy $T(q, \dot{q}, t)$, the potential energy $U(q, t)$, and the Lagrangian $L(q, \dot{q}, t) = T - U$. It is often helpful to first write the kinetic energy in Cartesian coordinates for each particle before converting to generalized coordinates.
3. Find the canonical momenta $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$ and the generalized forces $F_\sigma = \frac{\partial L}{\partial q_\sigma}$.
4. Evaluate the time derivatives $\dot{p}_\sigma$ and write the equations of motion $\dot{p}_\sigma = F_\sigma$. Be careful to differentiate properly, using the chain rule and the Leibniz rule where appropriate.
5. Identify any conserved quantities (more about this later).

6.6 Examples

6.6.1 One-dimensional motion

For a one-dimensional mechanical system with potential energy $U(x)$,

$$L = T - U = \tfrac{1}{2}m\dot{x}^2 - U(x) \ . \tag{6.40}$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{6.41}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m\ddot{x} = -U'(x) \ , \tag{6.42}$$

which is of course $F = ma$.

Note that we can multiply the equation of motion by $\dot{x}$ to get

$$0 = \dot{x}\,\Big\{m\ddot{x} + U'(x)\Big\} = \frac{d}{dt}\Big\{\tfrac{1}{2}m\dot{x}^2 + U(x)\Big\} = \frac{dE}{dt} \ , \tag{6.43}$$

where $E = T + U$.

6.6.2 Central force in two dimensions

Consider next a particle of mass m moving in two dimensions under the influence of a potential $U(\rho)$ which is a function of the distance from the origin $\rho = \sqrt{x^2+y^2}$. Clearly cylindrical ($2d$ polar) coordinates are called for:

$$L = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) - U(\rho)\ . \tag{6.44}$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = \frac{\partial L}{\partial \rho} \quad\Rightarrow\quad m\ddot{\rho} = m\rho\,\dot{\phi}^2 - U'(\rho) \tag{6.45}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad\Rightarrow\quad \frac{d}{dt}\left(m\rho^2\,\dot{\phi}\right) = 0\ . \tag{6.46}$$

Note that the canonical momentum conjugate to ϕ, which is to say the angular momentum, is conserved:

$$p_\phi = m\rho^2\,\dot{\phi} = \text{const.} \tag{6.47}$$

We can use this to eliminate $\dot{\phi}$ from the first Euler-Lagrange equation, obtaining

$$m\ddot{\rho} = \frac{p_\phi^2}{m\rho^3} - U'(\rho)\ . \tag{6.48}$$

We can also write the total energy as

$$\begin{aligned} E &= \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) + U(\rho) \\ &= \tfrac{1}{2}m\,\dot{\rho}^2 + \frac{p_\phi^2}{2m\rho^2} + U(\rho)\ , \end{aligned} \tag{6.49}$$

from which it may be shown that E is also a constant:

$$\frac{dE}{dt} = \left(m\,\ddot{\rho} - \frac{p_\phi^2}{m\rho^3} + U'(\rho)\right)\dot{\rho} = 0\ . \tag{6.50}$$

We shall discuss this case in much greater detail in the coming weeks.

6.6.3 A sliding point mass on a sliding wedge

Consider the situation depicted in Fig. 6.1, in which a point object of mass m slides frictionlessly along a wedge of opening angle α. The wedge itself slides frictionlessly along a horizontal surface, and its mass is M. We choose as generalized coordinates the horizontal position X of the left corner of the wedge, and the horizontal distance x from the left corner to the sliding point mass. The vertical coordinate of the sliding mass is then $y = x\,\tan\alpha$,

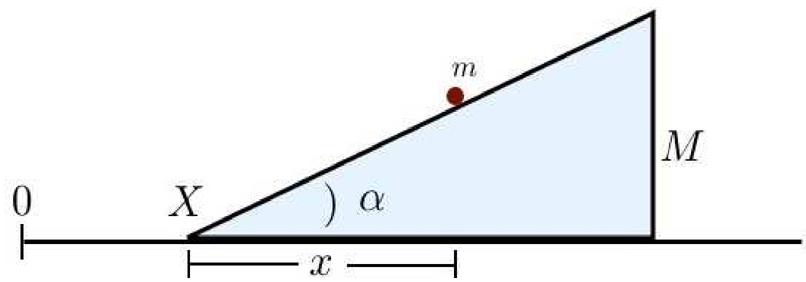


Figure 6.1: A wedge of mass M and opening angle α slides frictionlessly along a horizontal surface, while a small object of mass m slides frictionlessly along the wedge.

where the horizontal surface lies at $y = 0$. With these generalized coordinates, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2} M\dot{X}^2 + \tfrac{1}{2} m\,(\dot{X} + \dot{x})^2 + \tfrac{1}{2} m\dot{y}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2} m\,(1+\tan^2\!\alpha)\,\dot{x}^2 \ . \end{aligned} \tag{6.51}$$

The potential energy is simply

$$U = mgy = mg\,x\,\tan\alpha \ . \tag{6.52}$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2} m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha \ , \tag{6.53}$$

and the equations of motion are

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{X}}\bigg) = \frac{\partial L}{\partial X} \quad &\Rightarrow \quad (M+m)\ddot{X} + m\,\ddot{x} = 0 \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{x}}\bigg) = \frac{\partial L}{\partial x} \quad &\Rightarrow \quad m\ddot{X} + m\,(1+\tan^2\!\alpha)\,\ddot{x} = -mg\tan\alpha \ . \end{aligned} \tag{6.54}$$

At this point we can use the first of these equations to write

$$\ddot{X} = -\frac{m}{M+m}\,\ddot{x} \ . \tag{6.55}$$

Substituting this into the second equation, we obtain the constant accelerations

$$\ddot{x} = -\frac{(M+m)g\,\sin\alpha\,\cos\alpha}{M + m\sin^2\!\alpha} \quad , \quad \ddot{X} = \frac{mg\,\sin\alpha\,\cos\alpha}{M + m\sin^2\!\alpha} \ . \tag{6.56}$$

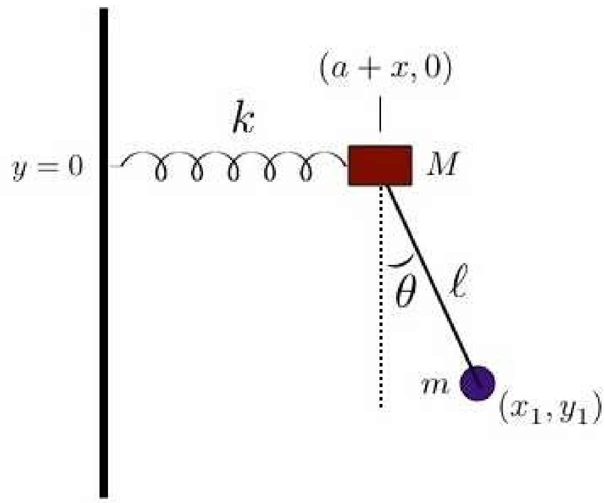


Figure 6.2: The spring–pendulum system.

6.6.4 A pendulum attached to a mass on a spring

Consider next the system depicted in Fig. 6.2 in which a mass M moves horizontally while attached to a spring of spring constant k. Hanging from this mass is a pendulum of arm length ℓ and bob mass m.

A convenient set of generalized coordinates is (x,θ), where x is the displacement of the mass M relative to the equilibrium extension a of the spring, and θ is the angle the pendulum arm makes with respect to the vertical. Let the Cartesian coordinates of the pendulum bob be (x_1,y_1). Then

$$x_1 = a + x + \ell\,\sin\theta \quad , \quad y_1 = -l\cos\theta\ . \tag{6.57}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\,(\dot{x}^2+\dot{y}^2) \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\Big[(\dot{x}+\ell\cos\theta\,\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2\Big] \\ &= \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta}\ , \end{aligned} \tag{6.58}$$

and the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kx^2 + mgy_1 \\ &= \tfrac{1}{2}kx^2 - mg\ell\cos\theta\ . \end{aligned} \tag{6.59}$$

Thus,

$$L = \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} - \tfrac{1}{2}kx^2 + mg\ell\cos\theta\ . \tag{6.60}$$

The canonical momenta are

$$p_x = \frac{\partial L}{\partial \dot{x}} = (M+m)\,\dot{x} + m\ell\cos\theta\,\dot{\theta}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\ell\cos\theta\,\dot{x} + m\ell^2\,\dot{\theta}\ , \tag{6.61}$$

and the canonical forces are

$$F_x = \frac{\partial L}{\partial x} = -kx$$

$$F_\theta = \frac{\partial L}{\partial \theta} = -m\ell\sin\theta\,\dot{x}\,\dot{\theta} - mg\ell\,\sin\theta\ . \tag{6.62}$$

The equations of motion then yield

$$(M+m)\,\ddot{x} + m\ell\,\cos\theta\,\ddot{\theta} - m\ell\,\sin\theta\,\dot{\theta}^2 = -kx \tag{6.63}$$

$$m\ell\,\cos\theta\,\ddot{x} + m\ell^2\,\ddot{\theta} = -mg\ell\,\sin\theta\ . \tag{6.64}$$

Small Oscillations : If we assume both x and θ are small, we may write $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, in which case the equations of motion may be linearized to

$$(M+m)\,\ddot{x} + m\ell\,\ddot{\theta} + kx = 0 \tag{6.65}$$

$$m\ell\,\ddot{x} + m\ell^2\,\ddot{\theta} + mg\ell\,\theta = 0\ . \tag{6.66}$$

If we define

$$u \equiv \frac{x}{\ell} \quad , \quad \alpha \equiv \frac{m}{M} \quad , \quad \omega_0^2 \equiv \frac{k}{M} \quad , \quad \omega_1^2 \equiv \frac{g}{\ell}\ , \tag{6.67}$$

then

$$(1+\alpha)\,\ddot{u} + \alpha\,\ddot{\theta} + \omega_0^2\,u = 0 \tag{6.68}$$

$$\ddot{u} + \ddot{\theta} + \omega_1^2\,\theta = 0\ . \tag{6.69}$$

We can solve by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-i\omega t}\ , \tag{6.70}$$

in which case

$$\begin{pmatrix} \omega_0^2 - (1+\alpha)\,\omega^2 & -\alpha\,\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ . \tag{6.71}$$

In order to have a nontrivial solution (*i.e.* without $a = b = 0$), the determinant of the above 2×2 matrix must vanish. This gives a condition on ω^2, with solutions

$$\omega_\pm^2 = \tfrac{1}{2}\big(\omega_0^2 + (1+\alpha)\,\omega_1^2\big) \pm \tfrac{1}{2}\sqrt{\big(\omega_0^2 - \omega_1^2\big)^2 + 2\alpha\,(\omega_0^2 + \omega_1^2)\,\omega_1^2}\ . \tag{6.72}$$

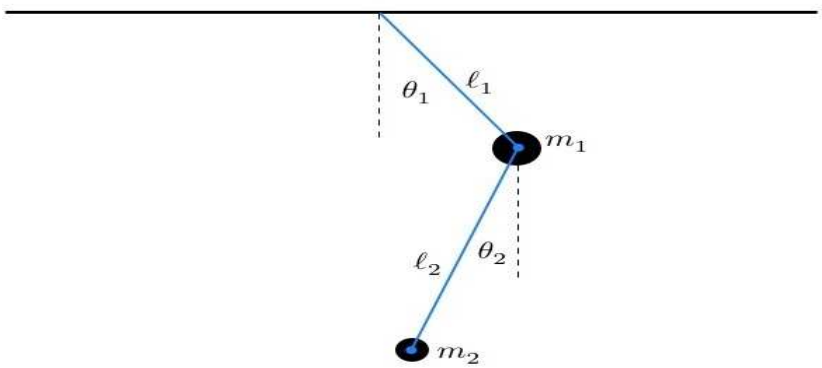


Figure 6.3: The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

6.6.5 The double pendulum

As yet another example of the generalized coordinate approach to Lagrangian dynamics, consider the double pendulum system, sketched in Fig. 6.3. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of the generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\dot\theta_1, \dot\theta_2\}$.

In Cartesian coordinates,

$$T = \tfrac{1}{2} m_1 \, (\dot x_1^2 + \dot y_1^2) + \tfrac{1}{2} m_2 \, (\dot x_2^2 + \dot y_2^2) \tag{6.73}$$

$$U = m_1 \, g \, y_1 + m_2 \, g \, y_2 \ . \tag{6.74}$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of the generalized coordinates $\{\theta_1, \theta_2\}$:

$$x_1 = \ell_1 \, \sin\theta_1 \qquad\qquad x_2 = \ell_1 \, \sin\theta_1 + \ell_2 \, \sin\theta_2 \tag{6.75}$$

$$y_1 = -\ell_1 \, \cos\theta_1 \qquad\qquad y_2 = -\ell_1 \, \cos\theta_1 - \ell_2 \, \cos\theta_2 \ . \tag{6.76}$$

Thus, the velocities are

$$\dot x_1 = \ell_1 \, \dot\theta_1 \, \cos\theta_1 \qquad\qquad \dot x_2 = \ell_1 \, \dot\theta_1 \, \cos\theta_1 + \ell_2 \, \dot\theta_2 \, \cos\theta_2 \tag{6.77}$$

$$\dot y_1 = \ell_1 \, \dot\theta_1 \, \sin\theta_1 \qquad\qquad \dot y_2 = \ell_1 \, \dot\theta_1 \, \sin\theta_1 + \ell_2 \, \dot\theta_2 \, \sin\theta_2 \ . \tag{6.78}$$

Thus,

$$T = \tfrac{1}{2} m_1 \, \ell_1^2 \, \dot\theta_1^2 + \tfrac{1}{2} m_2 \Big\{ \ell_1^2 \, \dot\theta_1^2 + 2\ell_1 \, \ell_2 \, \cos(\theta_1 - \theta_2) \, \dot\theta_1 \, \dot\theta_2 + \ell_2^2 \, \dot\theta_2^2 \Big\} \tag{6.79}$$

$$U = -m_1 \, g \, \ell_1 \, \cos\theta_1 - m_2 \, g \, \ell_1 \, \cos\theta_1 - m_2 \, g \, \ell_2 \, \cos\theta_2 \ , \tag{6.80}$$

and

$$L = T - U = \tfrac{1}{2}(m_1 + m_2)\,\ell_1^2\,\dot\theta_1^2 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 + \tfrac{1}{2} m_2\,\ell_2^2\,\dot\theta_2^2$$
$$+ (m_1 + m_2)\,g\,\ell_1\,\cos\theta_1 + m_2\,g\,\ell_2\,\cos\theta_2\ . \tag{6.81}$$

The generalized (canonical) momenta are

$$p_1 = \frac{\partial L}{\partial\dot\theta_1} = (m_1 + m_2)\,\ell_1^2\,\dot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_2 \tag{6.82}$$

$$p_2 = \frac{\partial L}{\partial\dot\theta_2} = m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\dot\theta_2\ , \tag{6.83}$$

and the equations of motion are

$$\dot p_1 = (m_1 + m_2)\,\ell_1^2\,\ddot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_2 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_2$$
$$= -(m_1 + m_2)\,g\,\ell_1\,\sin\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_1} \tag{6.84}$$

and

$$\dot p_2 = m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\ddot\theta_2$$
$$= -m_2\,g\,\ell_2\,\sin\theta_2 + m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_2}\ . \tag{6.85}$$

We therefore find

$$\ell_1\,\ddot\theta_1 + \frac{m_2\,\ell_2}{m_1 + m_2}\,\cos(\theta_1 - \theta_2)\,\ddot\theta_2 + \frac{m_2\,\ell_2}{m_1 + m_2}\,\sin(\theta_1 - \theta_2)\,\dot\theta_2^2 + g\,\sin\theta_1 = 0 \tag{6.86}$$

$$\ell_1\,\cos(\theta_1 - \theta_2)\,\ddot\theta_1 + \ell_2\,\ddot\theta_2 - \ell_1\,\sin(\theta_1 - \theta_2)\,\dot\theta_1^2 + g\,\sin\theta_2 = 0\ . \tag{6.87}$$

Small Oscillations : The equations of motion are coupled, nonlinear second order ODEs. When the system is close to equilibrium, the amplitudes of the motion are small, and we may expand in powers of the θ_1 and θ_2. The linearized equations of motion are then

$$\ddot\theta_1 + \alpha\beta\,\ddot\theta_2 + \omega_0^2\,\theta_1 = 0 \tag{6.88}$$

$$\ddot\theta_1 + \beta\,\ddot\theta_2 + \omega_0^2\,\theta_2 = 0\ , \tag{6.89}$$

where we have defined

$$\alpha \equiv \frac{m_2}{m_1 + m_2} \quad , \quad \beta \equiv \frac{\ell_2}{\ell_1} \quad , \quad \omega_0^2 \equiv \frac{g}{\ell_1}\ . \tag{6.90}$$

We can solve this coupled set of equations by a nifty trick. Let's take a linear combination of the first equation plus an undetermined coefficient, r, times the second:

$$(1+r)\,\ddot{\theta}_1 + (\alpha+r)\beta\,\ddot{\theta}_2 + \omega_0^2\,(\theta_1 + r\,\theta_2) = 0\ . \qquad (6.91)$$

We now demand that the ratio of the coefficients of θ_2 and θ_1 is the same as the ratio of the coefficients of $\ddot{\theta}_2$ and $\ddot{\theta}_1$:

$$\frac{(\alpha+r)\,\beta}{1+r} = r \qquad \Rightarrow \qquad r_\pm = \tfrac{1}{2}(\beta-1) \pm \tfrac{1}{2}\sqrt{(1-\beta)^2 + 4\alpha\beta} \qquad (6.92)$$

When $r = r_\pm$, the equation of motion may be written

$$\frac{d^2}{dt^2}\,\big(\theta_1 + r_\pm\,\theta_2\big) = -\frac{\omega_0^2}{1+r_\pm}\,\big(\theta_1 + r_\pm\,\theta_2\big) \qquad (6.93)$$

and defining the (unnormalized) *normal modes*

$$\xi_\pm \equiv \big(\theta_1 + r_\pm\,\theta_2\big)\ , \qquad (6.94)$$

we find

$$\ddot{\xi}_\pm + \omega_\pm^2\,\xi_\pm = 0\ , \qquad (6.95)$$

with

$$\omega_\pm = \frac{\omega_0}{\sqrt{1+r_\pm}}\ . \qquad (6.96)$$

Thus, by switching to the normal coordinates, we decoupled the equations of motion, and identified the two *normal frequencies of oscillation*. We shall have much more to say about small oscillations further below.

For example, with $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$, we have $\alpha = \frac{1}{2}$, and $\beta = 1$, in which case

$$r_\pm = \pm\tfrac{1}{\sqrt{2}} \quad , \quad \xi_\pm = \theta_1 \pm \tfrac{1}{\sqrt{2}}\,\theta_2 \quad , \quad \omega_\pm = \sqrt{2 \mp \sqrt{2}}\,\sqrt{\frac{g}{\ell}}\ . \qquad (6.97)$$

Note that the oscillation frequency for the 'in-phase' mode ξ_+ is low, and that for the 'out of phase' mode ξ_- is high.

6.6.6 The thingy

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached to each vertex. The opposite corners are joined by springs of spring constant k. In the square configuration, the strings are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about equilibrium.

Solution

The rhombus is depicted in figure 6.4. Let a be the equilibrium length of the springs; clearly $L = \frac{a}{\sqrt{2}}$. Let ϕ be half of one of the opening angles, as shown. Then the masses are located at $(\pm X, 0)$ and $(0, \pm Y)$, with $X = \frac{a}{\sqrt{2}} \cos\phi$ and $Y = \frac{a}{\sqrt{2}} \sin\phi$. The spring extensions are $\delta X = 2X - a$ and $\delta Y = 2Y - a$. The kinetic and potential energies are therefore

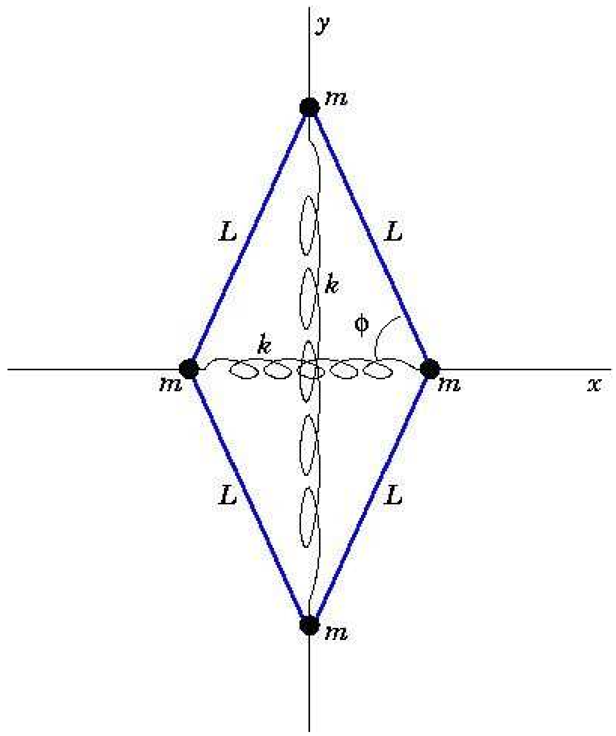


Figure 6.4: The thingy: a rhombus with opening angles 2ϕ and $\pi - 2\phi$.

$$\begin{aligned} T &= M\big(\dot{X}^2 + \dot{Y}^2\big) \\ &= \tfrac{1}{2} M a^2\, \dot{\phi}^2 \end{aligned}$$

and

$$\begin{aligned} U &= \tfrac{1}{2} k \big(\delta X\big)^2 + \tfrac{1}{2} k \big(\delta Y\big)^2 \\ &= \tfrac{1}{2} k a^2 \left\{ \big(\sqrt{2}\, \cos\phi - 1\big)^2 + \big(\sqrt{2}\, \sin\phi - 1\big)^2 \right\} \\ &= \tfrac{1}{2} k a^2 \left\{ 3 - 2\sqrt{2}(\cos\phi + \sin\phi) \right\} \ . \end{aligned}$$

Note that minimizing $U(\phi)$ gives $\sin\phi = \cos\phi$, *i.e.* $\phi_{\rm eq} = \frac{\pi}{4}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} M a^2\, \dot{\phi}^2 + \sqrt{2}\, k a^2 \big(\cos\phi + \sin\phi\big) + \text{const.}$$

The equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \qquad \Rightarrow \qquad Ma^2\,\ddot{\phi} = \sqrt{2}\,ka^2\,(\cos\phi - \sin\phi)$$

It's always smart to expand about equilibrium, so let's write $\phi = \frac{\pi}{4} + \delta$, which leads to

$$\ddot{\delta} + \omega_0^2\,\sin\delta = 0\ ,$$

with $\omega_0 = \sqrt{2k/M}$. This is the equation of a pendulum! Linearizing gives $\ddot{\delta} + \omega_0^2\,\delta = 0$, so the small oscillation frequency is just ω_0.

6.7 Appendix : Virial Theorem

The virial theorem is a statement about the time-averaged motion of a mechanical system. Define the *virial*,

$$G(q,p) = \sum_\sigma p_\sigma\, q_\sigma\ . \tag{6.98}$$

Then

$$\begin{aligned}\frac{dG}{dt} &= \sum_\sigma \big(\dot{p}_\sigma\, q_\sigma + p_\sigma\, \dot{q}_\sigma\big)\\ &= \sum_\sigma q_\sigma\, F_\sigma + \sum_\sigma \dot{q}_\sigma\, \frac{\partial L}{\partial \dot{q}_\sigma}\ .\end{aligned} \tag{6.99}$$

Now suppose that $T = \frac{1}{2}\sum_{\sigma,\sigma'} \mathrm{T}_{\sigma\sigma'}\,\dot{q}_\sigma \dot{q}_{\sigma'}$ is homogeneous of degree $k=2$ in $\dot{q}$, and that U is homogeneous of degree zero in $\dot{q}$. Then

$$\sum_\sigma \dot{q}_\sigma\, \frac{\partial L}{\partial \dot{q}_\sigma} = \sum_\sigma \dot{q}_\sigma\, \frac{\partial T}{\partial \dot{q}_\sigma} = 2\,T, \tag{6.100}$$

which follows from Euler's theorem on homogeneous functions.

Now consider the time average of $\dot{G}$ over a period τ:

$$\begin{aligned}\Big\langle \frac{dG}{dt}\Big\rangle &= \frac{1}{\tau}\int_0^\tau\! dt\, \frac{dG}{dt}\\ &= \frac{1}{\tau}\Big[G(\tau) - G(0)\Big]\ .\end{aligned} \tag{6.101}$$

If $G(t)$ is bounded, then in the limit $\tau \to \infty$ we must have $\langle \dot{G}\rangle = 0$. Any bounded motion, such as the orbit of the earth around the Sun, will result in $\langle \dot{G}\rangle_{\tau\to\infty} = 0$. But then

$$\Big\langle \frac{dG}{dt}\Big\rangle = 2\,\langle T\rangle + \Big\langle \sum_\sigma q_\sigma\, F_\sigma\Big\rangle = 0\ , \tag{6.102}$$

which implies

$$\begin{aligned}\langle T\rangle &= -\tfrac{1}{2}\Big\langle \sum_\sigma q_\sigma\, F_\sigma \Big\rangle = +\Big\langle \tfrac{1}{2}\sum_\sigma q_\sigma\, \frac{\partial U}{\partial q_\sigma}\Big\rangle \\ &= \Big\langle \tfrac{1}{2}\sum_i \boldsymbol{r}_i \cdot \boldsymbol{\nabla}_{\!i} U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big)\Big\rangle \qquad (6.103)\\ &= \tfrac{1}{2}k\,\langle U\rangle\ , \qquad (6.104)\end{aligned}$$

where the last line pertains to homogeneous potentials of degree k. Finally, since $T+U=E$ is conserved, we have

$$\langle T\rangle = \frac{k\,E}{k+2} \quad , \quad \langle U\rangle = \frac{2\,E}{k+2}\ . \qquad (6.105)$$

Chapter 7

Noether's Theorem

7.1 Continuous Symmetry Implies Conserved Charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector $\boldsymbol{r}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\phi}^2\right) - U(r) \ , \tag{7.1}$$

where we have chosen generalized coordinates (r, ϕ). The momentum conjugate to ϕ is $p_\phi = m\,r^2\,\dot{\phi}$. The generalized force F_ϕ clearly vanishes, since L does not depend on the coordinate ϕ. (One says that L is 'cyclic' in ϕ.) Thus, although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = m\,r^2\,\dot{\phi}$ is constant. This is the conserved angular momentum about the $\hat{\boldsymbol{z}}$ axis.

If instead the particle moved in a potential $U(y)$, independent of x, then writing

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(y) \ , \tag{7.2}$$

we have that the momentum $p_x = \partial L/\partial \dot{x} = m\dot{x}$ is conserved, because the generalized force $F_x = \partial L/\partial x = 0$ vanishes. This situation pertains in a uniform gravitational field, with $U(x, y) = mgy$, independent of x. The horizontal component of momentum is conserved.

In general, whenever the system exhibits a *continuous symmetry*, there is an associated *conserved charge*. (The terminology 'charge' is from field theory.) Indeed, this is a rigorous result, known as *Noether's Theorem*. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta) \ , \tag{7.3}$$

where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, *i.e.* $\tilde{q}_\sigma(q, 0) = q_\sigma$. The transformation may be nonlinear in the generalized coordinates. Suppose further that the Lagrangian L is invariant under the replacement $q \to \tilde{q}$. Then we must have

$$
\begin{aligned}
0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(\tilde{q}, \dot{\tilde{q}}, t) &= \frac{\partial L}{\partial q_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma} \frac{d}{dt}\left(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} .
\end{aligned} \tag{7.4}
$$

Thus, there is an associated conserved charge

$$
\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} . \tag{7.5}
$$

7.1.1 Examples of one-parameter families of transformations

Consider the Lagrangian

$$
L = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) - U\big(\sqrt{x^2 + y^2}\big) . \tag{7.6}
$$

In two-dimensional polar coordinates, we have

$$
L = \tfrac{1}{2} m(\dot{r}^2 + r^2 \dot{\phi}^2) - U(r) , \tag{7.7}
$$

and we may now define

$$
\tilde{r}(\zeta) = r \tag{7.8}
$$

$$
\tilde{\phi}(\zeta) = \phi + \zeta . \tag{7.9}
$$

Note that $\tilde{r}(0) = r$ and $\tilde{\phi}(0) = \phi$, *i.e.* the transformation is the identity when $\zeta = 0$. We now have

$$
\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = \frac{\partial L}{\partial \dot{r}} \frac{\partial \tilde{r}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} = m r^2 \dot{\phi} . \tag{7.10}
$$

Another way to derive the same result which is somewhat instructive is to work out the transformation in Cartesian coordinates. We then have

$$
\tilde{x}(\zeta) = x \, \cos\zeta - y \, \sin\zeta \tag{7.11}
$$

$$
\tilde{y}(\zeta) = x \, \sin\zeta + y \, \cos\zeta . \tag{7.12}
$$

Thus,

$$
\frac{\partial \tilde{x}}{\partial \zeta} = -\tilde{y} \quad , \quad \frac{\partial \tilde{y}}{\partial \zeta} = \tilde{x} \tag{7.13}
$$

and

$$\Lambda = \frac{\partial L}{\partial \dot{x}} \frac{\partial \tilde{x}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}} \frac{\partial \tilde{y}}{\partial \zeta}\bigg|_{\zeta=0} = m(x\dot{y} - y\dot{x}) \ . \tag{7.14}$$

But

$$m(x\dot{y} - y\dot{x}) = m\hat{\boldsymbol{z}} \cdot \boldsymbol{r} \times \dot{\boldsymbol{r}} = mr^2\dot{\phi} \ . \tag{7.15}$$

As another example, consider the potential

$$U(\rho, \phi, z) = V(\rho, a\phi + z) \ , \tag{7.16}$$

where (ρ, ϕ, z) are cylindrical coordinates for a particle of mass m, and where a is a constant with dimensions of length. The Lagrangian is

$$\tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\big) - V(\rho, a\phi + z) \ . \tag{7.17}$$

This model possesses a helical symmetry, with a one-parameter family

$$\tilde{\rho}(\zeta) = \rho \tag{7.18}$$

$$\tilde{\phi}(\zeta) = \phi + \zeta \tag{7.19}$$

$$\tilde{z}(\zeta) = z - \zeta a \ . \tag{7.20}$$

Note that

$$a\tilde{\phi} + \tilde{z} = a\phi + z \ , \tag{7.21}$$

so the potential energy, and the Lagrangian as well, is invariant under this one-parameter family of transformations. The conserved charge for this symmetry is

$$\Lambda = \frac{\partial L}{\partial \dot{\rho}} \frac{\partial \tilde{\rho}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}} \frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{z}} \frac{\partial \tilde{z}}{\partial \zeta}\bigg|_{\zeta=0} = m\rho^2\dot{\phi} - ma\dot{z} \ . \tag{7.22}$$

We can check explicitly that Λ is conserved, using the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) = \frac{\partial L}{\partial \phi} = -a\frac{\partial V}{\partial z} \tag{7.23}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = \frac{d}{dt}(m\dot{z}) = \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z} \ . \tag{7.24}$$

Thus,

$$\dot{\Lambda} = \frac{d}{dt}\big(m\rho^2\dot{\phi}\big) - a\,\frac{d}{dt}(m\dot{z}) = 0 \ . \tag{7.25}$$

7.2 Conservation of Linear and Angular Momentum

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol{n}}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol{x}}_a = \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}} \ , \tag{7.26}$$

and the associated conserved Noether charge is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{n}} \cdot \boldsymbol{P} \; , \tag{7.27}$$

where $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is the *total momentum* of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\begin{aligned} \tilde{\boldsymbol{x}}_a &= R(\zeta, \hat{\boldsymbol{n}})\, \boldsymbol{x}_a \\ &= \boldsymbol{x}_a + \zeta\, \hat{\boldsymbol{n}} \times \boldsymbol{x}_a + \mathcal{O}(\zeta^2) \; , \end{aligned} \tag{7.28}$$

where we have expanded the rotation matrix $R(\zeta, \hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial \dot{\boldsymbol{x}}_a} \cdot \hat{\boldsymbol{n}} \times \boldsymbol{x}_a = \hat{\boldsymbol{n}} \cdot \sum_a \boldsymbol{x}_a \times \boldsymbol{p}_a = \hat{\boldsymbol{n}} \cdot \boldsymbol{L} \; , \tag{7.29}$$

where $\boldsymbol{L}$ is the *total angular momentum* of the system.

7.3 Advanced Discussion : Invariance of L *vs.* Invariance of S

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action S[1]. Suppose S is invariant under

$$t \to \tilde{t}(q, t, \zeta) \tag{7.30}$$

$$q_\sigma(t) \to \tilde{q}_\sigma(q, t, \zeta) \; . \tag{7.31}$$

Then invariance of S means

$$S = \int\limits_{t_a}^{t_b} dt\, L(q, \dot{q}, t) = \int\limits_{\tilde{t}_a}^{\tilde{t}_b} dt\, L(\tilde{q}, \dot{\tilde{q}}, t) \; . \tag{7.32}$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde{t}$. The endpoints of the integral, however, do change under the transformation. Now consider an infinitesimal transformation, for which $\delta t = \tilde{t} - t$ and $\delta q = \tilde{q}\big(\tilde{t}\big) - q(t)$ are both small. Thus,

$$S = \int\limits_{t_a}^{t_b} dt\, L(q, \dot{q}, t) = \int\limits_{t_a+\delta t_a}^{t_b+\delta t_b} dt \left\{ L(q, \dot{q}, t) + \frac{\partial L}{\partial q_\sigma}\, \bar{\delta} q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}\, \bar{\delta} \dot{q}_\sigma + \ldots \right\} , \tag{7.33}$$

[1]Indeed, we should be demanding that S only change by a function of the endpoint values.

where

$$\begin{aligned}\bar{\delta}q_\sigma(t) &\equiv \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \tilde{q}_\sigma(\tilde{t}) - \tilde{q}_\sigma(\tilde{t}) + \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \delta q_\sigma - \dot{q}_\sigma\,\delta t + \mathcal{O}(\delta q\,\delta t)\end{aligned} \tag{7.34}$$

Subtracting eqn. 7.33 from eqn. 7.32, we obtain

$$\begin{aligned}0 &= L_b\,\delta t_b - L_a\,\delta t_a + \frac{\partial L}{\partial \dot{q}_\sigma}\bigg|_b \bar{\delta}q_{\sigma,b} - \frac{\partial L}{\partial \dot{q}_\sigma}\bigg|_a \bar{\delta}q_{\sigma,a} + \int\limits_{t_a+\delta t_a}^{t_b+\delta t_b} \!\! dt \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg)\right\} \bar{\delta}q_\sigma(t) \\ &= \int\limits_{t_a}^{t_b} \! dt\, \frac{d}{dt}\Bigg\{ \bigg(L - \frac{\partial L}{\partial \dot{q}_\sigma}\,\dot{q}_\sigma\bigg)\,\delta t + \frac{\partial L}{\partial \dot{q}_\sigma}\,\delta q_\sigma \Bigg\}\ ,\end{aligned} \tag{7.35}$$

where $L_{a,b}$ is $L(q, \dot{q}, t)$ evaluated at $t = t_{a,b}$. Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal, and

$$\delta t = A(q,t)\,\delta\zeta \tag{7.36}$$

$$\delta q_\sigma = B_\sigma(q,t)\,\delta\zeta\ , \tag{7.37}$$

then the conserved charge is

$$\begin{aligned}\Lambda &= \bigg(L - \frac{\partial L}{\partial \dot{q}_\sigma}\,\dot{q}_\sigma\bigg)\,A(q,t) + \frac{\partial L}{\partial \dot{q}_\sigma}\,B_\sigma(q,t) \\ &= -\,H(q,p,t)\,A(q,t) + p_\sigma\,B_\sigma(q,t)\ .\end{aligned} \tag{7.38}$$

Thus, when $A = 0$, we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance: $t \to t + \zeta$, for which $A = 1$ and $B_\sigma = 0$. Here we have written

$$H = p_\sigma\,\dot{q}_\sigma - L\ , \tag{7.39}$$

and expressed it in terms of the momenta p_σ, the coordinates q_σ, and time t. H is called the *Hamiltonian*.

7.3.1 The Hamiltonian

The Lagrangian is a function of generalized coordinates, velocities, and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}\ . \tag{7.40}$$

The Hamiltonian is a function of coordinates, *momenta*, and time. It is defined as the Legendre transform of L:

$$H(q,p,t) = \sum_\sigma p_\sigma \, \dot q_\sigma - L \ . \tag{7.41}$$

Let's examine the differential of H:

$$\begin{aligned} dH &= \sum_\sigma \bigg(\dot q_\sigma \, dp_\sigma + p_\sigma \, d\dot q_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma - \frac{\partial L}{\partial \dot q_\sigma} \, d\dot q_\sigma \bigg) - \frac{\partial L}{\partial t} \, dt \\ &= \sum_\sigma \bigg(\dot q_\sigma \, dp_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma \bigg) - \frac{\partial L}{\partial t} \, dt \ , \end{aligned} \tag{7.42}$$

where we have invoked the definition of p_σ to cancel the coefficients of $d\dot q_\sigma$. Since $\dot p_\sigma = \partial L / \partial q_\sigma$, we have *Hamilton's equations of motion*,

$$\dot q_\sigma = \frac{\partial H}{\partial p_\sigma} \quad , \quad \dot p_\sigma = -\frac{\partial H}{\partial q_\sigma} \ . \tag{7.43}$$

Thus, we can write

$$dH = \sum_\sigma \Big(\dot q_\sigma \, dp_\sigma - \dot p_\sigma \, dq_\sigma \Big) - \frac{\partial L}{\partial t} \, dt \ . \tag{7.44}$$

Dividing by dt, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \ , \tag{7.45}$$

which says that the Hamiltonian is *conserved* (*i.e.* it does not change with time) whenever there is no *explicit* time dependence to L.

Example #1 : For a simple $d = 1$ system with $L = \frac{1}{2} m\dot x^2 - U(x)$, we have $p = m\dot x$ and

$$H = p\,\dot x - L = \tfrac{1}{2} m\dot x^2 + U(x) = \frac{p^2}{2m} + U(x) \ . \tag{7.46}$$

Example #2 : Consider now the mass point – wedge system analyzed above, with

$$L = \tfrac{1}{2}(M+m)\dot X^2 + m\dot X \dot x + \tfrac{1}{2} m \, (1 + \tan^2\!\alpha) \, \dot x^2 - mg \, x \, \tan\alpha \ , \tag{7.47}$$

The canonical momenta are

$$P = \frac{\partial L}{\partial \dot X} = (M+m) \, \dot X + m \dot x \tag{7.48}$$

$$p = \frac{\partial L}{\partial \dot x} = m\dot X + m \, (1 + \tan^2\!\alpha) \, \dot x \ . \tag{7.49}$$

The Hamiltonian is given by

$$\begin{aligned} H &= P\,\dot X + p\,\dot x - L \\ &= \tfrac{1}{2}(M+m)\dot X^2 + m\dot X \dot x + \tfrac{1}{2} m \, (1 + \tan^2\!\alpha) \, \dot x^2 + mg \, x \, \tan\alpha \ . \end{aligned} \tag{7.50}$$

However, this is not quite H, since $H = H(X, x, P, p, t)$ must be expressed in terms of the coordinates and the *momenta* and not the coordinates and velocities. So we must eliminate $\dot{X}$ and $\dot{x}$ in favor of P and p. We do this by inverting the relations

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} M+m & m \\ m & m\,(1+\tan^2\!\alpha) \end{pmatrix} \begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} \tag{7.51}$$

to obtain

$$\begin{pmatrix} \dot{X} \\ \dot{x} \end{pmatrix} = \frac{1}{m\,\big(M + (M+m)\tan^2\!\alpha\big)} \begin{pmatrix} m\,(1+\tan^2\!\alpha) & -m \\ -m & M+m \end{pmatrix} \begin{pmatrix} P \\ p \end{pmatrix} \ . \tag{7.52}$$

Substituting into 7.50, we obtain

$$H = \frac{M+m}{2m}\,\frac{P^2\,\cos^2\!\alpha}{M+m\,\sin^2\!\alpha} - \frac{Pp\,\cos^2\!\alpha}{M+m\,\sin^2\!\alpha} + \frac{p^2}{2\,(M+m\,\sin^2\!\alpha)} + mg\,x\,\tan\alpha \ . \tag{7.53}$$

Notice that $\dot{P} = 0$ since $\frac{\partial L}{\partial X} = 0$. P is the total horizontal momentum of the system (wedge plus particle) and it is conserved.

7.3.2 Is $H = T + U$?

The most general form of the kinetic energy is

$$\begin{aligned} T &= T_2 + T_1 + T_0 \\ &= \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t) \ , \end{aligned} \tag{7.54}$$

where $T^{(n)}(q, \dot{q}, t)$ is homogeneous of degree n in the velocities[2]. We assume a potential energy of the form

$$\begin{aligned} U &= U_1 + U_0 \\ &= U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + U^{(0)}(q,t) \ , \end{aligned} \tag{7.55}$$

which allows for velocity-dependent forces, as we have with charged particles moving in an electromagnetic field. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} T^{(2)}_{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t)\,\dot{q}_\sigma + T^{(0)}(q,t) - U^{(1)}_\sigma(q,t)\,\dot{q}_\sigma - U^{(0)}(q,t) \ . \tag{7.56}$$

The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T^{(2)}_{\sigma\sigma'}\,\dot{q}_{\sigma'} + T^{(1)}_\sigma(q,t) - U^{(1)}_\sigma(q,t) \tag{7.57}$$

which is inverted to give

$$\dot{q}_\sigma = {T^{(2)}_{\sigma\sigma'}}^{-1} \Big(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'} \Big) \ . \tag{7.58}$$

[2] A homogeneous function of degree k satisfies $f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$. It is then easy to prove *Euler's theorem*, $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = kf$.

The Hamiltonian is then

$$
\begin{aligned}
H &= p_\sigma \, \dot q_\sigma - L \\
&= \tfrac{1}{2} \, T^{(2)\,-1}_{\sigma\sigma'} \Big(p_\sigma - T^{(1)}_\sigma + U^{(1)}_\sigma\Big) \Big(p_{\sigma'} - T^{(1)}_{\sigma'} + U^{(1)}_{\sigma'}\Big) - T_0 + U_0 \qquad (7.59) \\
&= T_2 - T_0 + U_0 \ . \qquad (7.60)
\end{aligned}
$$

If T_0, T_1, and U_1 vanish, *i.e.* if $T(q, \dot q, t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q,t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, or the potential is velocity-dependent, then $H \neq T + U$.

7.3.3 Example: A bead on a rotating hoop

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity $\dot\phi = \omega$ about the $\hat{\boldsymbol{z}}$-axis, as shown in Fig. 7.1.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$
\begin{aligned}
T &= \tfrac{1}{2} m \big(\dot r^2 + r^2 \, \dot\theta^2 + r^2 \sin^2\theta \, \dot\phi^2\big) \\
&= \tfrac{1}{2} m a^2 \big(\dot\theta^2 + \omega^2 \sin^2\theta\big) \ . \qquad (7.61)
\end{aligned}
$$

Thus, $T_2 = \frac{1}{2} m a^2 \dot\theta^2$ and $T_0 = \frac{1}{2} m a^2 \omega^2 \sin^2\theta$. The potential energy is $U(\theta) = mga(1 - \cos\theta)$. The momentum conjugate to θ is $p_\theta = m a^2 \dot\theta$, and thus

$$
\begin{aligned}
H(\theta, p) &= T_2 - T_0 + U \\
&= \tfrac{1}{2} m a^2 \dot\theta^2 - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1 - \cos\theta) \\
&= \frac{p_\theta^2}{2ma^2} - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1 - \cos\theta) \ . \qquad (7.62)
\end{aligned}
$$

For this problem, we can define the *effective potential*

$$
\begin{aligned}
U_{\rm eff}(\theta) \equiv U - T_0 &= mga(1 - \cos\theta) - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta \\
&= mga\bigg(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2} \, \sin^2\theta\bigg) \ , \qquad (7.63)
\end{aligned}
$$

where $\omega_0^2 \equiv g/a$. The Lagrangian may then be written

$$
L = \tfrac{1}{2} m a^2 \dot\theta^2 - U_{\rm eff}(\theta) \ , \qquad (7.64)
$$

and thus the equations of motion are

$$
m a^2 \ddot\theta = -\frac{\partial U_{\rm eff}}{\partial\theta} \ . \qquad (7.65)
$$

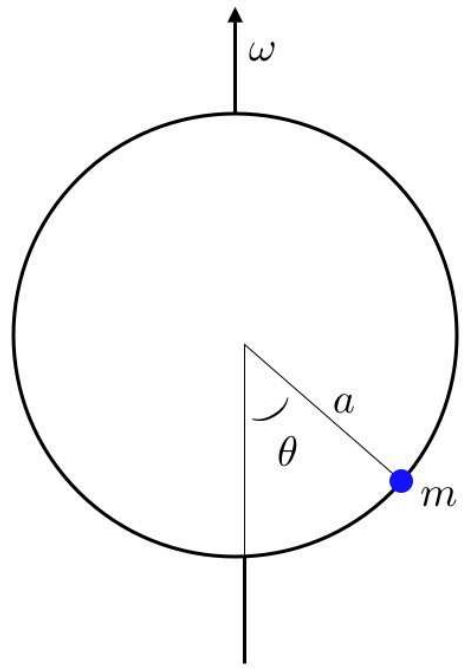


Figure 7.1: A bead of mass m on a rotating hoop of radius a.

Equilibrium is achieved when $U'_{\rm eff}(\theta) = 0$, which gives

$$\frac{\partial U_{\rm eff}}{\partial\theta} = mga\sin\theta\left\{1 - \frac{\omega^2}{\omega_0^2}\cos\theta\right\} = 0 \ , \tag{7.66}$$

i.e. $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibria is assessed by examining the sign of $U''_{\rm eff}(\theta^*)$. We have

$$U''_{\rm eff}(\theta) = mga\left\{\cos\theta - \frac{\omega^2}{\omega_0^2}\big(2\cos^2\theta - 1\big)\right\} \ . \tag{7.67}$$

Thus,

$$U''_{\rm eff}(\theta^*) = \begin{cases} mga\left(1 - \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = 0 \\ \\ -mga\left(1 + \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = \pi \\ \\ mga\left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & \text{at } \theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) \ . \end{cases} \tag{7.68}$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, *i.e.* when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in Fig. 7.2.

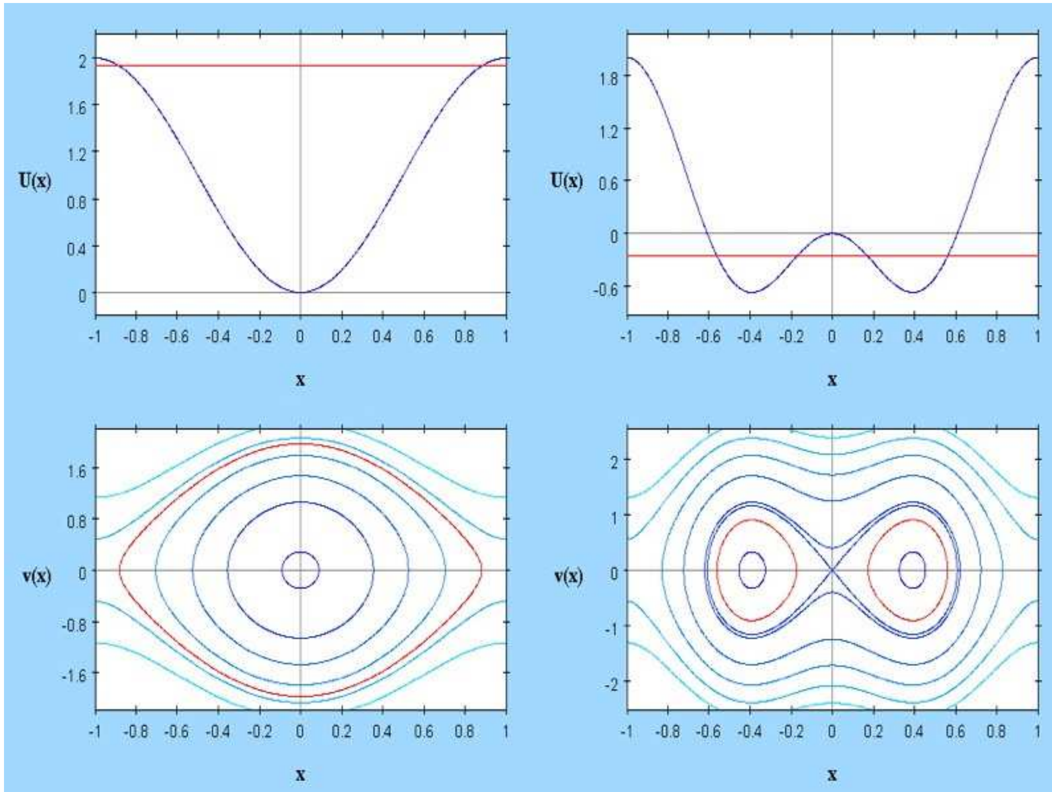


Figure 7.2: The effective potential $U_{\rm eff}(\theta) = mga\big[1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\big]$. (The dimensionless potential $\tilde U_{\rm eff}(x) = U_{\rm eff}/mga$ is shown, where $x = \theta/\pi$.) Left panels: $\omega = \frac{1}{2}\sqrt{3}\,\omega_0$. Right panels: $\omega = \sqrt{3}\,\omega_0$.

7.4 Charged Particle in a Magnetic Field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r},\dot{\boldsymbol{r}}) = q\,\phi(\boldsymbol{r},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{r}}\ , \tag{7.69}$$

which *is* velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\,\dot{\boldsymbol{r}}^2$, as usual. Here $\phi(\boldsymbol{r})$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r})$ the vector potential. The electric and magnetic fields are given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\,\frac{\partial\boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}\ . \tag{7.70}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial\dot{\boldsymbol{r}}} = m\,\dot{\boldsymbol{r}} + \frac{q}{c}\,\boldsymbol{A}\ , \tag{7.71}$$

and hence the Hamiltonian is

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{p},t) &= \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L \\ &= m\dot{\boldsymbol{r}}^2 + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\,\phi \\ &= \tfrac{1}{2}m\,\dot{\boldsymbol{r}}^2 + q\,\phi \\ &= \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\Big)^2 + q\,\phi(\boldsymbol{r},t)\ . \end{aligned} \tag{7.72}$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

Let's work out the equations of motion. We have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial\boldsymbol{r}} \tag{7.73}$$

which gives

$$m\,\ddot{\boldsymbol{r}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{r}})\ , \tag{7.74}$$

or, in component notation,

$$m\,\ddot{x}_i + \frac{q}{c}\,\frac{\partial A_i}{\partial x_j}\,\dot{x}_j + \frac{q}{c}\,\frac{\partial A_i}{\partial t} = -q\,\frac{\partial\phi}{\partial x_i} + \frac{q}{c}\,\frac{\partial A_j}{\partial x_i}\,\dot{x}_j\ , \tag{7.75}$$

which is to say

$$m\,\ddot{x}_i = -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j\ . \tag{7.76}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A_k}{\partial x_j}\ , \tag{7.77}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}\ , \tag{7.78}$$

we have $\epsilon_{ijk}\,B_i = \partial_j\,A_k - \partial_k\,A_j$, and

$$m\,\ddot{x}_i = -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}_j\,B_k\ , \tag{7.79}$$

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{r}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial\boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times(\boldsymbol{\nabla}\times\boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B}\ , \end{aligned} \tag{7.80}$$

which is, of course, the Lorentz force law.

7.5 Fast Perturbations : Rapidly Oscillating Fields

Consider a free particle moving under the influence of an oscillating force,

$$m\ddot{q} = F \sin \omega t \ . \tag{7.81}$$

The motion of the system is then

$$q(t) = q_{\rm h}(t) - \frac{F \, \sin \omega t}{m\omega^2} \ , \tag{7.82}$$

where $q_{\rm h}(t) = A + Bt$ is the solution to the homogeneous (unforced) equation of motion. Note that the amplitude of the response $q - q_{\rm h}$ goes as ω^{-2} and is therefore small when ω is large.

Now consider a general $n = 1$ system, with

$$H(q,p,t) = H_0(q,p) + V(q) \, \sin(\omega t + \delta) \ . \tag{7.83}$$

We assume that ω is much greater than any natural oscillation frequency associated with H_0. We separate the motion $q(t)$ and $p(t)$ into slow and fast components:

$$q(t) = \bar{q}(t) + \zeta(t) \tag{7.84}$$

$$p(t) = \bar{p}(t) + \pi(t) \ , \tag{7.85}$$

where $\zeta(t)$ and $\pi(t)$ oscillate with the driving frequency ω. Since ζ and π will be small, we expand Hamilton's equations in these quantities:

$$\dot{\bar{q}} + \dot{\zeta} = \frac{\partial H_0}{\partial \bar{p}} + \frac{\partial^2 H_0}{\partial \bar{p}^2}\, \pi + \frac{\partial^2 H_0}{\partial \bar{q}\, \partial \bar{p}}\, \zeta + \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}^2\, \partial \bar{p}}\, \zeta^2 + \frac{\partial^3 H_0}{\partial \bar{q}\, \partial \bar{p}^2}\, \zeta\pi + \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{p}^3}\, \pi^2 + \ldots \tag{7.86}$$

$$\begin{aligned} \dot{\bar{p}} + \dot{\pi} = &-\frac{\partial H_0}{\partial \bar{q}} - \frac{\partial^2 H_0}{\partial \bar{q}^2}\, \zeta - \frac{\partial^2 H_0}{\partial \bar{q}\, \partial \bar{p}}\, \pi - \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}^3}\, \zeta^2 - \frac{\partial^3 H_0}{\partial \bar{q}^2\, \partial \bar{p}}\, \zeta\pi - \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}\, \partial \bar{p}^2}\, \pi^2 \\ &- \frac{\partial V}{\partial \bar{q}}\, \sin(\omega t + \delta) - \frac{\partial^2 V}{\partial \bar{q}^2}\, \zeta \, \sin(\omega t + \delta) - \ldots \ . \end{aligned} \tag{7.87}$$

We now average over the fast degrees of freedom to obtain an equation of motion for the slow variables $\bar{q}$ and $\bar{p}$, which we here carry to lowest nontrivial order in averages of fluctuating quantities:

$$\dot{\bar{q}} = \frac{\partial H_0}{\partial \bar{p}} + \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}^2\, \partial \bar{p}} \big\langle \zeta^2 \big\rangle + \frac{\partial^3 H_0}{\partial \bar{q}\, \partial \bar{p}^2} \big\langle \zeta\pi \big\rangle + \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{p}^3} \big\langle \pi^2 \big\rangle \tag{7.88}$$

$$\dot{\bar{p}} = -\frac{\partial H_0}{\partial \bar{q}} - \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}^3} \big\langle \zeta^2 \big\rangle - \frac{\partial^3 H_0}{\partial \bar{q}^2\, \partial \bar{p}} \big\langle \zeta\pi \big\rangle - \frac{1}{2}\, \frac{\partial^3 H_0}{\partial \bar{q}\, \partial \bar{p}^2} \big\langle \pi^2 \big\rangle - \frac{\partial^2 V}{\partial \bar{q}^2} \big\langle \zeta \, \sin(\omega t + \delta) \big\rangle \ . \tag{7.89}$$

The fast degrees of freedom obey

$$\dot{\zeta} = \frac{\partial^2 H_0}{\partial \bar{q}\, \partial \bar{p}}\, \zeta + \frac{\partial^2 H_0}{\partial \bar{p}^2}\, \pi \tag{7.90}$$

$$\dot{\pi} = -\frac{\partial^2 H_0}{\partial \bar{q}^2}\, \zeta - \frac{\partial^2 H_0}{\partial \bar{q}\, \partial \bar{p}}\, \pi - \frac{\partial V}{\partial q}\, \sin(\omega t + \delta) \ . \tag{7.91}$$

Let us analyze the coupled equations[3]

$$\dot{\zeta} = A\,\zeta + B\,\pi \tag{7.92}$$

$$\dot{\pi} = -C\,\zeta - A\,\pi + F\,e^{-i\omega t}\ . \tag{7.93}$$

The solution is of the form

$$\begin{pmatrix} \zeta \\ \pi \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{-i\omega t}\ . \tag{7.94}$$

Plugging in, we find

$$\alpha = \frac{BF}{BC - A^2 - \omega^2} = -\frac{BF}{\omega^2} + \mathcal{O}\big(\omega^{-4}\big) \tag{7.95}$$

$$\beta = -\frac{(A + i\omega)F}{BC - A^2 - \omega^2} = \frac{iF}{\omega} + \mathcal{O}\big(\omega^{-3}\big)\ . \tag{7.96}$$

Taking the real part, and restoring the phase shift δ, we have

$$\zeta(t) = \frac{-BF}{\omega^2}\,\sin(\omega t + \delta) = \frac{1}{\omega^2}\,\frac{\partial V}{\partial \bar{q}}\,\frac{\partial^2 H_0}{\partial \bar{p}^2}\,\sin(\omega t + \delta) \tag{7.97}$$

$$\pi(t) = -\frac{F}{\omega}\,\cos(\omega t + \delta) = \frac{1}{\omega}\,\frac{\partial V}{\partial \bar{q}}\,\cos(\omega t + \delta)\ . \tag{7.98}$$

The desired averages, to lowest order, are thus

$$\big\langle \zeta^2 \big\rangle = \frac{1}{2\omega^4}\left(\frac{\partial V}{\partial \bar{q}}\right)^2 \left(\frac{\partial^2 H_0}{\partial \bar{p}^2}\right)^2 \tag{7.99}$$

$$\big\langle \pi^2 \big\rangle = \frac{1}{2\omega^2}\left(\frac{\partial V}{\partial \bar{q}}\right)^2 \tag{7.100}$$

$$\big\langle \zeta\,\sin(\omega t + \delta) \big\rangle = \frac{1}{2\omega^2}\,\frac{\partial V}{\partial \bar{q}}\,\frac{\partial^2 H_0}{\partial \bar{p}^2}\ , \tag{7.101}$$

along with $\big\langle \zeta\pi \big\rangle = 0$.

Finally, we substitute the averages into the equations of motion for the slow variables $\bar{q}$ and $\bar{p}$, resulting in the time-independent *effective Hamiltonian*

$$K(\bar{q},\bar{p}) = H_0(\bar{q},\bar{p}) + \frac{1}{4\omega^2}\,\frac{\partial^2 H_0}{\partial \bar{p}^2}\left(\frac{\partial V}{\partial \bar{q}}\right)^2 , \tag{7.102}$$

and the equations of motion

$$\dot{\bar{q}} = \frac{\partial K}{\partial \bar{p}} \quad , \quad \dot{\bar{p}} = -\frac{\partial K}{\partial \bar{q}}\ . \tag{7.103}$$

[3]With real coefficients A, B, and C, one can always take the real part to recover the fast variable equations of motion.

7.5.1 Example : pendulum with oscillating support

Consider a pendulum with a vertically oscillating point of support. The coordinates of the pendulum bob are

$$x = \ell \sin\theta \quad , \quad y = a(t) - \ell \cos\theta \; . \tag{7.104}$$

The Lagrangian is easily obtained:

$$L = \tfrac{1}{2} m\ell^2 \, \dot\theta^2 + m\ell\dot a \, \dot\theta \sin\theta + mg\ell\cos\theta + \tfrac{1}{2} m\dot a^2 - mga \tag{7.105}$$

$$= \tfrac{1}{2} m\ell^2 \, \dot\theta^2 + m(g+\ddot a)\ell\cos\theta + \overbrace{\tfrac{1}{2} m\dot a^2 - mga - \frac{d}{dt}\big(m\ell\dot a \, \sin\theta\big)}^{\text{these may be dropped}} \; . \tag{7.106}$$

Thus we may take the Lagrangian to be

$$\bar L = \tfrac{1}{2} m\ell^2 \, \dot\theta^2 + m(g+\ddot a)\ell\cos\theta \; , \tag{7.107}$$

from which we derive the Hamiltonian

$$H(\theta, p_\theta, t) = \frac{p_\theta^2}{2m\ell^2} - mg\ell\cos\theta - m\ell\ddot a \, \cos\theta \tag{7.108}$$

$$= H_0(\theta, p_\theta, t) + V_1(\theta) \, \sin\omega t \; . \tag{7.109}$$

We have assumed $a(t) = a_0 \sin\omega t$, so

$$V_1(\theta) = m\ell a_0 \, \omega^2 \cos\theta \; . \tag{7.110}$$

The effective Hamiltonian, per eqn. 7.102, is

$$K(\bar\theta, \bar p_\theta) = \frac{\bar p_\theta}{2m\ell^2} - mg\ell\cos\bar\theta + \tfrac{1}{4} m \, a_0^2 \, \omega^2 \sin^2\bar\theta \; . \tag{7.111}$$

Let's define the dimensionless parameter

$$\epsilon \equiv \frac{2g\ell}{\omega^2 a_0^2} \; . \tag{7.112}$$

The slow variable $\bar\theta$ executes motion in the *effective potential* $V_{\rm eff}(\bar\theta) = mg\ell \, v(\bar\theta)$, with

$$v(\bar\theta) = -\cos\bar\theta + \frac{1}{2\epsilon} \, \sin^2\bar\theta \; . \tag{7.113}$$

Differentiating, and dropping the bar on θ, we find that $V_{\rm eff}(\theta)$ is stationary when

$$v'(\theta) = 0 \quad \Rightarrow \quad \sin\theta\cos\theta = -\epsilon\sin\theta \; . \tag{7.114}$$

Thus, $\theta = 0$ and $\theta = \pi$, where $\sin\theta = 0$, are equilibria. When $\epsilon < 1$ (note $\epsilon > 0$ always), there are two new solutions, given by the roots of $\cos\theta = -\epsilon$.

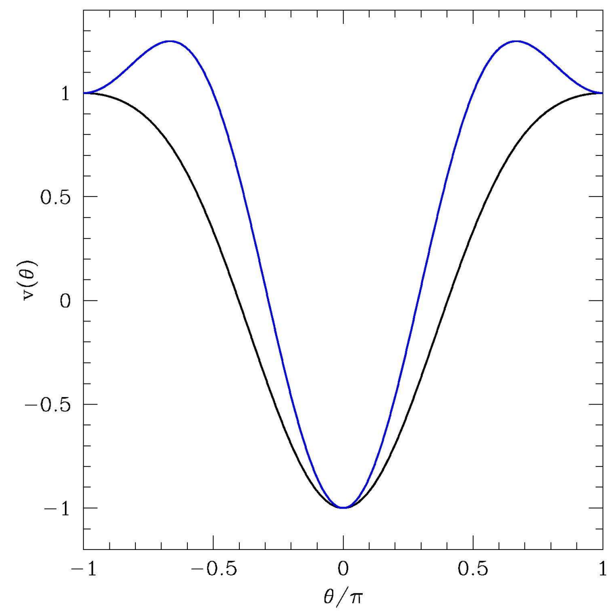


Figure 7.3: Dimensionless potential $v(\theta)$ for $\epsilon = 1.5$ (black curve) and $\epsilon = 0.5$ (blue curve).

To assess stability of these equilibria, we compute the second derivative:

$$v''(\theta) = \cos\theta + \frac{1}{\epsilon}\cos 2\theta \ . \tag{7.115}$$

From this, we see that $\theta = 0$ is stable (*i.e.* $v''(\theta = 0) > 0$) always, but $\theta = \pi$ is stable for $\epsilon < 1$ and unstable for $\epsilon > 1$. When $\epsilon < 1$, two new solutions appear, at $\cos\theta = -\epsilon$, for which

$$v''(\cos^{-1}(-\epsilon)) = \epsilon - \frac{1}{\epsilon} \ , \tag{7.116}$$

which is always negative since $\epsilon < 1$ in order for these equilibria to exist. The situation is sketched in fig. 7.3, showing $v(\theta)$ for two representative values of the parameter ϵ. For $\epsilon > 1$, the equilibrium at $\theta = \pi$ is unstable, but as ϵ decreases, a subcritical pitchfork bifurcation is encountered at $\epsilon = 1$, and $\theta = \pi$ becomes stable, while the outlying $\theta = \cos^{-1}(-\epsilon)$ solutions are unstable.

7.6 Field Theory: Systems with Several Independent Variables

Suppose $\phi_a(\boldsymbol{x})$ depends on several independent variables: $\{x^1, x^2, \ldots, x^n\}$. Furthermore, suppose

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_{\Omega} d\boldsymbol{x}\, \mathcal{L}(\phi_a\, \partial_\mu \phi_a, \boldsymbol{x}) \ , \tag{7.117}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^K. Then the first variation of S is

$$\begin{aligned} \delta S &= \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial\, \delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \delta\phi_a + \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a \ , \end{aligned} \tag{7.118}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) . \tag{7.119}$$

As an example, consider the case of a stretched string of linear mass density μ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t} \right)^2 - \tfrac{1}{2}\tau \left(\frac{\partial y}{\partial x} \right)^2 , \tag{7.120}$$

whence the Euler-Lagrange equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'} \right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \\ &= \tau\, \frac{\partial^2 y}{\partial x^2} - \mu\, \frac{\partial^2 y}{\partial t^2} \ , \end{aligned} \tag{7.121}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. Thus, $\mu\ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a,t) = \delta y(x_b,t) = \delta y(x,t_a) = \delta y(x,t_b) = 0$.

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} - \tfrac{1}{c}\, j_\mu\, A^\mu \ . \tag{7.122}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)} \right) = 0 \quad \Rightarrow \quad \partial_\mu\, F^{\mu\nu} = \frac{4\pi}{c}\, j^\nu \ , \tag{7.123}$$

which are Maxwell's equations.

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\, \frac{\partial \tilde{q}_\sigma}{\partial \zeta} \right)_{\zeta=0} = 0 \ , \tag{7.124}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t) \ , \tag{7.125}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $d\Lambda/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\,\frac{\partial\tilde{\phi}_a}{\partial\zeta}\right)_{\zeta=0} = 0 \ , \tag{7.126}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\,\frac{\partial\tilde{\phi}_a}{\partial\zeta}\bigg|_{\zeta=0} \ . \tag{7.127}$$

We call $\Lambda = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu\, J^\mu$ over the spatial region Ω gives

$$\frac{d\Lambda}{dt} = \int\limits_\Omega d^3\!x\,\partial_0\,J^0 = -\int\limits_\Omega d^3\!x\,\boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\,\hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \ , \tag{7.128}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[4]

$$\mathcal{L}(\psi,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \ . \tag{7.129}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \ , \tag{7.130}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)}\cdot(-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \ . \end{aligned} \tag{7.131}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

[4]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

7.6.1 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^* \frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\,\big(|\psi|^2 - n_0\big)^2 \ . \tag{7.132}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x},t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned}
\delta S[\psi^*,\psi] &= S[\psi^*+\delta\psi^*,\psi+\delta\psi] \\
&= \int\! dt \int\! d^d\!x\,\bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^*\cdot\boldsymbol{\nabla}\psi \\
&\qquad\qquad - 2g\,\big(|\psi|^2 - n_0\big)\,(\psi^*\delta\psi + \psi\delta\psi^*)\bigg\} \\
&= \int\! dt \int\! d^d\!x\,\Bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\,\big(|\psi|^2 - n_0\big)\,\psi^*\bigg]\,\delta\psi \\
&\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi\bigg]\,\delta\psi^*\Bigg\}\ ,
\end{aligned} \tag{7.133}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\,\big(|\psi|^2 - n_0\big)\,\psi \tag{7.134}$$

as well as its complex conjugate,

$$-i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \ . \tag{7.135}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\frac{\delta S}{\delta\psi} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right) \tag{7.136}$$

$$\frac{\delta S}{\delta\psi^*} = \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right)\ , \tag{7.137}$$

with $x^\mu = (t,\boldsymbol{x})$[5] Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi} = -2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{7.138}$$

and

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\,\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi\ , \tag{7.139}$$

[5] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) \ . \tag{7.140}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\,\frac{\partial\tilde{\psi}}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\,\frac{\partial\tilde{\psi}^*}{\partial\zeta}\bigg|_{\zeta=0}$$

$$J^0 = -\hbar\,|\psi|^2 \tag{7.141}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{7.142}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \ , \tag{7.143}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{7.144}$$

are the particle density and the particle current, respectively.

Chapter 8

Constraints

A mechanical system of N point particles in d dimensions possesses $n = dN$ degrees of freedom[1]. To specify these degrees of freedom, we can choose any independent set of generalized coordinates $\{q_1, \ldots, q_K\}$. Oftentimes, however, not all n coordinates are independent.

Consider, for example, the situation in Fig. 8.1, where a cylinder of radius a rolls over a half-cylinder of radius R. If there is no slippage, then the angles θ_1 and θ_2 are not independent, and they obey the *equation of constraint*,

$$R\,\theta_1 = a\,(\theta_2 - \theta_1) \ . \tag{8.1}$$

In this case, we can easily solve the constraint equation and substitute $\theta_2 = \left(1 + \frac{R}{a}\right)\theta_1$. In other cases, though, the equation of constraint might not be so easily solved (*e.g.* it may be nonlinear). How then do we proceed?

8.1 Constraints and Variational Calculus

Before addressing the subject of constrained dynamical systems, let's consider the issue of constraints in the broader context of variational calculus. Suppose we have a functional

$$F[y(x)] = \int\limits_{x_a}^{x_b} \! dx\, L(y, y', x) \ , \tag{8.2}$$

which we want to extremize subject to some constraints. Here y may stand for a set of functions $\{y_\sigma(x)\}$. There are two classes of constraints we will consider:

[1]For N rigid bodies, the number of degrees of freedom is $n' = \frac{1}{2}d(d+1)N$, corresponding to d center-of-mass coordinates and $\frac{1}{2}d(d-1)$ angles of orientation for each particle. The dimension of the group of rotations in d dimensions is $\frac{1}{2}d(d-1)$, corresponding to the number of parameters in a general rank-d orthogonal matrix (*i.e.* an element of the group $O(d)$).

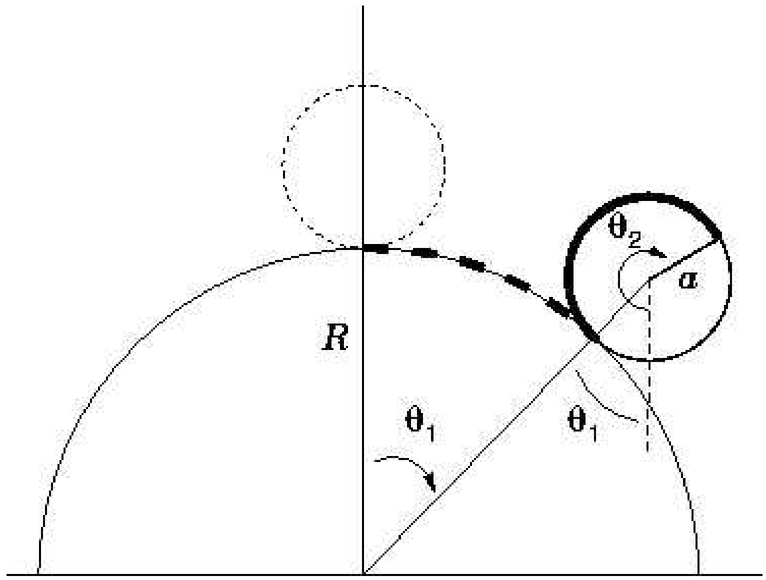


Figure 8.1: A cylinder of radius a rolls along a half-cylinder of radius R. When there is no slippage, the angles θ_1 and θ_2 obey the constraint equation $R\theta_1 = a(\theta_2 - \theta_1)$.

1. *Integral constraints:* These are of the form
$$\int\limits_{x_a}^{x_b} dx\, N_j(y, y', x) = C_j \ , \tag{8.3}$$
where j labels the constraint.

2. *Holonomic constraints:* These are of the form
$$G_j(y, x) = 0 \ . \tag{8.4}$$

The cylinders system in Fig. 8.1 provides an example of a holonomic constraint. There, $G(\theta, t) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0$. As an example of a problem with an integral constraint, suppose we want to know the shape of a hanging rope of fixed length C. This means we minimize the rope's potential energy,
$$U[y(x)] = \lambda g \int\limits_{x_a}^{x_b} ds\, y(x) = \lambda g \int\limits_{x_a}^{x_b} dx\, y\sqrt{1 + y'^2} \ , \tag{8.5}$$
where λ is the linear mass density of the rope, subject to the fixed-length constraint
$$C = \int\limits_{x_a}^{x_b} ds = \int\limits_{x_a}^{x_b} dx\, \sqrt{1 + y'^2} \ . \tag{8.6}$$
Note $ds = \sqrt{dx^2 + dy^2}$ is the differential element of arc length along the rope. To solve problems like these, we turn to Lagrange's method of *undetermined multipliers*.

8.2 Constrained Extremization of Functions

Given $F(x_1, \ldots, x_n)$ to be extremized subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ where $j = 1, \ldots, k$, construct

$$F^*\big(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j \, G_j(x_1, \ldots, x_n) \tag{8.7}$$

which is a function of the $(n + k)$ variables $\big\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big\}$. Now freely extremize the extended function F^*:

$$dF^* = \sum_{\sigma=1}^{n} \frac{\partial F^*}{\partial x_\sigma} \, dx_\sigma + \sum_{j=1}^{k} \frac{\partial F^*}{\partial \lambda_j} \, d\lambda_j \tag{8.8}$$

$$= \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j = 0 \tag{8.9}$$

This results in the $(n + k)$ equations

$$\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, n) \tag{8.10}$$

$$G_j = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.11}$$

The interpretation of all this is as follows. The n equations in 8.10 can be written in vector form as

$$\boldsymbol{\nabla} F + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{\nabla} G_j = 0 \ . \tag{8.12}$$

This says that the (n-component) vector $\boldsymbol{\nabla} F$ is linearly dependent upon the k vectors $\boldsymbol{\nabla} G_j$. Thus, any movement in the direction of $\boldsymbol{\nabla} F$ must necessarily entail movement along one or more of the directions $\boldsymbol{\nabla} G_j$. This would require violating the constraints, since movement along $\boldsymbol{\nabla} G_j$ takes us off the level set $G_j = 0$. Were $\boldsymbol{\nabla} F$ linearly *independent* of the set $\{\boldsymbol{\nabla} G_j\}$, this would mean that we could find a differential displacement $d\boldsymbol{x}$ which has finite overlap with $\boldsymbol{\nabla} F$ but zero overlap with each $\boldsymbol{\nabla} G_j$. Thus $\boldsymbol{x} + d\boldsymbol{x}$ would still satisfy $G_j(\boldsymbol{x} + d\boldsymbol{x}) = 0$, but F would change by the finite amount $dF = \boldsymbol{\nabla} F(\boldsymbol{x}) \cdot d\boldsymbol{x}$.

8.3 Extremization of Functionals : Integral Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int\limits_{x_a}^{x_b} dx \, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{8.13}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$\int_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l = 1,\ldots,k)\ , \tag{8.14}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j\}\big] \equiv \int_{x_a}^{x_b} dx \left\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{l=1}^{k} \lambda_l\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big)\right\} - \sum_{l=1}^{k} \lambda_l\, C_l \tag{8.15}$$

and freely extremize over $\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\}$. This results in $(n+k)$ equations

$$\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{l=1}^{k} \lambda_l \left\{ \frac{\partial N_l}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial N_l}{\partial y'_\sigma}\right)\right\} = 0 \qquad (\sigma = 1,\ldots,n) \tag{8.16}$$

$$\int_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\},\{y'_\sigma\},x\big) = C_l \qquad (l = 1,\ldots,k)\ . \tag{8.17}$$

8.4 Extremization of Functionals : Holonomic Constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int_{x_a}^{x_b} dx\, L\big(\{y_\sigma\},\{y'_\sigma\},x\big) \qquad (\sigma = 1,\ldots,n) \tag{8.18}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$G_j\big(\{y_\sigma(x)\},x\big) = 0 \qquad (j = 1,\ldots,k)\ , \tag{8.19}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j(x)\}\big] \equiv \int_{x_a}^{x_b} dx \left\{ L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{j=1}^{k} \lambda_j\, G_j\big(\{y_\sigma\}\big)\right\} \tag{8.20}$$

and freely extremize over $\big\{y_1,\ldots,y_n;\lambda_1,\ldots,\lambda_k\big\}$:

$$\delta F^* = \int_{x_a}^{x_b} dx \left\{ \sum_{\sigma=1}^{n} \left(\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma}\right)\delta y_\sigma + \sum_{j=1}^{k} G_j\, \delta\lambda_j \right\} = 0\ , \tag{8.21}$$

resulting in the $(n+k)$ equations

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial y_\sigma} \qquad (\sigma = 1,\ldots,n) \tag{8.22}$$

$$G_j\big(\{y_\sigma\},x\big) = 0 \qquad (j = 1,\ldots,k)\ . \tag{8.23}$$

8.4.1 Examples of extremization with constraints

Volume of a cylinder : As a warm-up problem, let's maximize the volume $V = \pi a^2 h$ of a cylinder of radius a and height h, subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \tag{8.24}$$

We therefore define

$$V^*(a,h,\lambda) \equiv V(a,h) + \lambda\, G(a,h) \ , \tag{8.25}$$

and set

$$\frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0 \tag{8.26}$$

$$\frac{\partial V^*}{\partial h} = \pi a^2 + 2\lambda\, \frac{h}{b} = 0 \tag{8.27}$$

$$\frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0 \ . \tag{8.28}$$

Solving these three equations simultaneously gives

$$a = \frac{2\ell}{5\pi} \quad , \quad h = \sqrt{\frac{b\ell}{5}} \quad , \quad \lambda = \frac{2\pi}{5^{3/2}}\, b^{1/2}\, \ell^{3/2} \quad , \quad V = \frac{4}{5^{5/2}\,\pi}\, \ell^{5/2}\, b^{1/2} \ . \tag{8.29}$$

Hanging rope : We minimize the energy functional

$$E\big[y(x)\big] = \mu g \int\limits_{x_1}^{x_2} \! dx\, y \sqrt{1 + y'^2} \ , \tag{8.30}$$

where μ is the linear mass density, subject to the constraint of fixed total length,

$$C\big[y(x)\big] = \int\limits_{x_1}^{x_2} \! dx\, \sqrt{1 + y'^2} \ . \tag{8.31}$$

Thus,

$$E^*\big[y(x),\lambda\big] = E\big[y(x)\big] + \lambda C\big[y(x)\big] = \int\limits_{x_1}^{x_2} \! dx\, L^*(y,y',x) \ , \tag{8.32}$$

with

$$L^*(y,y',x) = (\mu g y + \lambda)\, \sqrt{1 + y'^2} \ . \tag{8.33}$$

Since $\frac{\partial L^*}{\partial x} = 0$ we have that

$$\mathcal{J} = y'\, \frac{\partial L^*}{\partial y'} - L^* = -\frac{\mu g y + \lambda}{\sqrt{1 + y'^2}} \tag{8.34}$$

is constant. Thus,

$$\frac{dy}{dx} = \pm\mathcal{J}^{-1}\sqrt{(\mu g y + \lambda)^2 - \mathcal{J}^2} \ , \tag{8.35}$$

with solution

$$y(x) = -\frac{\lambda}{\mu g} + \frac{\mathcal{J}}{\mu g}\cosh\left(\frac{\mu g}{\mathcal{J}}\,(x-a)\right) \ . \tag{8.36}$$

Here, $\mathcal{J}$, a, and λ are constants to be determined by demanding $y(x_i) = y_i$ $(i = 1, 2)$, and that the total length of the rope is C.

Geodesic on a curved surface : Consider next the problem of a geodesic on a curved surface. Let the equation for the surface be

$$G(x, y, z) = 0 \ . \tag{8.37}$$

We wish to extremize the distance,

$$D = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2 + dz^2} \ . \tag{8.38}$$

We introduce a parameter t defined on the unit interval: $t \in [0, 1]$, such that $x(0) = x_a$, $x(1) = x_b$, *etc.* Then D may be regarded as a functional, *viz.*

$$D\big[x(t), y(t), z(t)\big] = \int_0^1 dt\,\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} \ . \tag{8.39}$$

We impose the constraint by forming the extended functional, D^*:

$$D^*\big[x(t), y(t), z(t), \lambda(t)\big] \equiv \int_0^1 dt\,\left\{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} + \lambda\,G(x, y, z)\right\} \ , \tag{8.40}$$

and we demand that the first functional derivatives of D^* vanish:

$$\frac{\delta D^*}{\delta x(t)} = -\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial x} = 0 \tag{8.41}$$

$$\frac{\delta D^*}{\delta y(t)} = -\frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial y} = 0 \tag{8.42}$$

$$\frac{\delta D^*}{\delta z(t)} = -\frac{d}{dt}\left(\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}\right) + \lambda\,\frac{\partial G}{\partial z} = 0 \tag{8.43}$$

$$\frac{\delta D^*}{\delta \lambda(t)} = G(x, y, z) = 0 \ . \tag{8.44}$$

Thus,

$$\lambda(t) = \frac{v\ddot{x} - \dot{x}\dot{v}}{v^2\,\partial_x G} = \frac{v\ddot{y} - \dot{y}\dot{v}}{v^2\,\partial_y G} = \frac{v\ddot{z} - \dot{z}\dot{v}}{v^2\,\partial_z G} \ , \tag{8.45}$$

with $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and $\partial_x \equiv \frac{\partial}{\partial x}$, *etc.* These three equations are supplemented by $G(x, y, z) = 0$, which is the fourth.

8.5 Application to Mechanics

Let us write our system of constraints in the differential form

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, dq_\sigma + h_j(q,t)dt = 0 \qquad (j = 1,\ldots,k) \ . \tag{8.46}$$

If the partial derivatives satisfy

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma} \ , \tag{8.47}$$

then the differential can be integrated to give $dG(q,t) = 0$, where

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad , \quad h_j = \frac{\partial G_j}{\partial t} \ . \tag{8.48}$$

The action functional is

$$S[\{q_\sigma(t)\}] = \int\limits_{t_a}^{t_b} dt\, L\big(\{q_\sigma\},\{\dot q_\sigma\},t\big) \qquad (\sigma = 1,\ldots,n) \ , \tag{8.49}$$

subject to boundary conditions $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. The first variation of S is given by

$$\delta S = \int\limits_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) \right\} \delta q_\sigma \ . \tag{8.50}$$

Since the $\{q_\sigma(t)\}$ are no longer independent, we cannot infer that the term in brackets vanishes for each σ. What are the constraints on the variations $\delta q_\sigma(t)$? The constraints are expressed in terms of *virtual displacements* which take no time: $\delta t = 0$. Thus,

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \delta q_\sigma(t) = 0 \ , \tag{8.51}$$

where $j = 1,\ldots,k$ is the constraint index. We may now relax the constraint by introducing k undetermined functions $\lambda_j(t)$, by adding integrals of the above equations with undetermined coefficient functions to δS:

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0 \ . \tag{8.52}$$

Now we can demand that the term in brackets vanish for all σ. Thus, we obtain a set of $(n+k)$ equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \equiv Q_\sigma \tag{8.53}$$

$$g_{j\sigma}(q,t)\, \dot q_\sigma + h_j(q,t) = 0 \ , \tag{8.54}$$

in $(n+k)$ unknowns $\{q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k\}$. Here, Q_σ is the *force of constraint conjugate to the generalized coordinate* q_σ. Thus, with

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad F_\sigma = \frac{\partial L}{\partial q_\sigma} \quad , \quad Q_\sigma = \sum_{j=1}^{k} \lambda_j \, g_{j\sigma} \ , \tag{8.55}$$

we write Newton's second law as

$$\dot{p}_\sigma = F_\sigma + Q_\sigma \ . \tag{8.56}$$

Note that we can write

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \tag{8.57}$$

and that the *instantaneous* constraints may be written

$$\boldsymbol{g}_j \cdot \delta \boldsymbol{q} = 0 \qquad (j = 1, \ldots, k) \ . \tag{8.58}$$

Thus, by demanding

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} + \sum_{j=1}^{k} \lambda_j \, \boldsymbol{g}_j = 0 \tag{8.59}$$

we require that the functional derivative be linearly dependent on the k vectors $\boldsymbol{g}_j$.

8.5.1 Constraints and conservation laws

We have seen how invariance of the Lagrangian with respect to a one-parameter family of coordinate transformations results in an associated conserved quantity Λ, and how a lack of explicit time dependence in L results in the conservation of the Hamiltonian H. In deriving both these results, however, we used the equations of motion $\dot{p}_\sigma = F_\sigma$. What happens when we have constraints, in which case $\dot{p}_\sigma = F_\sigma + Q_\sigma$?

Let's begin with the Hamiltonian. We have $H = \dot{q}_\sigma \, p_\sigma - L$, hence

$$\begin{aligned}\frac{dH}{dt} &= \left(p_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\right)\ddot{q}_\sigma + \left(\dot{p}_\sigma - \frac{\partial L}{\partial q_\sigma}\right)\dot{q}_\sigma - \frac{\partial L}{\partial t} \\ &= Q_\sigma \, \dot{q}_\sigma - \frac{\partial L}{\partial t} \ .\end{aligned} \tag{8.60}$$

We now use

$$Q_\sigma \, \dot{q}_\sigma = \lambda_j \, g_{j\sigma} \, \dot{q}_\sigma = -\lambda_j \, h_j \tag{8.61}$$

to obtain

$$\frac{dH}{dt} = -\lambda_j \, h_j - \frac{\partial L}{\partial t} \ . \tag{8.62}$$

We therefore conclude that *in a system with constraints of the form* $g_{j\sigma} \, \dot{q}_\sigma + h_j = 0$, *the Hamiltonian is conserved if each* $h_j = 0$ *and if* L *is not explicitly dependent on time.* In the case of holonomic constraints, $h_j = \frac{\partial G_j}{\partial t}$, so H is conserved if neither L nor any of the constraints G_j is explicitly time-dependent.

Next, let us rederive Noether's theorem when constraints are present. We assume a one-parameter family of transformations $q_\sigma \to \tilde{q}_\sigma(\zeta)$ leaves L invariant. Then

$$\begin{aligned} 0 = \frac{dL}{d\zeta} &= \frac{\partial L}{\partial \tilde{q}_\sigma}\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \frac{\partial L}{\partial \dot{\tilde{q}}_\sigma}\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta} \\ &= \big(\dot{\tilde{p}}_\sigma - \tilde{Q}_\sigma\big)\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \tilde{p}_\sigma \frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) \\ &= \frac{d}{dt}\bigg(\tilde{p}_\sigma \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) - \lambda_j\, \tilde{g}_{j\sigma}\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\ . \end{aligned} \tag{8.63}$$

Now let us write the constraints in differential form as

$$\tilde{g}_{j\sigma}\, d\tilde{q}_\sigma + \tilde{h}_j\, dt + \tilde{k}_j\, d\zeta = 0\ . \tag{8.64}$$

We now have

$$\frac{d\Lambda}{dt} = \lambda_j\, \tilde{k}_j\ , \tag{8.65}$$

which says that *if the constraints are independent of* ζ *then* Λ *is conserved.* For holonomic constraints, this means that

$$G_j\big(\tilde{q}(\zeta), t\big) = 0 \quad \Rightarrow \quad \tilde{k}_j = \frac{\partial G_j}{\partial \zeta} = 0\ , \tag{8.66}$$

i.e. $G_j(\tilde{q}, t)$ has no explicit ζ dependence.

8.6 Worked Examples

Here we consider several example problems of constrained dynamics, and work each out in full detail.

8.6.1 One cylinder rolling off another

As an example of the constraint formalism, consider the system in Fig. 8.1, where a cylinder of radius a rolls atop a cylinder of radius R. We have two constraints:

$$G_1(r,\theta_1,\theta_2) = r - R - a = 0 \qquad \text{(cylinders in contact)} \tag{8.67}$$

$$G_2(r,\theta_1,\theta_2) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0 \qquad \text{(no slipping)}\ , \tag{8.68}$$

from which we obtain the $g_{j\sigma}$:

$$g_{j\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix}\ , \tag{8.69}$$

which is to say

$$\frac{\partial G_1}{\partial r} = 1 \qquad \frac{\partial G_1}{\partial \theta_1} = 0 \qquad \frac{\partial G_1}{\partial \theta_2} = 0 \tag{8.70}$$

$$\frac{\partial G_2}{\partial r} = 0 \qquad \frac{\partial G_2}{\partial \theta_1} = R + a \qquad \frac{\partial G_2}{\partial \theta_2} = -a \ . \tag{8.71}$$

The Lagrangian is

$$L = T - U = \tfrac{1}{2} M \big(\dot{r}^2 + r^2\,\dot{\theta}_1^2\big) + \tfrac{1}{2} I\,\dot{\theta}_2^2 - Mgr\,\cos\theta_1 \ , \tag{8.72}$$

where M and I are the mass and rotational inertia of the rolling cylinder, respectively. Note that the kinetic energy is a sum of center-of-mass translation $T_{\rm tr} = \frac{1}{2} M \big(\dot{r}^2 + r^2\,\dot{\theta}_1^2\big)$ and rotation about the center-of-mass, $T_{\rm rot} = \frac{1}{2} I\,\dot{\theta}_2^2$. The equations of motion are

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{r}}\bigg) - \frac{\partial L}{\partial r} = M\ddot{r} - Mr\,\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1 \equiv Q_r \tag{8.73}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\theta}_1}\bigg) - \frac{\partial L}{\partial \theta_1} = Mr^2\ddot{\theta}_1 + 2Mr\dot{r}\,\dot{\theta}_1 - Mgr\sin\theta_1 = (R+a)\,\lambda_2 \equiv Q_{\theta_1} \tag{8.74}$$

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\theta}_2}\bigg) - \frac{\partial L}{\partial \theta_2} = I\ddot{\theta}_2 = -a\,\lambda_2 \equiv Q_{\theta_2} \ . \tag{8.75}$$

To these three equations we add the two constraints, resulting in five equations in the five unknowns $\big\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\big\}$.

We solve by first implementing the constraints, which give $r = (R+a)$ a constant (*i.e.* $\dot{r} = 0$), and $\dot{\theta}_2 = \big(1 + \frac{R}{a}\big)\,\dot{\theta}_1$. Substituting these into the above equations gives

$$-M(R+a)\,\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1 \tag{8.76}$$

$$M(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = (R+a)\,\lambda_2 \tag{8.77}$$

$$I\bigg(\frac{R+a}{a}\bigg)\ddot{\theta}_1 = -a\lambda_2 \ . \tag{8.78}$$

From eqn. 8.78 we obtain

$$\lambda_2 = -\frac{I}{a}\,\ddot{\theta}_2 = -\frac{R+a}{a^2}\,I\,\ddot{\theta}_1 \ , \tag{8.79}$$

which we substitute into eqn. 8.77 to obtain

$$\bigg(M + \frac{I}{a^2}\bigg)(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = 0 \ . \tag{8.80}$$

Multiplying by $\dot{\theta}_1$, we obtain an exact differential, which may be integrated to yield

$$\tfrac{1}{2} M\bigg(1 + \frac{I}{Ma^2}\bigg)\dot{\theta}_1^2 + \frac{Mg}{R+a}\,\cos\theta_1 = \frac{Mg}{R+a}\cos\theta_1^\circ \ . \tag{8.81}$$

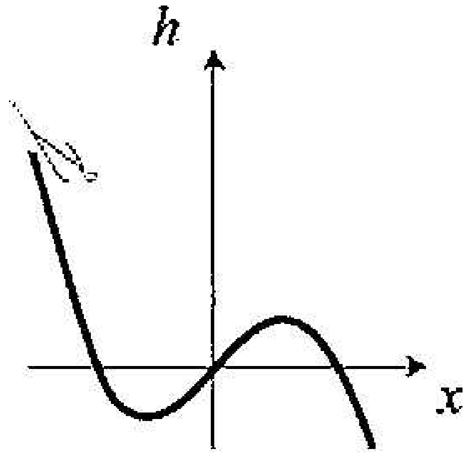


Figure 8.2: Frictionless motion under gravity along a curved surface. The skier flies off the surface when the normal force vanishes.

Here, we have assumed that $\dot{\theta}_1 = 0$ when $\theta_1 = \theta_1^\circ$, *i.e.* the rolling cylinder is released from rest at $\theta_1 = \theta_1^\circ$. Finally, inserting this result into eqn. 8.76, we obtain the radial force of constraint,

$$Q_r = \frac{Mg}{1+\alpha}\Big\{(3+\alpha)\,\cos\theta_1 - 2\,\cos\theta_1^\circ\Big\}\ , \tag{8.82}$$

where $\alpha = I/Ma^2$ is a dimensionless parameter ($0 \le \alpha \le 1$). This is the radial component of the normal force between the two cylinders. When Q_r vanishes, the cylinders lose contact – the rolling cylinder flies off. Clearly this occurs at an angle $\theta_1 = \theta_1^*$, where

$$\theta_1^* = \cos^{-1}\left(\frac{2\cos\theta_1^\circ}{3+\alpha}\right)\ . \tag{8.83}$$

The detachment angle θ_1^* is an increasing function of α, which means that larger I delays detachment. This makes good sense, since when I is larger the gain in kinetic energy is split between translational and rotational motion of the rolling cylinder.

8.6.2 Frictionless motion along a curve

Consider the situation in Fig. 8.2 where a skier moves frictionlessly under the influence of gravity along a general curve $y = h(x)$. The Lagrangian for this problem is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy \tag{8.84}$$

and the (holonomic) constraint is

$$G(x,y) = y - h(x) = 0\ . \tag{8.85}$$

Accordingly, the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \lambda\,\frac{\partial G}{\partial q_\sigma}\ , \tag{8.86}$$

where $q_1 = x$ and $q_2 = y$. Thus, we obtain

$$m\ddot{x} = -\lambda\, h'(x) = Q_x \tag{8.87}$$

$$m\ddot{y} + mg = \lambda = Q_y \ . \tag{8.88}$$

We eliminate y in favor of x by invoking the constraint. Since we need $\ddot{y}$, we must differentiate the constraint, which gives

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.89}$$

Using the second Euler-Lagrange equation, we then obtain

$$\frac{\lambda}{m} = g + h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \ . \tag{8.90}$$

Finally, we substitute this into the first E-L equation to obtain an equation for x alone:

$$\Big(1 + \big[h'(x)\big]^2\Big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 + g\,h'(x) = 0 \ . \tag{8.91}$$

Had we started by eliminating $y = h(x)$ at the outset, writing

$$L(x,\dot{x}) = \tfrac{1}{2}m\Big(1 + \big[h'(x)\big]^2\Big)\,\dot{x}^2 - mg\,h(x) \ , \tag{8.92}$$

we would also have obtained this equation of motion.

The skier flies off the curve when the vertical force of constraint $Q_y = \lambda$ starts to become negative, because the curve can only supply a positive normal force. Suppose the skier starts from rest at a height y_0. We may then determine the point x at which the skier detaches from the curve by setting $\lambda(x) = 0$. To do so, we must eliminate $\dot{x}$ and $\ddot{x}$ in terms of x. For $\ddot{x}$, we may use the equation of motion to write

$$\ddot{x} = -\bigg(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\bigg) \ , \tag{8.93}$$

which allows us to write

$$\lambda = m\bigg(\frac{g + h''\,\dot{x}^2}{1 + h'^2}\bigg) \ . \tag{8.94}$$

To eliminate $\dot{x}$, we use conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot{x}^2 + mgh \ , \tag{8.95}$$

which fixes

$$\dot{x}^2 = 2g\bigg(\frac{y_0 - h}{1 + h'^2}\bigg) \ . \tag{8.96}$$

Putting it all together, we have

$$\lambda(x) = \frac{mg}{\big(1 + h'^2\big)^2}\,\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\} \ . \tag{8.97}$$

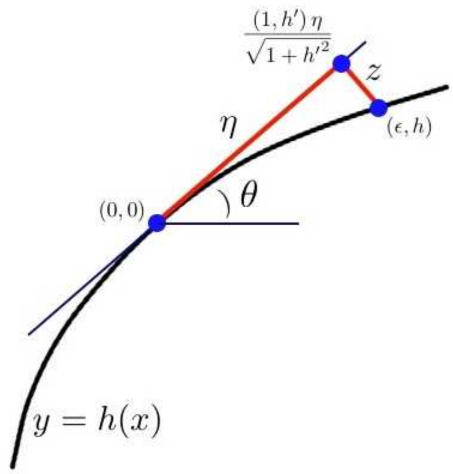


Figure 8.3: Finding the local radius of curvature: $z = \eta^2/2R$.

The skier detaches from the curve when $\lambda(x) = 0$, *i.e.* when

$$1 + h'^2 + 2(y_0 - h)\, h'' = 0 \ . \tag{8.98}$$

There is a somewhat easier way of arriving at the same answer. This is to note that the skier must fly off when the local centripetal force equals the gravitational force normal to the curve, *i.e.*

$$\frac{m\, v^2(x)}{R(x)} = mg\, \cos\theta(x) \ , \tag{8.99}$$

where $R(x)$ is the local radius of curvature. Now $\tan\theta = h'$, so $\cos\theta = \big(1 + h'^2\big)^{-1/2}$. The square of the velocity is $v^2 = \dot{x}^2 + \dot{y}^2 = \big(1 + h'^2\big)\, \dot{x}^2$. What is the local radius of curvature $R(x)$? This can be determined from the following argument, and from the sketch in Fig. 8.3. Writing $x = x^* + \epsilon$, we have

$$y = h(x^*) + h'(x^*)\, \epsilon + \tfrac{1}{2} h''(x^*)\, \epsilon^2 + \ldots \ . \tag{8.100}$$

We now drop a perpendicular segment of length z from the point (x, y) to the line which is tangent to the curve at $\big(x^*, h(x^*)\big)$. According to Fig. 8.3, this means

$$\begin{pmatrix} \epsilon \\ y \end{pmatrix} = \eta \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} 1 \\ h' \end{pmatrix} - z \cdot \frac{1}{\sqrt{1+h'^2}} \begin{pmatrix} -h' \\ 1 \end{pmatrix} \ . \tag{8.101}$$

Thus, we have

$$
\begin{aligned}
y &= h'\,\epsilon + \tfrac{1}{2} h''\,\epsilon^2 \\
&= h' \left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}} \right) + \tfrac{1}{2} h'' \left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}} \right)^2 \\
&= \frac{\eta\,h' + z\,h'^2}{\sqrt{1+h'^2}} + \frac{h''\,\eta^2}{2\big(1+h'^2\big)} + \mathcal{O}(\eta z) \\
&= \frac{\eta\,h' - z}{\sqrt{1+h'^2}} \ ,
\end{aligned}
\tag{8.102}
$$

from which we obtain

$$
z = -\frac{h''\,\eta^2}{2\big(1+h'^2\big)^{3/2}} + \mathcal{O}(\eta^3)
\tag{8.103}
$$

and therefore

$$
R(x) = -\frac{1}{h''(x)} \cdot \Big(1 + \big[h'(x)\big]^2 \Big)^{3/2} \ .
\tag{8.104}
$$

Thus, the detachment condition,

$$
\frac{mv^2}{R} = -\frac{m\,h''\,\dot x^2}{\sqrt{1+h'^2}} = \frac{mg}{\sqrt{1+h'^2}} = mg\,\cos\theta
\tag{8.105}
$$

reproduces the result from eqn. 8.94.

8.6.3 Disk rolling down an inclined plane

A hoop of mass m and radius R rolls without slipping down an inclined plane. The inclined plane has opening angle α and mass M, and itself slides frictionlessly along a horizontal surface. Find the motion of the system.

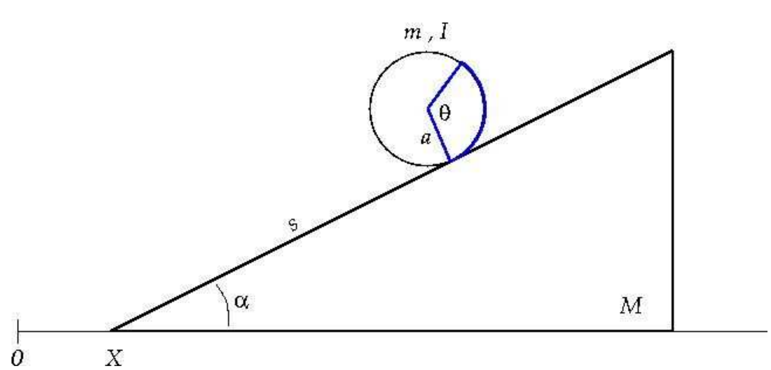


Figure 8.4: A hoop rolling down an inclined plane lying on a frictionless surface.

Solution : Referring to the sketch in Fig. 8.4, the center of the hoop is located at

$$\begin{aligned} x &= X + s\,\cos\alpha - a\,\sin\alpha \\ y &= s\,\sin\alpha + a\,\cos\alpha\ , \end{aligned}$$

where X is the location of the lower left corner of the wedge, and s is the distance along the wedge to the bottom of the hoop. If the hoop rotates through an angle θ, the no-slip condition is $a\,\dot{\theta} + \dot{s} = 0$. Thus,

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \\ &= \tfrac{1}{2}\bigg(m + \frac{I}{a^2}\bigg)\dot{s}^2 + \tfrac{1}{2}(M+m)\dot{X}^2 + m\cos\alpha\,\dot{X}\,\dot{s} - mgs\,\sin\alpha - mga\,\cos\alpha\ . \end{aligned}$$

Since X is cyclic in L, the momentum

$$P_X = (M+m)\dot{X} + m\cos\alpha\,\dot{s}\ ,$$

is preserved: $\dot{P}_X = 0$. The second equation of motion, corresponding to the generalized coordinate s, is

$$\bigg(1 + \frac{I}{ma^2}\bigg)\ddot{s} + \cos\alpha\,\ddot{X} = -g\,\sin\alpha\ .$$

Using conservation of P_X, we eliminate $\ddot{s}$ in favor of $\ddot{X}$, and immediately obtain

$$\ddot{X} = \frac{g\sin\alpha\cos\alpha}{\Big(1+\frac{M}{m}\Big)\Big(1+\frac{I}{ma^2}\Big) - \cos^2\alpha} \equiv a_X\ .$$

The result

$$\ddot{s} = -\frac{g\Big(1+\frac{M}{m}\Big)\sin\alpha}{\Big(1+\frac{M}{m}\Big)\Big(1+\frac{I}{ma^2}\Big) - \cos^2\alpha} \equiv a_s$$

follows immediately. Thus,

$$\begin{aligned} X(t) &= X(0) + \dot{X}(0)\,t + \tfrac{1}{2}a_X t^2 \\ s(t) &= s(0) + \dot{s}(0)\,t + \tfrac{1}{2}a_s t^2\ . \end{aligned}$$

Note that $a_s < 0$ while $a_X > 0$, *i.e.* the hoop rolls down and to the left as the wedge slides to the right. Note that $I = ma^2$ for a hoop; we've computed the answer here for general I.

8.6.4 Pendulum with nonrigid support

A particle of mass m is suspended from a flexible string of length ℓ in a uniform gravitational field. While hanging motionless in equilibrium, it is struck a horizontal blow resulting in an initial angular velocity ω_0. Treating the system as one with *two* degrees of freedom and a constraint, answer the following:

(a) Compute the Lagrangian, the equation of constraint, and the equations of motion.

Solution : The Lagrangian is

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + mgr\,\cos\theta\ .$$

The constraint is $r = \ell$. The equations of motion are

$$\begin{aligned} m\ddot{r} - mr\,\dot{\theta}^2 - mg\,\cos\theta &= \lambda \\ mr^2\,\ddot{\theta} + 2mr\,\dot{r}\,\dot{\theta} - mg\sin\theta &= 0\ . \end{aligned}$$

(b) Compute the tension in the string as a function of angle θ.

Solution : Energy is conserved, hence

$$\tfrac{1}{2}m\ell^2\,\dot{\theta}^2 - mg\ell\cos\theta = \tfrac{1}{2}m\ell^2\,\dot{\theta}_0^2 - mg\ell\cos\theta_0\ .$$

We take $\theta_0 = 0$ and $\dot{\theta}_0 = \omega_0$. Thus,

$$\dot{\theta}^2 = \omega_0^2 - 2\,\Omega^2\left(1 - \cos\theta\right)\ ,$$

with $\Omega = \sqrt{g/\ell}$. Substituting this into the equation for λ, we obtain

$$\lambda = mg\left\{2 - 3\cos\theta - \frac{\omega_0^2}{\Omega^2}\right\}\ .$$

(c) Show that if $\omega_0^2 < 2g/\ell$ then the particle's motion is confined below the horizontal and that the tension in the string is always positive (defined such that positive means exerting a pulling force and negative means exerting a pushing force). Note that the difference between a string and a rigid rod is that the string can only pull but the rod can pull or push. Thus, *the string tension must always be positive or else the string goes "slack"*.

Solution : Since $\dot{\theta}^2 \geq 0$, we must have

$$\frac{\omega_0^2}{2\Omega^2} \geq 1 - \cos\theta\ .$$

The condition for slackness is $\lambda = 0$, or

$$\frac{\omega_0^2}{2\Omega^2} = 1 - \tfrac{3}{2}\cos\theta\ .$$

Thus, if $\omega_0^2 < 2\Omega^2$, we have

$$1 > \frac{\omega_0^2}{2\Omega^2} > 1 - \cos\theta > 1 - \tfrac{3}{2}\cos\theta\ ,$$

and the string never goes slack. Note the last equality follows from $\cos\theta > 0$. The string rises to a maximum angle

$$\theta_{\rm max} = \cos^{-1}\left(1 - \frac{\omega_0^2}{2\Omega^2}\right)\ .$$

(d) Show that if $2g/\ell < \omega_0^2 < 5g/\ell$ the particle rises above the horizontal and the string becomes slack (the tension vanishes) at an angle θ^*. Compute θ^*.

Solution : When $\omega^2 > 2\Omega^2$, the string rises above the horizontal and goes slack at an angle

$$\theta^* = \cos^{-1}\left(\tfrac{2}{3} - \frac{\omega_0^2}{3\Omega^2}\right) .$$

This solution craps out when the string is still taut at $\theta = \pi$, which means $\omega_0^2 = 5\Omega^2$.

(e) Show that if $\omega_0^2 > 5g/\ell$ the tension is always positive and the particle executes circular motion.

Solution : For $\omega_0^2 > 5\Omega^2$, the string never goes slack. Furthermore, $\dot{\theta}$ never vanishes. Therefore, the pendulum undergoes circular motion, albeit not with constant angular velocity.

8.6.5 Falling ladder

A uniform ladder of length ℓ and mass m has one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest and makes an angle θ_0 with respect to the horizontal.

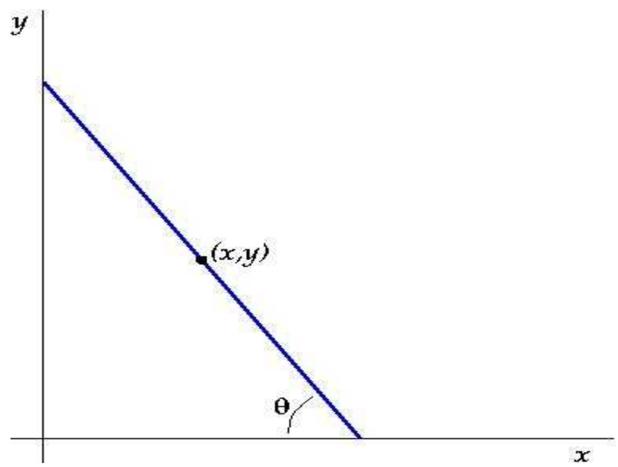


Figure 8.5: A ladder sliding down a wall and across a floor.

(a) Make a convenient choice of generalized coordinates and find the Lagrangian.

Solution : I choose as generalized coordinates the Cartesian coordinates (x, y) of the ladder's center of mass, and the angle θ it makes with respect to the floor. The Lagrangian is then

$$L = \tfrac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\, I\, \dot{\theta}^2 + mgy .$$

There are two constraints: one enforcing contact along the wall, and the other enforcing contact along the floor. These are written

$$G_1(x,y,\theta) = x - \tfrac{1}{2}\,\ell\,\cos\theta = 0$$
$$G_2(x,y,\theta) = y - \tfrac{1}{2}\,\ell\,\sin\theta = 0\ .$$

(b) Prove that the ladder leaves the wall when its upper end has fallen to a height $\frac{2}{3}L\sin\theta_0$.
The equations of motion are

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot q_\sigma}\bigg) - \frac{\partial L}{\partial q_\sigma} = \sum_j \lambda_j\,\frac{\partial G_j}{\partial q_\sigma}\ .$$

Thus, we have

$$\begin{aligned} m\,\ddot x &= \lambda_1 = Q_x \\ m\,\ddot y + mg &= \lambda_2 = Q_y \\ I\,\ddot\theta &= \tfrac{1}{2}\ell\big(\lambda_1\sin\theta - \lambda_2\cos\theta\big) = Q_\theta\ . \end{aligned}$$

We now implement the constraints to eliminate x and y in terms of θ. We have

$$\begin{aligned} \dot x &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot\theta & \qquad \ddot x &= -\tfrac{1}{2}\,\ell\,\cos\theta\,\dot\theta^2 - \tfrac{1}{2}\,\ell\,\sin\theta\,\ddot\theta \\ \dot y &= \tfrac{1}{2}\,\ell\,\cos\theta\,\dot\theta & \qquad \ddot y &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot\theta^2 + \tfrac{1}{2}\,\ell\,\cos\theta\,\ddot\theta\ . \end{aligned}$$

We can now obtain the forces of constraint in terms of the function $\theta(t)$:

$$\begin{aligned} \lambda_1 &= -\tfrac{1}{2}m\ell\,\big(\sin\theta\,\ddot\theta + \cos\theta\,\dot\theta^2\big) \\ \lambda_2 &= +\tfrac{1}{2}m\ell\,\big(\cos\theta\,\ddot\theta - \sin\theta\,\dot\theta^2\big) + mg\ . \end{aligned}$$

We substitute these into the last equation of motion to obtain the result

$$I\,\ddot\theta = -I_0\,\ddot\theta - \tfrac{1}{2}mg\ell\,\cos\theta\ ,$$

or

$$(1+\alpha)\,\ddot\theta = -2\omega_0^2\,\cos\theta\ ,$$

with $I_0 = \frac{1}{4}m\ell^2$, $\alpha \equiv I/I_0$ and $\omega_0 = \sqrt{g/\ell}$. This may be integrated once (multiply by $\dot\theta$ to convert to a total derivative) to yield

$$\tfrac{1}{2}(1+\alpha)\,\dot\theta^2 + 2\,\omega_0^2\,\sin\theta = 2\,\omega_0^2\,\sin\theta_0\ ,$$

which is of course a statement of energy conservation. This,

$$\begin{aligned} \dot\theta^2 &= \frac{4\,\omega_0^2\,(\sin\theta_0 - \sin\theta)}{1+\alpha} \\ \ddot\theta &= -\frac{2\,\omega_0^2\,\cos\theta}{1+\alpha}\ . \end{aligned}$$

We may now obtain $\lambda_1(\theta)$ and $\lambda_2(\theta)$:

$$\lambda_1(\theta) = -\frac{mg}{1+\alpha}\big(3\sin\theta - 2\sin\theta_0\big)\,\cos\theta$$

$$\lambda_2(\theta) = \frac{mg}{1+\alpha}\Big\{\big(3\sin\theta - 2\sin\theta_0\big)\,\sin\theta + \alpha\Big\}\ .$$

Demanding $\lambda_1(\theta) = 0$ gives the detachment angle $\theta = \theta_{\rm d}$, where

$$\sin\theta_{\rm d} = \tfrac{2}{3}\sin\theta_0\ .$$

Note that $\lambda_2(\theta_{\rm d}) = mg\alpha/(1+\alpha) > 0$, so the normal force from the floor is always positive for $\theta > \theta_{\rm d}$. The time to detachment is

$$T_1(\theta_0) = \int\frac{d\theta}{\dot\theta} = \frac{\sqrt{1+\alpha}}{2\,\omega_0}\int\limits_{\theta_{\rm d}}^{\theta_0}\frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}}\ .$$

(c) Show that the subsequent motion can be reduced to quadratures (*i.e.* explicit integrals).

Solution : After the detachment, there is no longer a constraint G_1. The equations of motion are

$$\begin{aligned} m\,\ddot x &= 0 \qquad \text{(conservation of x-momentum)}\\ m\,\ddot y + m\,g &= \lambda\\ I\,\ddot\theta &= -\tfrac{1}{2}\,\ell\,\lambda\,\cos\theta\ , \end{aligned}$$

along with the constraint $y = \frac{1}{2}\,\ell\,\sin\theta$. Eliminating y in favor of θ using the constraint, the second equation yields

$$\lambda = mg - \tfrac{1}{2}m\ell\,\sin\theta\,\dot\theta^2 + \tfrac{1}{2}m\ell\,\cos\theta\,\ddot\theta\ .$$

Plugging this into the third equation of motion, we find

$$I\,\ddot\theta = -2\,I_0\,\omega_0^2\,\cos\theta + I_0\,\sin\theta\,\cos\theta\,\dot\theta^2 - I_0\,\cos^2\theta\,\ddot\theta\ .$$

Multiplying by $\dot\theta$ one again obtains a total time derivative, which is equivalent to rediscovering energy conservation:

$$E = \tfrac{1}{2}\big(I + I_0\,\cos^2\theta\big)\,\dot\theta^2 + 2\,I_0\,\omega_0^2\,\sin\theta\ .$$

By continuity with the first phase of the motion, we obtain the initial conditions for this second phase:

$$\theta = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big)$$

$$\dot\theta = -2\,\omega_0\sqrt{\frac{\sin\theta_0}{3\,(1+\alpha)}}\ .$$

```
In[37]:= T[x_] := NIntegrate[Sqrt[(4 / 3) / (x - Sin[y])], {y, ArcSin[2 x / 3], ArcSin[x] - 10^-9}] / 2

In[38]:= S[x_] := NIntegrate[
           Sqrt[(1 + (4 / 3) (Cos[y]) ^2) / ((1 - (x / 3) ^2) x - Sin[y])], {y, 0, ArcSin[2 x / 3]}] / 2

In[39]:= Q[x_] := T[x] + S[x]

In[43]:= Plot[Q[x], {x, 0, 1}]
```

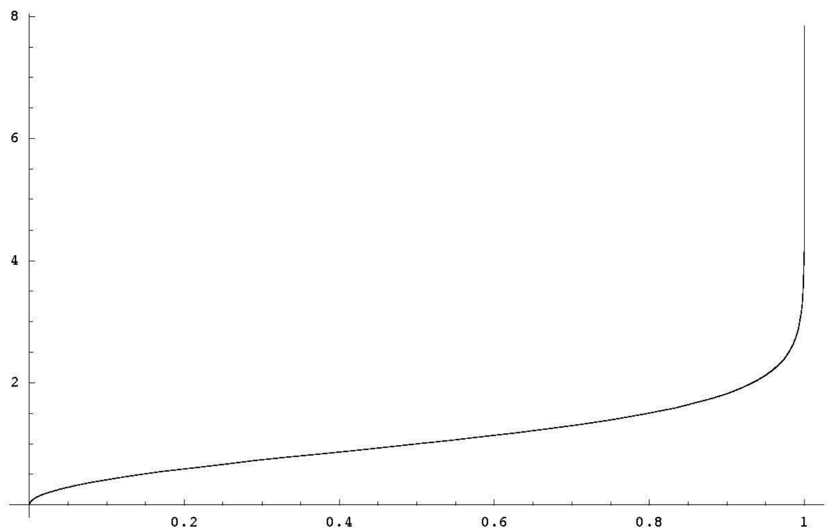


Figure 8.6: Plot of time to fall for the slipping ladder. Here $x = \sin\theta_0$.

Thus,

$$
\begin{aligned}
E &= \tfrac{1}{2}\big(I + I_0 - \tfrac{4}{9}\, I_0 \sin^2\theta_0\big) \cdot \frac{4\,\omega_0^2 \sin\theta_0}{3\,(1+\alpha)} + \tfrac{1}{3}\, mg\ell \sin\theta_0 \\
&= 2\, I_0\, \omega_0^2 \cdot \left\{ 1 + \tfrac{4}{27}\, \frac{\sin^2\theta_0}{1+\alpha} \right\} \sin\theta_0 \ .
\end{aligned}
$$

(d) Find an expression for the time $T(\theta_0)$ it takes the ladder to smack against the floor. Note that, expressed in units of the time scale $\sqrt{L/g}$, T is a dimensionless function of θ_0. Numerically integrate this expression and plot T *versus* θ_0.

Solution : The time from detachment to smack is

$$
T_2(\theta_0) = \int \frac{d\theta}{\dot\theta} = \frac{1}{2\,\omega_0} \int\limits_0^{\theta_{\rm d}} d\theta \sqrt{\frac{1 + \alpha \cos^2\theta}{\big(1 - \frac{4}{27} \frac{\sin^2\theta_0}{1+\alpha}\big) \sin\theta_0 - \sin\theta}} \ .
$$

The total time is then $T(\theta_0) = T_1(\theta_0) + T_2(\theta_0)$. For a uniformly dense ladder, $I = \frac{1}{12}\, m\ell^2 = \frac{1}{3}\, I_0$, so $\alpha = \frac{1}{3}$.

(e) What is the horizontal velocity of the ladder at long times?

Solution : From the moment of detachment, and thereafter,

$$\dot{x} = -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta} = \sqrt{\frac{4\,g\,\ell}{27\,(1+\alpha)}}\;\sin^{3/2}\!\theta_0\ .$$

(f) Describe in words the motion of the ladder subsequent to it slapping against the floor.

Solution : Only a fraction of the ladder's initial potential energy is converted into kinetic energy of horizontal motion. The rest is converted into kinetic energy of vertical motion and of rotation. The slapping of the ladder against the floor is an elastic collision. After the collision, the ladder must rise again, and continue to rise and fall *ad infinitum*, as it slides along with constant horizontal velocity.

8.6.6 Point mass inside rolling hoop

Consider the point mass m inside the hoop of radius R, depicted in Fig. 8.7. We choose as generalized coordinates the Cartesian coordinates (X, Y) of the center of the hoop, the Cartesian coordinates (x, y) for the point mass, the angle ϕ through which the hoop turns, and the angle θ which the point mass makes with respect to the vertical. These six coordinates are not all independent. Indeed, there are only two independent coordinates for this system, which can be taken to be θ and ϕ. Thus, there are *four* constraints:

$$X - R\phi \equiv G_1 = 0 \tag{8.106}$$

$$Y - R \equiv G_2 = 0 \tag{8.107}$$

$$x - X - R\sin\theta \equiv G_3 = 0 \tag{8.108}$$

$$y - Y + R\cos\theta \equiv G_4 = 0\ . \tag{8.109}$$

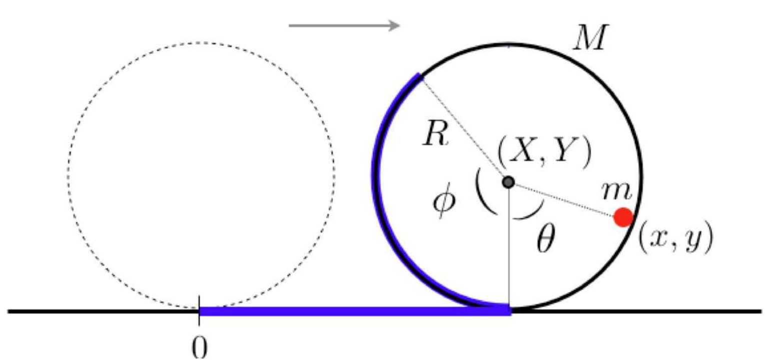


Figure 8.7: A point mass m inside a hoop of mass M, radius R, and moment of inertia I.

The kinetic and potential energies are easily expressed in terms of the Cartesian coordinates, aside from the energy of rotation of the hoop about its CM, which is expressed in terms of

$\dot{\phi}$:

$$T = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 \tag{8.110}$$

$$U = MgY + mgy\ . \tag{8.111}$$

The moment of inertia of the hoop about its CM is $I = MR^2$, but we could imagine a situation in which I were different. For example, we could instead place the point mass inside a very short cylinder with two solid end caps, in which case $I = \frac{1}{2}MR^2$. The Lagrangian is then

$$L = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 - MgY - mgy\ . \tag{8.112}$$

Note that L as written is completely independent of θ and $\dot{\theta}$!

Continuous symmetry

Note that there is an continuous symmetry to L which is satisfied by all the constraints, under

$$\tilde{X}(\zeta) = X + \zeta \qquad \tilde{Y}(\zeta) = Y \tag{8.113}$$

$$\tilde{x}(\zeta) = x + \zeta \qquad \tilde{y}(\zeta) = y \tag{8.114}$$

$$\tilde{\phi}(\zeta) = \phi + \frac{\zeta}{R} \qquad \tilde{\theta}(\zeta) = \theta\ . \tag{8.115}$$

Thus, according to Noether's theorem, there is a conserved quantity

$$\begin{aligned} \Lambda &= \frac{\partial L}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{x}} + \frac{1}{R}\frac{\partial L}{\partial \dot{\phi}} \\ &= M\dot{X} + m\dot{x} + \frac{I}{R}\,\dot{\phi}\ . \end{aligned} \tag{8.116}$$

This means $\dot{\Lambda} = 0$. This reflects the overall conservation of momentum in the x-direction.

Energy conservation

Since neither L nor any of the constraints are explicitly time-dependent, the Hamiltonian is conserved. And since T is homogeneous of degree two in the generalized velocities, we have $H = E = T + U$:

$$E = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 + MgY + mgy\ . \tag{8.117}$$

Equations of motion

We have $n = 6$ generalized coordinates and $k = 4$ constraints. Thus, there are four undetermined multipliers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ used to impose the constraints. This makes for ten unknowns:

$$X \ , \ Y \ , \ x \ , \ y \ , \ \phi \ , \ \theta \ , \ \lambda_1 \ , \ \lambda_2 \ , \ \lambda_3 \ , \ \lambda_4 \ . \tag{8.118}$$

Accordingly, we have ten equations: six equations of motion plus the four equations of constraint. The equations of motion are obtained from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \ . \tag{8.119}$$

Taking each generalized coordinate in turn, the equations of motion are thus

$$M\ddot{X} = \lambda_1 - \lambda_3 \tag{8.120}$$

$$M\ddot{Y} = -Mg + \lambda_2 - \lambda_4 \tag{8.121}$$

$$m\ddot{x} = \lambda_3 \tag{8.122}$$

$$m\ddot{y} = -mg + \lambda_4 \tag{8.123}$$

$$I\,\ddot{\phi} = -R\,\lambda_1 \tag{8.124}$$

$$0 = -R\cos\theta\,\lambda_3 - R\sin\theta\,\lambda_4 \ . \tag{8.125}$$

Along with the four constraint equations, these determine the motion of the system. Note that the last of the equations of motion, for the generalized coordinate $q_\sigma = \theta$, says that $Q_\theta = 0$, which means that the force of constraint on the point mass is radial. Were the point mass replaced by a rolling object, there would be an angular component to this constraint in order that there be no slippage.

Implementation of constraints

We now use the constraint equations to eliminate X, Y, x, and y in terms of θ and ϕ:

$$X = R\phi \quad , \quad Y = R \quad , \quad x = R\phi + R\sin\theta \quad , \quad y = R(1-\cos\theta) \ . \tag{8.126}$$

We also need the derivatives:

$$\dot{x} = R\,\dot{\phi} + R\cos\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\,\ddot{\phi} + R\cos\theta\,\ddot{\theta} - R\sin\theta\,\dot{\theta}^2 \ , \tag{8.127}$$

and

$$\dot{y} = R\sin\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\sin\theta\,\ddot{\theta} + R\cos\theta\,\dot{\theta}^2 \ , \tag{8.128}$$

as well as

$$\dot{X} = R\,\dot{\phi} \quad , \quad \ddot{X} = R\,\ddot{\phi} \quad , \quad \dot{Y} = 0 \quad , \quad \ddot{Y} = 0 \ . \tag{8.129}$$

We now may write the conserved charge as

$$\Lambda = \frac{1}{R}(I + MR^2 + mR^2)\,\dot{\phi} + mR\cos\theta\,\dot{\theta} \ . \tag{8.130}$$

This, in turn, allows us to eliminate $\dot{\phi}$ in terms of $\dot{\theta}$ and the constant Λ:

$$\dot{\phi} = \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR} - \dot{\theta}\cos\theta\right) \ , \tag{8.131}$$

where

$$\gamma = \frac{mR^2}{I + MR^2} \ . \tag{8.132}$$

The energy is then

$$\begin{aligned} E &= \tfrac{1}{2}(I + MR^2)\,\dot{\phi}^2 + \tfrac{1}{2}m\big(R^2\,\dot{\phi}^2 + R^2\,\dot{\theta}^2 + 2R^2\cos\theta\,\dot{\phi}\,\dot{\theta}\big) + MgR + mgR(1-\cos\theta) \\ &= \tfrac{1}{2}mR^2\left\{\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) + \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR}\right)^2 + \frac{2Mg}{mR}\right\} \ . \end{aligned} \tag{8.133}$$

The last two terms inside the big bracket are constant, so we can write this as

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) = \frac{4gk}{R} \ . \tag{8.134}$$

Here, k is a dimensionless measure of the energy of the system, after subtracting the aforementioned constants. If $k > 1$, then $\dot{\theta}^2 > 0$ for all θ, which would result in 'loop-the-loop' motion of the point mass inside the hoop – provided, that is, the normal force of the hoop doesn't vanish and the point mass doesn't detach from the hoop's surface.

Equation motion for $\theta(t)$

The equation of motion for θ obtained by eliminating all other variables from the original set of ten equations is the same as $\dot{E} = 0$, and may be written

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\ddot{\theta} + \left(\frac{\gamma\sin\theta\cos\theta}{1+\gamma}\right)\dot{\theta}^2 = -\frac{g}{R} \ . \tag{8.135}$$

We can use this to write $\ddot{\theta}$ in terms of $\dot{\theta}^2$, and, after invoking eqn. 17.51, in terms of θ itself. We find

$$\dot{\theta}^2 = \frac{4g}{R}\cdot\left(\frac{1+\gamma}{1+\gamma\sin^2\!\theta}\right)\big(k - \sin^2\!\tfrac{1}{2}\theta\big) \tag{8.136}$$

$$\ddot{\theta} = -\frac{g}{R}\cdot\frac{(1+\gamma)\sin\theta}{\big(1+\gamma\sin^2\!\theta\big)^2}\Big[4\gamma\,\big(k - \sin^2\!\tfrac{1}{2}\theta\big)\cos\theta + 1 + \gamma\sin^2\!\theta\Big] \ . \tag{8.137}$$

Forces of constraint

We can solve for the λ_j, and thus obtain the forces of constraint $Q_\sigma = \sum_j \lambda_j \frac{\partial G_j}{\partial q_\sigma}$.

$$\begin{aligned}\lambda_3 &= m\ddot{x} = mR\,\ddot{\phi} + mR\cos\theta\,\ddot{\theta} - mR\sin\theta\,\dot{\theta}^2 \\ &= \frac{mR}{1+\gamma}\Big[\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta\Big]\end{aligned} \tag{8.138}$$

$$\begin{aligned}\lambda_4 &= m\ddot{y} + mg = mg + mR\sin\theta\,\ddot{\theta} + mR\cos\theta\,\dot{\theta}^2 \\ &= mR\Big[\ddot{\theta}\,\sin\theta + \dot{\theta}^2\sin\theta + \frac{g}{R}\Big]\end{aligned} \tag{8.139}$$

$$\lambda_1 = -\frac{I}{R}\,\ddot{\phi} = \frac{(1+\gamma)I}{mR^2}\,\lambda_3 \tag{8.140}$$

$$\lambda_2 = (M+m)g + m\ddot{y} = \lambda_4 + Mg\ . \tag{8.141}$$

One can check that $\lambda_3\cos\theta + \lambda_4\sin\theta = 0$.

The condition that the normal force of the hoop on the point mass vanish is $\lambda_3 = 0$, which entails $\lambda_4 = 0$. This gives

$$-(1+\gamma\sin^2\!\theta)\,\cos\theta = 4(1+\gamma)\left(k - \sin^2\!\tfrac{1}{2}\theta\right)\ . \tag{8.142}$$

Note that this requires $\cos\theta < 0$, *i.e.* the point of detachment lies above the horizontal diameter of the hoop. Clearly if k is sufficiently large, the equality cannot be satisfied, and the point mass executes a periodic 'loop-the-loop' motion. In particular, setting $\theta = \pi$, we find that

$$k_{\rm c} = 1 + \frac{1}{4(1+\gamma)}\ . \tag{8.143}$$

If $k > k_{\rm c}$, then there is periodic 'loop-the-loop' motion. If $k < k_{\rm c}$, then the point mass may detach at a critical angle θ^*, but only if the motion allows for $\cos\theta < 0$. From the energy conservation equation, we have that the maximum value of θ achieved occurs when $\dot{\theta} = 0$, which means

$$\cos\theta_{\rm max} = 1 - 2k\ . \tag{8.144}$$

If $\frac{1}{2} < k < k_{\rm c}$, then, we have the possibility of detachment. This means the energy must be large enough but not too large.

Chapter 9

Central Forces and Orbital Mechanics

9.1 Reduction to a one-body problem

Consider two particles interacting via a potential $U(\boldsymbol{r}_1, \boldsymbol{r}_2) = U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big)$. Such a potential, which depends only on the relative distance between the particles, is called a *central* potential. The Lagrangian of this system is then

$$L = T - U = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \ . \tag{9.1}$$

9.1.1 Center-of-mass (CM) and relative coordinates

The two-body central force problem may always be reduced to two independent one-body problems, by transforming to center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates (see Fig. 9.1), *viz.*

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad \boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\, \boldsymbol{r} \tag{9.2}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\, \boldsymbol{r} \tag{9.3}$$

We then have

$$L = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1{}^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2{}^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \tag{9.4}$$

$$= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r) \ . \tag{9.5}$$

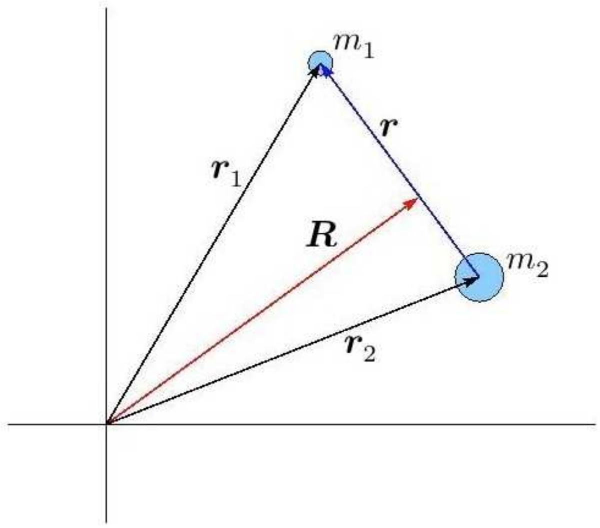


Figure 9.1: Center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates.

where

$$M = m_1 + m_2 \qquad \text{(total mass)} \tag{9.6}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad \text{(reduced mass)} \ . \tag{9.7}$$

9.1.2 Solution to the CM problem

We have $\partial L/\partial \boldsymbol{R} = 0$, which gives $\dot{\boldsymbol{R}}d = 0$ and hence

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)\, t \ . \tag{9.8}$$

Thus, the CM problem is trivial. The center-of-mass moves at constant velocity.

9.1.3 Solution to the relative coordinate problem

`Angular momentum conservation:` We have that $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant of the motion. This means that the motion $\boldsymbol{r}(t)$ is confined to a plane perpendicular to $\boldsymbol{\ell}$. It is convenient to adopt two-dimensional polar coordinates (r, ϕ). The magnitude of $\boldsymbol{\ell}$ is

$$\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}} \tag{9.9}$$

where $d\mathcal{A} = \frac{1}{2} r^2 d\phi$ is the differential element of area subtended relative to the force center. *The relative coordinate vector for a central force problem subtends equal areas in equal times.* This is known as *Kepler's Second Law.*

`Energy conservation:` The equation of motion for the relative coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right)=\frac{\partial L}{\partial \boldsymbol{r}} \quad \Rightarrow \quad \mu\ddot{\boldsymbol{r}}=-\frac{\partial U}{\partial \boldsymbol{r}} \ . \tag{9.10}$$

Taking the dot product with $\dot{\boldsymbol{r}}$, we have

$$\begin{aligned} 0 &= \mu\ddot{\boldsymbol{r}}\cdot\dot{\boldsymbol{r}}+\frac{\partial U}{\partial \boldsymbol{r}}\cdot\dot{\boldsymbol{r}} \\ &= \frac{d}{dt}\Big\{\tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2+U(r)\Big\}=\frac{dE}{dt} \ . \end{aligned} \tag{9.11}$$

Thus, the relative coordinate contribution to the total energy is itself conserved. The total energy is of course $E_{\rm tot}=E+\frac{1}{2}M\dot{\boldsymbol{R}}^2$.

Since $\boldsymbol{\ell}$ is conserved, and since $\boldsymbol{r}\cdot\boldsymbol{\ell}=0$, all motion is confined to a plane perpendicular to $\boldsymbol{\ell}$. Choosing coordinates such that $\hat{\boldsymbol{z}}=\hat{\boldsymbol{\ell}}$, we have

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2+U(r)=\tfrac{1}{2}\mu\dot{r}^2+\frac{\ell^2}{2\mu r^2}+U(r) \\ &= \tfrac{1}{2}\mu\dot{r}^2+U_{\rm eff}(r) \end{aligned} \tag{9.12}$$

$$U_{\rm eff}(r)=\frac{\ell^2}{2\mu r^2}+U(r) \ . \tag{9.13}$$

`Integration of the Equations of Motion, Step I:` The second order equation for $r(t)$ is

$$\frac{dE}{dt}=0 \quad \Rightarrow \quad \mu\ddot{r}=\frac{\ell^2}{\mu r^3}-\frac{dU(r)}{dr}=-\frac{dU_{\rm eff}(r)}{dr} \ . \tag{9.14}$$

However, conservation of energy reduces this to a first order equation, via

$$\dot{r}=\pm\sqrt{\frac{2}{\mu}\Big(E-U_{\rm eff}(r)\Big)} \quad \Rightarrow \quad dt=\pm\frac{\sqrt{\frac{\mu}{2}}\,dr}{\sqrt{E-\frac{\ell^2}{2\mu r^2}-U(r)}} \ . \tag{9.15}$$

This gives $t(r)$, which must be inverted to obtain $r(t)$. In principle this is possible. Note that a constant of integration also appears at this stage – call it $r_0=r(t=0)$.

`Integration of the Equations of Motion, Step II:` After finding $r(t)$ one can integrate to find $\phi(t)$ using the conservation of ℓ:

$$\dot{\phi}=\frac{\ell}{\mu r^2} \quad \Rightarrow \quad d\phi=\frac{\ell}{\mu r^2(t)}\,dt \ . \tag{9.16}$$

This gives $\phi(t)$, and introduces another constant of integration – call it $\phi_0=\phi(t=0)$.

`Pause to Reflect on the Number of Constants:` Confined to the plane perpendicular to $\boldsymbol{\ell}$, the relative coordinate vector has two degrees of freedom. The equations of motion are second order in time, leading to *four* constants of integration. Our four constants are E, ℓ, r_0, and ϕ_0.

The original problem involves two particles, hence six positions and six velocities, making for 12 initial conditions. Six constants are associated with the CM system: $\boldsymbol{R}(0)$ and $\dot{\boldsymbol{R}}(0)$. The six remaining constants associated with the relative coordinate system are $\boldsymbol{\ell}$ (three components), E, r_0, and ϕ_0.

`Geometric Equation of the Orbit:` From $\ell = \mu r^2 \dot{\phi}$, we have

$$\frac{d}{dt} = \frac{\ell}{\mu r^2} \frac{d}{d\phi} \ , \tag{9.17}$$

leading to

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2} F(r) + r \tag{9.18}$$

where $F(r) = -dU(r)/dr$ is the magnitude of the central force. This second order equation may be reduced to a first order one using energy conservation:

$$\begin{aligned} E &= \tfrac{1}{2}\mu \dot{r}^2 + U_{\rm eff}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\rm eff}(r) \ . \end{aligned} \tag{9.19}$$

Thus,

$$d\phi = \pm \frac{\ell}{\sqrt{2\mu}} \cdot \frac{dr}{r^2 \sqrt{E - U_{\rm eff}(r)}} \ , \tag{9.20}$$

which can be integrated to yield $\phi(r)$, and then inverted to yield $r(\phi)$. Note that only one integration need be performed to obtain the geometric shape of the orbit, while two integrations – one for $r(t)$ and one for $\phi(t)$ – must be performed to obtain the full motion of the system.

It is sometimes convenient to rewrite this equation in terms of the variable $s = 1/r$:

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} F\big(s^{-1}\big) \ . \tag{9.21}$$

As an example, suppose the geometric orbit is $r(\phi) = k\, e^{\alpha\phi}$, known as a logarithmic spiral. What is the force? We invoke (9.18), with $s''(\phi) = \alpha^2\, s$, yielding

$$F\big(s^{-1}\big) = -(1+\alpha^2)\,\frac{\ell^2}{\mu}\, s^3 \quad \Rightarrow \quad F(r) = -\frac{C}{r^3} \tag{9.22}$$

with

$$\alpha^2 = \frac{\mu C}{\ell^2} - 1 \ . \tag{9.23}$$

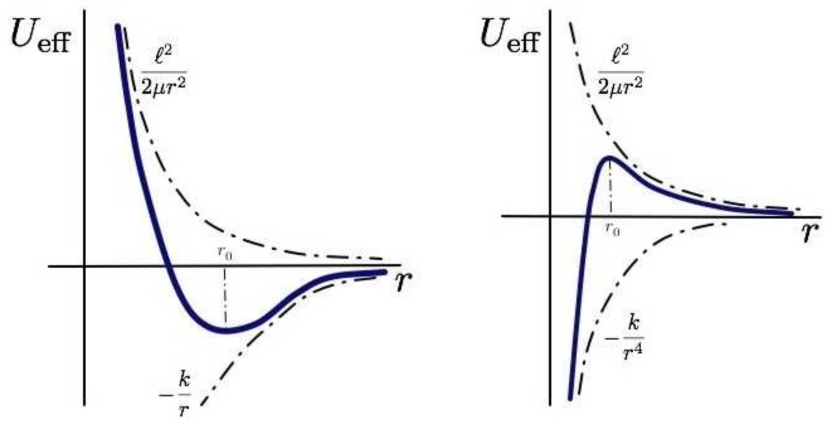


Figure 9.2: Stable and unstable circular orbits. Left panel: $U(r) = -k/r$ produces a stable circular orbit. Right panel: $U(r) = -k/r^4$ produces an unstable circular orbit.

The general solution for $s(\phi)$ for this force law is

$$s(\phi) = \begin{cases} A\cosh(\alpha\phi) + B\sinh(-\alpha\phi) & \text{if } \ell^2 > \mu C \\ \\ A'\,\cos\big(|\alpha|\phi\big) + B'\,\sin\big(|\alpha|\phi\big) & \text{if } \ell^2 < \mu C\ . \end{cases} \tag{9.24}$$

The logarithmic spiral shape is a special case of the first kind of orbit.

9.2 Almost Circular Orbits

A circular orbit with $r(t) = r_0$ satisfies $\ddot{r} = 0$, which means that $U'_{\rm eff}(r_0) = 0$, which says that $F(r_0) = -\ell^2/\mu r_0^3$. This is negative, indicating that a circular orbit is possible only if the force is attractive over some range of distances. Since $\dot{r} = 0$ as well, we must also have $E = U_{\rm eff}(r_0)$. An almost circular orbit has $r(t) = r_0 + \eta(t)$, where $|\eta/r_0| \ll 1$. To lowest order in η, one derives the equations

$$\frac{d^2\eta}{dt^2} = -\omega^2\,\eta \qquad , \qquad \omega^2 = \frac{1}{\mu}\,U''_{\rm eff}(r_0)\ . \tag{9.25}$$

If $\omega^2 > 0$, the circular orbit is *stable* and the perturbation oscillates harmonically. If $\omega^2 < 0$, the circular orbit is *unstable* and the perturbation grows exponentially. For the geometric shape of the perturbed orbit, we write $r = r_0 + \eta$, and from (9.18) we obtain

$$\frac{d^2\eta}{d\phi^2} = \left(\frac{\mu r_0^4}{\ell^2}\,F'(r_0) - 3\right)\eta = -\beta^2\,\eta\ , \tag{9.26}$$

with

$$\beta^2 = 3 + \frac{d\ln F(r)}{d\ln r}\bigg|_{r_0}\ . \tag{9.27}$$

The solution here is

$$\eta(\phi) = \eta_0 \cos\beta(\phi - \delta_0) \ , \tag{9.28}$$

where η_0 and δ_0 are initial conditions. Setting $\eta = \eta_0$, we obtain the sequence of ϕ values

$$\phi_n = \delta_0 + \frac{2\pi n}{\beta} \ , \tag{9.29}$$

at which $\eta(\phi)$ is a local maximum, *i.e.* at *apoapsis*, where $r = r_0 + \eta_0$. Setting $r = r_0 - \eta_0$ is the condition for closest approach, *i.e. periapsis*. This yields the identical set if angles, just shifted by π. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi\big(\beta^{-1} - 1\big) \ , \tag{9.30}$$

is the amount by which the apsides (*i.e.* periapsis and apoapsis) *precess* during each cycle. If $\beta > 1$, the apsides advance, *i.e.* it takes less than a complete revolution $\Delta\phi = 2\pi$ between successive periapses. If $\beta < 1$, the apsides retreat, and it takes longer than a complete revolution between successive periapses. The situation is depicted in Fig. 9.3 for the case $\beta = 1.1$. Below, we will exhibit a soluble model in which the precessing orbit may be determined exactly. Finally, note that if $\beta = p/q$ is a rational number, then the orbit is *closed*, *i.e.* it eventually retraces itself, after every q revolutions.

As an example, let $F(r) = -kr^{-\alpha}$. Solving for a circular orbit, we write

$$U'_{\text{eff}}(r) = \frac{k}{r^\alpha} - \frac{\ell^2}{\mu r^3} = 0 \ , \tag{9.31}$$

which has a solution only for $k > 0$, corresponding to an attractive potential. We then find

$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/(3-\alpha)} \ , \tag{9.32}$$

and $\beta^2 = 3 - \alpha$. The shape of the perturbed orbits follows from $\eta'' = -\beta^2\,\eta$. Thus, while circular orbits exist whenever $k > 0$, small perturbations about these orbits are stable only for $\beta^2 > 0$, *i.e.* for $\alpha < 3$. One then has $\eta(\phi) = A\cos\beta(\phi - \phi_0)$. The perturbed orbits are closed, at least to lowest order in η, for $\alpha = 3 - (p/q)^2$, *i.e.* for $\beta = p/q$. The situation is depicted in Fig. 9.2, for the potentials $U(r) = -k/r$ ($\alpha = 2$) and $U(r) = -k/r^4$ ($\alpha = 5$).

9.3 Precession in a Soluble Model

Let's start with the answer and work backwards. Consider the geometrical orbit,

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos\beta\phi} \ . \tag{9.33}$$

Our interest is in bound orbits, for which $0 \le \epsilon < 1$ (see Fig. 9.3). What sort of potential gives rise to this orbit? Writing $s = 1/r$ as before, we have

$$s(\phi) = s_0\,(1 - \varepsilon\cos\beta\phi) \ . \tag{9.34}$$

Substituting into (9.21), we have

$$\begin{aligned} -\frac{\mu}{\ell^2 s^2}\, F\big(s^{-1}\big) &= \frac{d^2 s}{d\phi^2} + s \\ &= \beta^2\, s_0\, \epsilon \cos\beta\phi + s \\ &= (1-\beta^2)\, s + \beta^2\, s_0 \ , \end{aligned} \tag{9.35}$$

from which we conclude

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \ , \tag{9.36}$$

with

$$k = \beta^2 s_0\, \frac{\ell^2}{\mu} \quad , \quad C = (\beta^2 - 1)\, \frac{\ell^2}{\mu} \ . \tag{9.37}$$

The corresponding potential is

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} + U_\infty \ , \tag{9.38}$$

where U_∞ is an arbitrary constant, conveniently set to zero. If μ and C are given, we have

$$r_0 = \frac{\ell^2}{\mu k} + \frac{C}{k} \quad , \quad \beta = \sqrt{1 + \frac{\mu C}{\ell^2}} \ . \tag{9.39}$$

When $C = 0$, these expressions recapitulate those from the Kepler problem. Note that when $\ell^2 + \mu C < 0$ that the effective potential is monotonically increasing as a function of r. In this case, the angular momentum barrier is overwhelmed by the (attractive, $C < 0$) inverse square part of the potential, and $U_{\rm eff}(r)$ is monotonically increasing. The orbit then passes through the force center. It is a useful exercise to derive the total energy for the orbit,

$$E = (\varepsilon^2 - 1)\, \frac{\mu k^2}{2(\ell^2 + \mu C)} \qquad \Longleftrightarrow \qquad \varepsilon^2 = 1 + \frac{2E(\ell^2 + \mu C)}{\mu k^2} \ . \tag{9.40}$$

9.4 The Kepler Problem: $U(r) = -k\,r^{-1}$

9.4.1 Geometric shape of orbits

The force is $F(r) = -kr^{-2}$, hence the equation for the geometric shape of the orbit is

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\, F(s^{-1}) = \frac{\mu k}{\ell^2} \ , \tag{9.41}$$

with $s = 1/r$. Thus, the most general solution is

$$s(\phi) = s_0 - C \cos(\phi - \phi_0) \ , \tag{9.42}$$

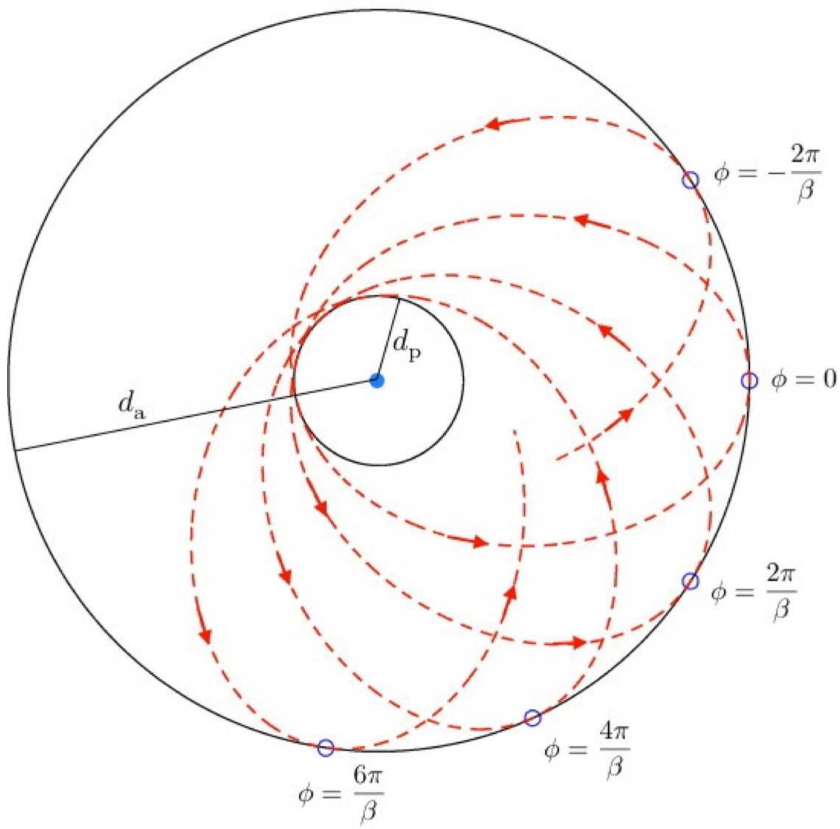


Figure 9.3: Precession in a soluble model, with geometric orbit $r(\phi) = r_0/(1 - \varepsilon \cos \beta\phi)$, shown here with $\beta = 1.1$. Periapsis and apoapsis advance by $\Delta\phi = 2\pi(1 - \beta^{-1})$ per cycle.

where C and ϕ_0 are constants. Thus,

$$r(\phi) = \frac{r_0}{1 - \varepsilon \cos(\phi - \phi_0)} \; , \tag{9.43}$$

where $r_0 = \ell^2/\mu k$ and where we have defined a new constant $\varepsilon \equiv C r_0$.

9.4.2 Laplace-Runge-Lenz vector

Consider the vector

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{\ell} - \mu k \, \hat{\boldsymbol{r}} \; , \tag{9.44}$$

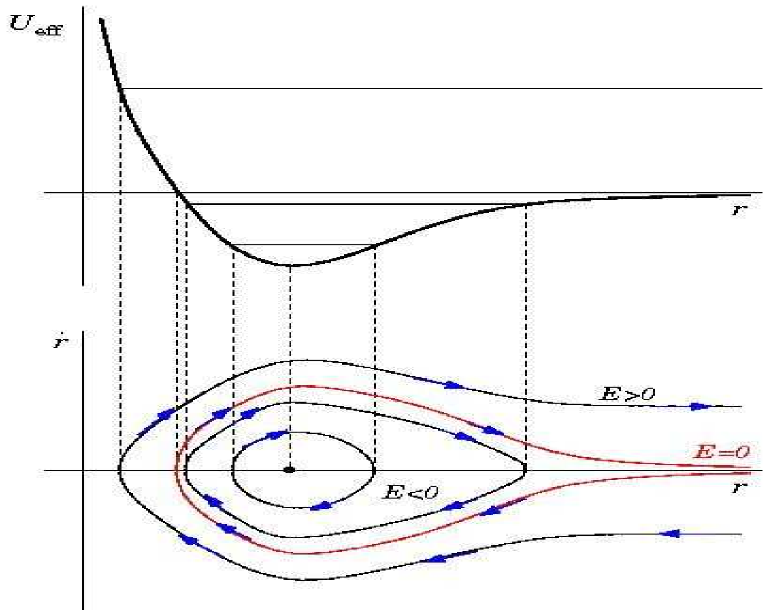


Figure 9.4: The effective potential for the Kepler problem, and associated phase curves. The orbits are geometrically described as conic sections: hyperbolae ($E > 0$), parabolae ($E = 0$), ellipses ($E_{\rm min} < E < 0$), and circles ($E = E_{\rm min}$).

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector pointing in the direction of $\boldsymbol{r}$. We may now show that $\boldsymbol{A}$ is conserved:

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} &= \frac{d}{dt}\Big\{\boldsymbol{p}\times\boldsymbol{\ell} - \mu k\frac{\boldsymbol{r}}{r}\Big\} \\ &= \dot{\boldsymbol{p}}\times\boldsymbol{\ell} + \boldsymbol{p}\times\dot{\boldsymbol{\ell}} - \mu k\,\frac{r\dot{\boldsymbol{r}} - \boldsymbol{r}\dot r}{r^2} \\ &= -\frac{k\boldsymbol{r}}{r^3}\times(\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot r\boldsymbol{r}}{r^2} \\ &= -\mu k\,\frac{\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})}{r^3} + \mu k\,\frac{\dot{\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{r})}{r^3} - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot r\boldsymbol{r}}{r^2} = 0\ .\end{aligned} \tag{9.45}$$

So $\boldsymbol{A}$ is a conserved vector which clearly lies in the plane of the motion. $\boldsymbol{A}$ points toward periapsis, *i.e.* toward the point of closest approach to the force center.

Let's assume *apo*apsis occurs at $\phi = \phi_0$. Then

$$\boldsymbol{A}\cdot\boldsymbol{r} = -Ar\cos(\phi - \phi_0) = \ell^2 - \mu k r \tag{9.46}$$

giving

$$r(\phi) = \frac{\ell^2}{\mu k - A\cos(\phi-\phi_0)} = \frac{a(1-\varepsilon^2)}{1 - \varepsilon\cos(\phi-\phi_0)}\ , \tag{9.47}$$

where

$$\varepsilon = \frac{A}{\mu k} \qquad , \qquad a(1-\varepsilon^2) = \frac{\ell^2}{\mu k}\ . \tag{9.48}$$

The orbit is a *conic section* with eccentricity ε. Squaring $\boldsymbol{A}$, one finds

$$\begin{aligned}\boldsymbol{A}^2 &= (\boldsymbol{p}\times\boldsymbol{\ell})^2 - 2\mu k\,\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\times\boldsymbol{\ell} + \mu^2k^2 \\ &= p^2\ell^2 - 2\mu\ell^2\,\frac{k}{r} + \mu^2k^2 \\ &= 2\mu\ell^2\bigg(\frac{p^2}{2\mu} - \frac{k}{r} + \frac{\mu k^2}{2\ell^2}\bigg) = 2\mu\ell^2\bigg(E + \frac{\mu k^2}{2\ell^2}\bigg)\end{aligned} \tag{9.49}$$

and thus

$$a = -\frac{k}{2E} \quad , \quad \varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} \ . \tag{9.50}$$

9.4.3 Kepler orbits are conic sections

There are four classes of conic sections:

- *Circle:* $\varepsilon = 0$, $E = -\mu k^2/2\ell^2$, radius $a = \ell^2/\mu k$. The force center lies at the center of circle.
- *Ellipse:* $0 < \varepsilon < 1$, $-\mu k^2/2\ell^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1-\varepsilon^2}$. The force center is at one of the foci.
- *Parabola:* $\varepsilon = 1$, $E = 0$, force center is the focus.
- *Hyperbola:* $\varepsilon > 1$, $E > 0$, force center is closest focus (attractive) or farthest focus (repulsive).

To see that the Keplerian orbits are indeed conic sections, consider the ellipse of Fig. 9.6. The law of cosines gives

$$\rho^2 = r^2 + 4f^2 - 4rf\cos\phi \ , \tag{9.51}$$

where $f = \varepsilon a$ is the focal distance. Now for any point on an ellipse, the sum of the distances to the left and right foci is a constant, and taking $\phi = 0$ we see that this constant is $2a$. Thus, $\rho = 2a - r$, and we have

$$\begin{aligned}(2a-r)^2 = 4a^2 - 4ar + r^2 &= r^2 + 4\varepsilon^2a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(1-\varepsilon\cos\phi) &= a(1-\varepsilon^2) \ .\end{aligned} \tag{9.52}$$

Thus, we obtain

$$r(\phi) = \frac{a\,(1-\varepsilon^2)}{1-\varepsilon\cos\phi} \ , \tag{9.53}$$

and we therefore conclude that

$$r_0 = \frac{\ell^2}{\mu k} = a\,(1-\varepsilon^2) \ . \tag{9.54}$$

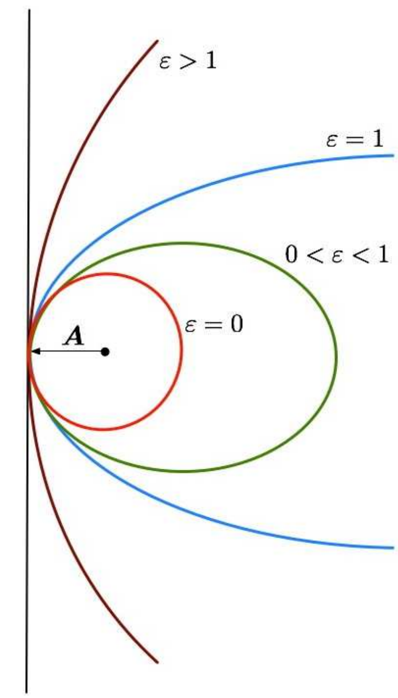


Figure 9.5: Keplerian orbits are conic sections, classified according to eccentricity: hyperbola ($\epsilon > 1$), parabola ($\epsilon = 1$), ellipse ($0 < \epsilon < 1$), and circle ($\epsilon = 0$). The Laplace-Runge-Lenz vector, $\boldsymbol{A}$, points toward periapsis.

Next let us examine the energy,

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \tfrac{1}{2}\mu\bigg(\frac{\ell}{\mu r^2}\,\frac{dr}{d\phi}\bigg)^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r} \\ &= \frac{\ell^2}{2\mu}\bigg(\frac{ds}{d\phi}\bigg)^2 + \frac{\ell^2}{2\mu}\,s^2 - ks\ , \end{aligned} \tag{9.55}$$

with

$$s = \frac{1}{r} = \frac{\mu k}{\ell^2}\Big(1 - \varepsilon\cos\phi\Big)\ . \tag{9.56}$$

Thus,

$$\frac{ds}{d\phi} = \frac{\mu k}{\ell^2}\,\varepsilon\,\sin\phi\ , \tag{9.57}$$

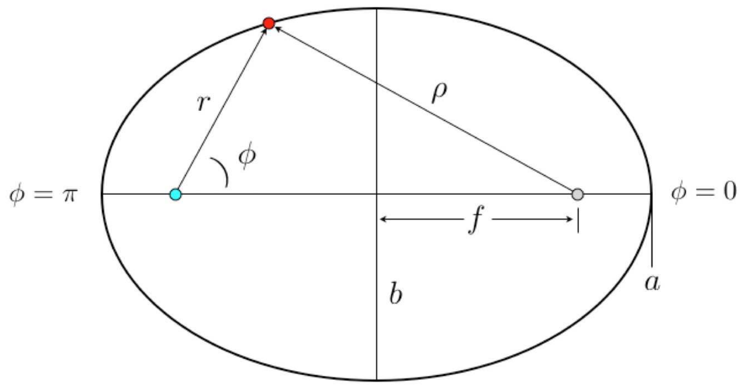


Figure 9.6: The Keplerian ellipse, with the force center at the left focus. The focal distance is $f = \varepsilon a$, where a is the semimajor axis length. The length of the semiminor axis is $b = \sqrt{1-\varepsilon^2}\,a$.

and

$$\begin{aligned}\left(\frac{ds}{d\phi}\right)^2 &= \frac{\mu^2 k^2}{\ell^4}\,\varepsilon^2\,\sin^2\!\phi \\ &= \frac{\mu^2 k^2 \varepsilon^2}{\ell^4} - \left(\frac{\mu k}{\ell^2} - s\right)^2 \\ &= -s^2 + \frac{2\mu k}{\ell^2}\,s + \frac{\mu^2 k^2}{\ell^4}\,\big(\varepsilon^2 - 1\big) \ . \end{aligned} \tag{9.58}$$

Substituting this into eqn. 9.55, we obtain

$$E = \frac{\mu k^2}{2\ell^2}\,\big(\varepsilon^2 - 1\big) \ . \tag{9.59}$$

For the hyperbolic orbit, depicted in Fig. 9.7, we have $r - \rho = \mp 2a$, depending on whether we are on the attractive or repulsive branch, respectively. We then have

$$\begin{aligned}(r \pm 2a)^2 = 4a^2 \pm 4ar + r^2 &= r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \\ \Rightarrow \quad r(\pm 1 + \varepsilon\cos\phi) &= a(\varepsilon^2 - 1) \ . \end{aligned} \tag{9.60}$$

This yields

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon\cos\phi} \ . \tag{9.61}$$

9.4.4 Period of bound Kepler orbits

From $\ell = \mu r^2\dot\phi = 2\mu\dot{\mathcal{A}}$, the period is $\tau = 2\mu\mathcal{A}/\ell$, where $\mathcal{A} = \pi a^2\sqrt{1-\varepsilon^2}$ is the area enclosed by the orbit. This gives

$$\tau = 2\pi\left(\frac{\mu a^3}{k}\right)^{1/2} = 2\pi\left(\frac{a^3}{GM}\right)^{1/2} \tag{9.62}$$

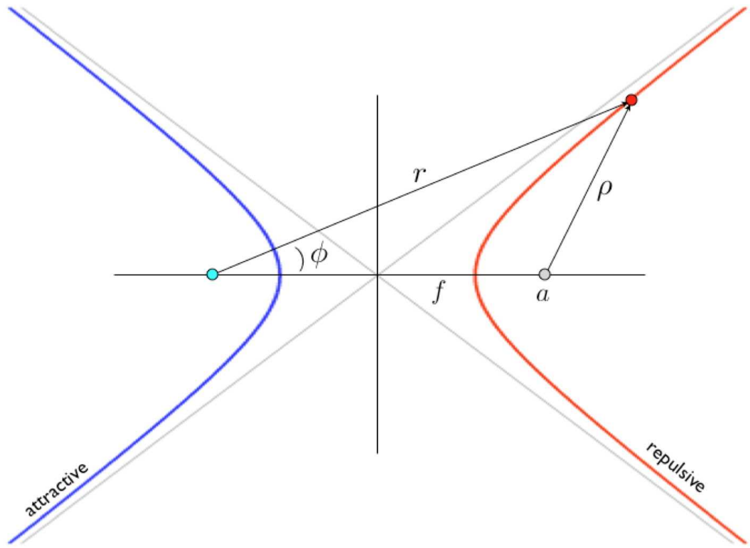


Figure 9.7: The Keplerian hyperbolae, with the force center at the left focus. The left (blue) branch corresponds to an attractive potential, while the right (red) branch corresponds to a repulsive potential. The equations of these branches are $r = \rho = \mp 2a$, where the top sign corresponds to the left branch and the bottom sign to the right branch.

as well as

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} \ , \tag{9.63}$$

where $k = Gm_1 m_2$ and $M = m_1 + m_2$ is the total mass. For planetary orbits, $m_1 = M_\odot$ is the solar mass and $m_2 = m_\text{p}$ is the planetary mass. We then have

$$\frac{a^3}{\tau^2} = \left(1 + \frac{m_\text{p}}{M_\odot}\right)\frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2} \ , \tag{9.64}$$

which is to an excellent approximation independent of the planetary mass. (Note that $m_\text{p}/M_\odot \approx 10^{-3}$ even for Jupiter.) This analysis also holds, *mutatis mutandis,* for the case of satellites orbiting the earth, and indeed in any case where the masses are grossly disproportionate in magnitude.

9.4.5 Escape velocity

The threshold for escape from a gravitational potential occurs at $E = 0$. Since $E = T + U$ is conserved, we determine the *escape velocity* for a body a distance r from the force center by setting

$$E = 0 = \tfrac{1}{2}\mu v_\text{esc}^2(t) - \frac{GMm}{r} \quad \Rightarrow \quad v_\text{esc}(r) = \sqrt{\frac{2G(M+m)}{r}} \ . \tag{9.65}$$

When $M \gg m$, $v_\text{esc}(r) = \sqrt{2GM/r}$. Thus, for an object at the surface of the earth, $v_\text{esc} = \sqrt{2gR_\text{E}} = 11.2\,\text{km/s}$.

9.4.6 Satellites and spacecraft

A satellite in a circular orbit a distance h above the earth's surface has an orbital period

$$\tau = \frac{2\pi}{\sqrt{GM_{\rm E}}}\,(R_{\rm E}+h)^{3/2} \ , \tag{9.66}$$

where we take $m_{\rm satellite} \ll M_{\rm E}$. For low earth orbit (LEO), $h \ll R_{\rm E} = 6.37\times 10^6$ m, in which case $\tau_{\rm LEO} = 2\pi\sqrt{R_{\rm E}/g} = 1.4\,{\rm hr}$.

Consider a weather satellite in an elliptical orbit whose closest approach to the earth (perigee) is 200 km above the earth's surface and whose farthest distance (apogee) is 7200 km above the earth's surface. What is the satellite's orbital period? From Fig. 9.6, we see that

$$\begin{aligned} d_{\rm apogee} &= R_{\rm E} + 7200\,{\rm km} = 13571\,{\rm km} \\ d_{\rm perigee} &= R_{\rm E} + 200\,{\rm km} = 6971\,{\rm km} \\ a &= \tfrac{1}{2}(d_{\rm apogee} + d_{\rm perigee}) = 10071\,{\rm km} \ . \end{aligned} \tag{9.67}$$

We then have

$$\tau = \left(\frac{a}{R_{\rm E}}\right)^{3/2} \cdot \tau_{\rm LEO} \approx 2.65\,{\rm hr} \ . \tag{9.68}$$

What happens if a spacecraft in orbit about the earth fires its rockets? Clearly the energy and angular momentum of the orbit will change, and this means the shape will change. If the rockets are fired (in the direction of motion) at perigee, then perigee itself is unchanged, because $\boldsymbol{v}\cdot\boldsymbol{r} = 0$ is left unchanged at this point. However, E is increased, hence the eccentricity $\varepsilon = \sqrt{1+\frac{2E\ell^2}{\mu k^2}}$ increases. This is the most efficient way of boosting a satellite into an orbit with higher eccentricity. Conversely, and somewhat paradoxically, when a satellite in LEO loses energy due to frictional drag of the atmosphere, the energy E decreases. Initially, because the drag is weak and the atmosphere is isotropic, the orbit remains circular. Since E decreases, $\langle T\rangle = -E$ must *increase*, which means that the frictional forces cause the satellite to speed up!

9.4.7 Two examples of orbital mechanics

- Problem #1: At perigee of an elliptical Keplerian orbit, a satellite receives an impulse $\Delta\boldsymbol{p} = p_0\hat{\boldsymbol{r}}$. Describe the resulting orbit.

- ◦ Solution #1: Since the impulse is radial, the angular momentum $\boldsymbol{\ell} = \boldsymbol{r}\times\boldsymbol{p}$ is unchanged. The energy, however, does change, with $\Delta E = p_0^2/2\mu$. Thus,

$$\varepsilon_{\rm f}^2 = 1 + \frac{2E_{\rm f}\ell^2}{\mu k^2} = \varepsilon_{\rm i}^2 + \left(\frac{\ell p_0}{\mu k}\right)^2 \ . \tag{9.69}$$

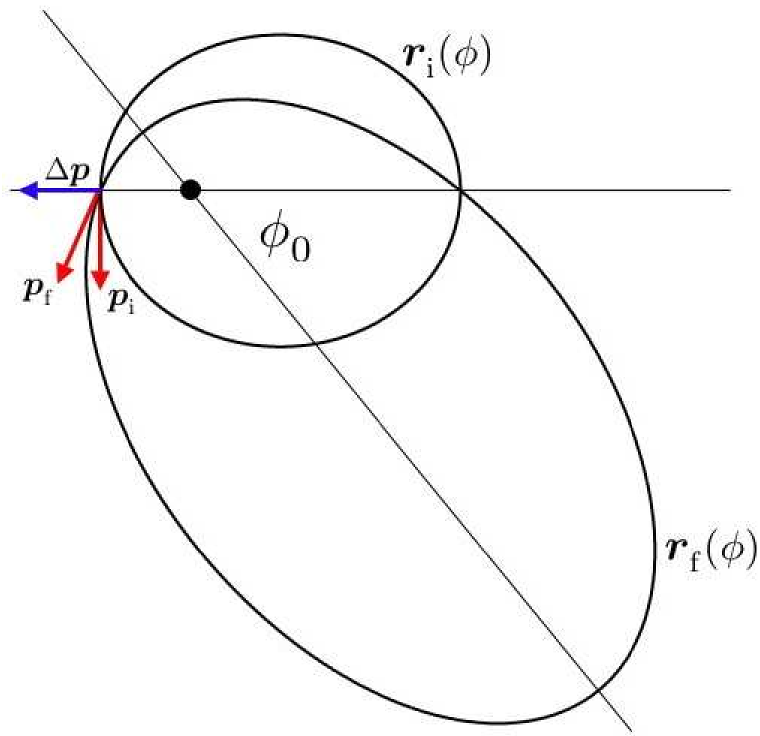


Figure 9.8: At perigee of an elliptical orbit $r_{\rm i}(\phi)$, a radial impulse $\Delta\boldsymbol{p}$ is applied. The shape of the resulting orbit $r_{\rm f}(\phi)$ is shown.

The new semimajor axis length is

$$\begin{aligned} a_{\rm f} &= \frac{\ell^2/\mu k}{1-\varepsilon_{\rm f}^2} = a_{\rm i} \cdot \frac{1-\varepsilon_{\rm i}^2}{1-\varepsilon_{\rm f}^2} \\ &= \frac{a_{\rm i}}{1-(a_{\rm i} p_0^2/\mu k)} \ . \end{aligned} \tag{9.70}$$

The shape of the final orbit must also be a Keplerian ellipse, described by

$$r_{\rm f}(\phi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1-\varepsilon_{\rm f}\cos(\phi+\delta)} \ , \tag{9.71}$$

where the phase shift δ is determined by setting

$$r_{\rm i}(\pi) = r_{\rm f}(\pi) = \frac{\ell^2}{\mu k} \cdot \frac{1}{1+\varepsilon_{\rm i}} \ . \tag{9.72}$$

Solving for δ, we obtain

$$\delta = \cos^{-1}\big(\varepsilon_{\rm i}/\varepsilon_{\rm f}\big) \ . \tag{9.73}$$

The situation is depicted in Fig. 9.8.

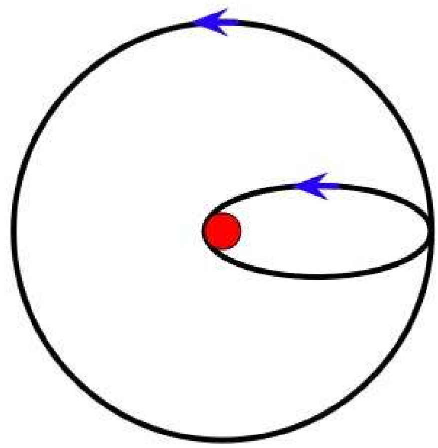


Figure 9.9: The larger circular orbit represents the orbit of the earth. The elliptical orbit represents that for an object orbiting the Sun with distance at perihelion equal to the Sun's radius.

- Problem #2: Which is more energy efficient – to send nuclear waste outside the solar system, or to send it into the Sun?

∘ Solution #2: Escape velocity for the solar system is $v_{\rm esc,\odot}(r) = \sqrt{GM_\odot/r}$. At a distance $a_{\rm E}$, we then have $v_{\rm esc,\odot}(a_{\rm E}) = \sqrt{2}\,v_{\rm E}$, where $v_{\rm E} = \sqrt{GM_\odot/a_{\rm E}} = 2\pi a_{\rm E}/\tau_{\rm E} = 29.9\,{\rm km/s}$ is the velocity of the earth in its orbit. The satellite is launched from earth, and clearly the most energy efficient launch will be one in the direction of the earth's motion, in which case the velocity after escape from earth must be $u = \big(\sqrt{2}-1\big)v_{\rm E} = 12.4\,{\rm km/s}$. The speed just above the earth's atmosphere must then be $\tilde{u}$, where

$$\tfrac{1}{2}m\tilde{u}^2 - \frac{GM_{\rm E}m}{R_{\rm E}} = \tfrac{1}{2}mu^2 \ , \tag{9.74}$$

or, in other words,

$$\tilde{u}^2 = u^2 + v^2_{\rm esc,E} \ . \tag{9.75}$$

We compute $\tilde{u} = 16.7\,{\rm km/s}$.

The second method is to place the trash ship in an elliptical orbit whose perihelion is the Sun's radius, $R_\odot = 6.98\times10^8\,{\rm m}$, and whose aphelion is $a_{\rm E}$. Using the general equation $r(\phi) = (\ell^2/\mu k)/(1-\varepsilon\cos\phi)$ for a Keplerian ellipse, we therefore solve the two equations

$$r(\phi=\pi) = R_\odot = \frac{1}{1+\varepsilon}\cdot\frac{\ell^2}{\mu k} \tag{9.76}$$

$$r(\phi=0) = a_{\rm E} = \frac{1}{1-\varepsilon}\cdot\frac{\ell^2}{\mu k} \ . \tag{9.77}$$

We thereby obtain

$$\varepsilon = \frac{a_{\rm E}-R_\odot}{a_{\rm E}+R_\odot} = 0.991 \ , \tag{9.78}$$

which is a very eccentric ellipse, and

$$\frac{\ell^2}{\mu k} = \frac{a_{\rm E}^2\, v^2}{G(M_\odot + m)} \approx a_{\rm E} \cdot \frac{v^2}{v_{\rm E}^2}$$
$$= (1-\varepsilon)\, a_{\rm E} = \frac{2a_{\rm E} R_\odot}{a_{\rm E} + R_\odot} \ . \tag{9.79}$$

Hence,

$$v^2 = \frac{2R_\odot}{a_{\rm E} + R_\odot}\, v_{\rm E}^2 \ , \tag{9.80}$$

and the necessary velocity relative to earth is

$$u = \left(\sqrt{\frac{2R_\odot}{a_{\rm E} + R_\odot}} - 1\right) v_{\rm E} \approx -0.904\, v_{\rm E} \ , \tag{9.81}$$

i.e. $u = -27.0\,{\rm km/s}$. Launch is in the opposite direction from the earth's orbital motion, and from $\tilde{u}^2 = u^2 + v_{\rm esc,E}^2$ we find $\tilde{u} = -29.2\,{\rm km/s}$, which is larger (in magnitude) than in the first scenario. Thus, it is cheaper to ship the trash out of the solar system than to send it crashing into the Sun, by a factor $\tilde{u}_{\rm I}^2/\tilde{u}_{\rm II}^2 = 0.327$.

9.5 Appendix I : Mission to Neptune

Four earth-launched spacecraft have escaped the solar system: *Pioneer 10* (launch 3/3/72), *Pioneer 11* (launch 4/6/73), *Voyager 1* (launch 9/5/77), and *Voyager 2* (launch 8/20/77).[1] The latter two are still functioning, and each are moving away from the Sun at a velocity of roughly 3.5 AU/yr.

As the first objects of earthly origin to leave our solar system, both *Pioneer* spacecraft featured a graphic message in the form of a 6" x 9" gold anodized plaque affixed to the spacecrafts' frame. This plaque was designed in part by the late astronomer and popular science writer Carl Sagan. The humorist Dave Barry, in an essay entitled *Bring Back Carl's Plaque*, remarks,

> But the really bad part is what they put on the plaque. I mean, if we're going to have a plaque, it ought to at least show the aliens what we're really like, right? Maybe a picture of people eating cheeseburgers and watching "The Dukes of Hazzard." Then if aliens found it, they'd say, "Ah. Just plain folks."
>
> But no. Carl came up with this incredible science-fair-wimp plaque that features drawings of – you are not going to believe this – a hydrogen atom and naked people. To represent the entire Earth! This is crazy! Walk the streets of any town on this planet, and the two things you will almost never see are hydrogen atoms and naked people.

[1]There is a very nice discussion in the Barger and Olsson book on 'Grand Tours of the Outer Planets'. Here I reconstruct and extend their discussion.

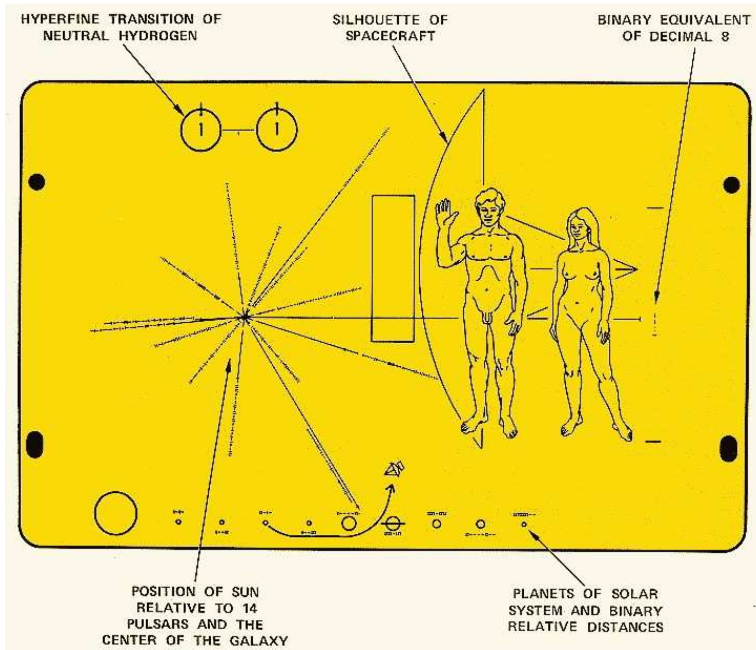


Figure 9.10: The unforgivably dorky *Pioneer 10* and *Pioneer 11* plaque.

During August, 1989, *Voyager 2* investigated the planet Neptune. A direct trip to Neptune along a Keplerian ellipse with $r_{\rm p} = a_{\rm E} = 1\,{\rm AU}$ and $r_{\rm a} = a_{\rm N} = 30.06\,{\rm AU}$ would take 30.6 years. To see this, note that $r_{\rm p} = a\,(1-\varepsilon)$ and $r_{\rm a} = a\,(1+\varepsilon)$ yield

$$a = \tfrac{1}{2}\big(a_{\rm E} + a_{\rm N}\big) = 15.53\,{\rm AU} \quad , \quad \varepsilon = \frac{a_{\rm N} - a_{\rm E}}{a_{\rm N} + a_{\rm E}} = 0.9356 \ . \tag{9.82}$$

Thus,

$$\tau = \tfrac{1}{2}\,\tau_{\rm E} \cdot \left(\frac{a}{a_{\rm E}}\right)^{3/2} = 30.6\,{\rm yr} \ . \tag{9.83}$$

The energy cost per kilogram of such a mission is computed as follows. Let the speed of the probe after its escape from earth be $v_{\rm p} = \lambda v_{\rm E}$, and the speed just above the atmosphere (*i.e.* neglecting atmospheric friction) is v_0. For the most efficient launch possible, the probe is shot in the direction of earth's instantaneous motion about the Sun. Then we must have

$$\tfrac{1}{2} m\, v_0^2 - \frac{G M_{\rm E} m}{R_{\rm E}} = \tfrac{1}{2} m\, (\lambda - 1)^2\, v_{\rm E}^2 \ , \tag{9.84}$$

since the speed of the probe in the frame of the earth is $v_{\rm p} - v_{\rm E} = (\lambda - 1)\, v_{\rm E}$. Thus,

$$\frac{E}{m} = \tfrac{1}{2} v_0^2 = \Big[\tfrac{1}{2}(\lambda - 1)^2 + h\Big]\, v_{\rm E}^2 \tag{9.85}$$

$$v_{\rm E}^2 = \frac{G M_\odot}{a_{\rm E}} = 6.24 \times 10^7\, R_{\rm J}/{\rm kg} \ ,$$

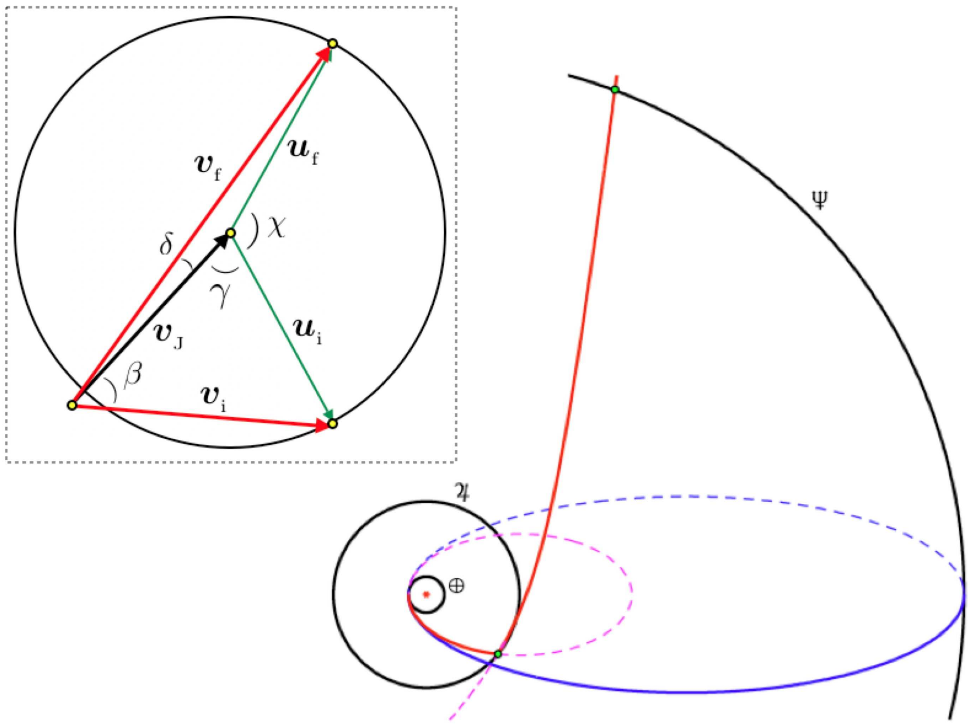


Figure 9.11: Mission to Neptune. The figure at the lower right shows the orbits of Earth, Jupiter, and Neptune in black. The cheapest (in terms of energy) direct flight to Neptune, shown in blue, would take 30.6 years. By swinging past the planet Jupiter, the satellite can pick up great speed and with even less energy the mission time can be cut to 8.5 years (red curve). The inset in the upper left shows the scattering event with Jupiter.

where

$$h \equiv \frac{M_{\rm E}}{M_\odot} \cdot \frac{a_{\rm E}}{R_{\rm E}} = 7.050 \times 10^{-2} \ . \tag{9.86}$$

Therefore, a convenient dimensionless measure of the energy is

$$\eta \equiv \frac{2E}{mv_{\rm E}^2} = \frac{v_0^2}{v_{\rm E}^2} = (\lambda - 1)^2 + 2h \ . \tag{9.87}$$

As we shall derive below, a direct mission to Neptune requires

$$\lambda \geq \sqrt{\frac{2a_{\rm N}}{a_{\rm N} + a_{\rm E}}} = 1.3913 \ , \tag{9.88}$$

which is close to the criterion for escape from the solar system, $\lambda_{\rm esc} = \sqrt{2}$. Note that about 52% of the energy is expended after the probe escapes the Earth's pull, and 48% is expended in liberating the probe from Earth itself.

This mission can be done much more economically by taking advantage of a Jupiter flyby, as shown in Fig. 9.11. The idea of a flyby is to steal some of Jupiter's momentum and then fly away very fast before Jupiter realizes and gets angry. The CM frame of the probe-Jupiter system is of course the rest frame of Jupiter, and in this frame conservation of energy means that the final velocity $\boldsymbol{u}_{\rm f}$ is of the same magnitude as the initial velocity $\boldsymbol{u}_{\rm i}$. However, in the frame of the Sun, the initial and final velocities are $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm i}$ and $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm f}$, respectively, where $\boldsymbol{v}_{\rm J}$ is the velocity of Jupiter in the rest frame of the Sun. If, as shown in the inset to Fig. 9.11, $\boldsymbol{u}_{\rm f}$ is roughly parallel to $\boldsymbol{v}_{\rm J}$, the probe's velocity in the Sun's frame will be enhanced. Thus, the motion of the probe is broken up into three segments:

I : Earth to Jupiter
II : Scatter off Jupiter's gravitational pull
III : Jupiter to Neptune

We now analyze each of these segments in detail. In so doing, it is useful to recall that the general form of a Keplerian orbit is

$$r(\phi) = \frac{d}{1 - \varepsilon\cos\phi} \quad , \quad d = \frac{\ell^2}{\mu k} = \big|\varepsilon^2 - 1\big|\, a \ . \tag{9.89}$$

The energy is

$$E = (\varepsilon^2 - 1)\,\frac{\mu k^2}{2\ell^2} \ , \tag{9.90}$$

with $k = GMm$, where M is the mass of either the Sun or a planet. In either case, M dominates, and $\mu = Mm/(M+m) \simeq m$ to extremely high accuracy. The time for the trajectory to pass from $\phi = \phi_1$ to $\phi = \phi_2$ is

$$T = \int\! dt = \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\dot\phi} = \frac{\mu}{\ell}\int\limits_{\phi_1}^{\phi_2} \! d\phi\, r^2(\phi) = \frac{\ell^3}{\mu k^2} \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\big[1 - \varepsilon\cos\phi\big]^2} \ . \tag{9.91}$$

For reference,

$$a_{\rm E} = 1\,{\rm AU} \qquad a_{\rm J} = 5.20\,{\rm AU} \qquad a_{\rm N} = 30.06\,{\rm AU}$$
$$M_{\rm E} = 5.972\times 10^{24}\,{\rm kg} \qquad M_{\rm J} = 1.900\times 10^{27}\,{\rm kg} \qquad M_\odot = 1.989\times 10^{30}\,{\rm kg}$$

with $1\,{\rm AU} = 1.496\times 10^8\,{\rm km}$. Here $a_{\rm E,J,N}$ and $M_{\rm E,J,\odot}$ are the orbital radii and masses of Earth, Jupiter, and Neptune, and the Sun. The last thing we need to know is the radius of Jupiter,

$$R_{\rm J} = 9.558\times 10^{-4}\,{\rm AU} \ .$$

We need $R_{\rm J}$ because the distance of closest approach to Jupiter, or *perijove*, must be $R_{\rm J}$ or greater, or else the probe crashes into Jupiter!

9.5.1 I. Earth to Jupiter

The probe's velocity at perihelion is $v_{\rm p} = \lambda v_{\rm E}$. The angular momentum is $\ell = \mu a_{\rm E}\cdot\lambda v_{\rm E}$, whence

$$d = \frac{(a_{\rm E}\lambda v_{\rm E})^2}{GM_\odot} = \lambda^2\, a_{\rm E} \ . \tag{9.92}$$

From $r(\pi) = a_{\rm E}$, we obtain

$$\varepsilon_{\rm I} = \lambda^2 - 1 \ . \tag{9.93}$$

This orbit will intersect the orbit of Jupiter if $r_{\rm a} \geq a_{\rm J}$, which means

$$\frac{d}{1-\varepsilon_{\rm I}} \geq a_{\rm J} \quad \Rightarrow \quad \lambda \geq \sqrt{\frac{2a_{\rm J}}{a_{\rm J}+a_{\rm E}}} = 1.2952 \ . \tag{9.94}$$

If this inequality holds, then intersection of Jupiter's orbit will occur for

$$\phi_{\rm J} = 2\pi - \cos^{-1}\left(\frac{a_{\rm J} - \lambda^2 a_{\rm E}}{(\lambda^2-1)\, a_{\rm J}}\right) \ . \tag{9.95}$$

Finally, the time for this portion of the trajectory is

$$\tau_{\rm EJ} = \tau_{\rm E} \cdot \lambda^3 \int\limits_{\pi}^{\phi_{\rm J}} \frac{d\phi}{2\pi}\, \frac{1}{\big[1-(\lambda^2-1)\cos\phi\big]^2} \ . \tag{9.96}$$

9.5.2 II. Encounter with Jupiter

We are interested in the final speed $\boldsymbol{v}_{\rm f}$ of the probe after its encounter with Jupiter. We will determine the speed $v_{\rm f}$ and the angle δ which the probe makes with respect to Jupiter after its encounter. According to the geometry of Fig. 9.11,

$$v_{\rm f}^2 = v_{\rm J}^2 + u^2 - 2uv_{\rm J}\cos(\chi+\gamma) \tag{9.97}$$

$$\cos\delta = \frac{v_{\rm J}^2 + v_{\rm f}^2 - u^2}{2v_{\rm f} v_{\rm J}} \tag{9.98}$$

Note that

$$v_{\rm J}^2 = \frac{GM_\odot}{a_{\rm J}} = \frac{a_{\rm E}}{a_{\rm J}} \cdot v_{\rm E}^2 \ . \tag{9.99}$$

But what are u, χ, and γ?

To determine u, we invoke

$$u^2 = v_{\rm J}^2 + v_{\rm i}^2 - 2v_{\rm J} v_{\rm i} \cos\beta \ . \tag{9.100}$$

The initial velocity (in the frame of the Sun) when the probe crosses Jupiter's orbit is given by energy conservation:

$$\tfrac{1}{2}m(\lambda v_{\rm E})^2 - \frac{GM_\odot m}{a_{\rm E}} = \tfrac{1}{2}mv_{\rm i}^2 - \frac{GM_\odot m}{a_{\rm J}} \ , \tag{9.101}$$

which yields

$$v_i^2 = \left(\lambda^2 - 2 + \frac{2a_{\rm E}}{a_{\rm J}}\right) v_{\rm E}^2 \ . \tag{9.102}$$

As for β, we invoke conservation of angular momentum:

$$\mu(v_{\rm i}\cos\beta)a_{\rm J} = \mu(\lambda v_{\rm E})a_{\rm E} \quad \Rightarrow \quad v_{\rm i}\cos\beta = \lambda\,\frac{a_{\rm E}}{a_{\rm J}}\,v_{\rm E} \ . \tag{9.103}$$

The angle γ is determined from

$$v_{\rm J} = v_{\rm i} \cos\beta + u\cos\gamma \ . \tag{9.104}$$

Putting all this together, we obtain

$$v_{\rm i} = v_{\rm E}\sqrt{\lambda^2 - 2 + 2x} \tag{9.105}$$

$$u = v_{\rm E}\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}} \tag{9.106}$$

$$\cos\gamma = \frac{\sqrt{x} - \lambda x}{\sqrt{\lambda^2 - 2 + 3x - 2\lambda x^{3/2}}} \ , \tag{9.107}$$

where

$$x \equiv \frac{a_{\rm E}}{a_{\rm J}} = 0.1923 \ . \tag{9.108}$$

We next consider the scattering of the probe by the planet Jupiter. In the Jovian frame, we may write

$$r(\phi) = \frac{\kappa R_{\rm J}\,(1 + \varepsilon_{\rm J})}{1 + \varepsilon_{\rm J}\cos\phi} \ , \tag{9.109}$$

where perijove occurs at

$$r(0) = \kappa R_{\rm J} \ . \tag{9.110}$$

Here, κ is a dimensionless quantity, which is simply perijove in units of the Jovian radius. Clearly we require $\kappa > 1$ or else the probe crashes into Jupiter! The probe's energy in this frame is simply $E = \frac{1}{2}mu^2$, which means the probe enters into a hyperbolic orbit about Jupiter. Next, from

$$E = \frac{k}{2}\,\frac{\varepsilon^2 - 1}{\ell^2/\mu k} \tag{9.111}$$

$$\frac{\ell^2}{\mu k} = (1 + \varepsilon)\,\kappa R_{\rm J} \tag{9.112}$$

we find

$$\varepsilon_{\rm J} = 1 + \kappa\left(\frac{R_{\rm J}}{a_{\rm E}}\right)\left(\frac{M_\odot}{M_{\rm J}}\right)\left(\frac{u}{v_{\rm E}}\right)^2 \ . \tag{9.113}$$

The opening angle of the Keplerian hyperbola is then $\phi_{\rm c} = \cos^{-1}\big(\varepsilon_{\rm J}^{-1}\big)$, and the angle χ is related to $\phi_{\rm c}$ through

$$\chi = \pi - 2\phi_{\rm c} = \pi - 2\cos^{-1}\left(\frac{1}{\varepsilon_{\rm J}}\right) \ . \tag{9.114}$$

Therefore, we may finally write

$$v_{\rm f} = \sqrt{x\,v_{\rm E}^2 + u^2 + 2\,u\,v_{\rm E}\sqrt{x}\cos(2\phi_{\rm c} - \gamma)} \tag{9.115}$$

$$\cos\delta = \frac{x\,v_{\rm E}^2 + v_{\rm f}^2 - u^2}{2\,v_{\rm f}\,v_{\rm E}\sqrt{x}} \ . \tag{9.116}$$

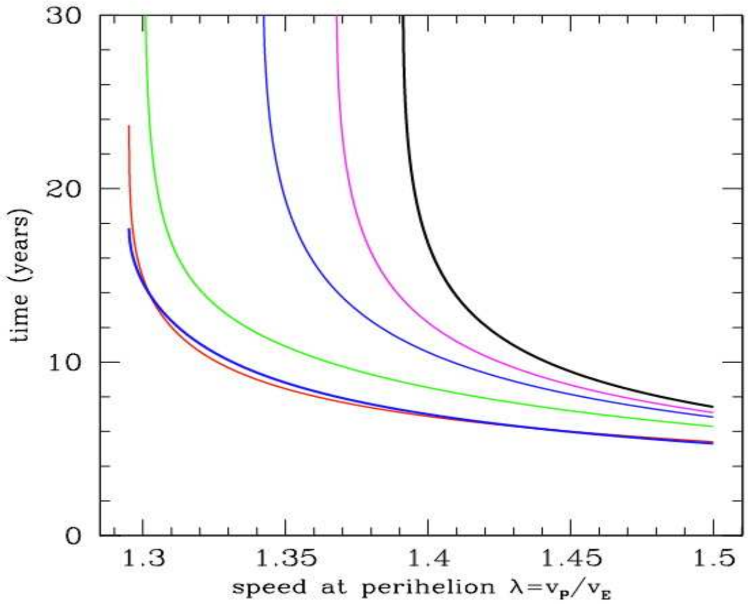


Figure 9.12: Total time for Earth-Neptune mission as a function of dimensionless velocity at perihelion, $\lambda = v_{\rm p}/v_{\rm E}$. Six different values of κ, the value of perijove in units of the Jovian radius, are shown: $\kappa = 1.0$ (thick blue), $\kappa = 5.0$ (red), $\kappa = 20$ (green), $\kappa = 50$ (blue), $\kappa = 100$ (magenta), and $\kappa = \infty$ (thick black).

9.5.3 III. Jupiter to Neptune

Immediately after undergoing gravitational scattering off Jupiter, the energy and angular momentum of the probe are

$$E = \tfrac{1}{2} m v_{\rm f}^2 - \frac{GM_\odot m}{a_{\rm J}} \tag{9.117}$$

and

$$\ell = \mu \, v_{\rm f} \, a_{\rm J} \cos\delta \ . \tag{9.118}$$

We write the geometric equation for the probe's orbit as

$$r(\phi) = \frac{d}{1 + \varepsilon\cos(\phi - \phi_{\rm J} - \alpha)} \ , \tag{9.119}$$

where

$$d = \frac{\ell^2}{\mu k} = \left(\frac{v_{\rm f} \, a_{\rm J} \cos\delta}{v_{\rm E} \, a_{\rm E}} \right)^2 a_{\rm E} \ . \tag{9.120}$$

Setting $E = (\mu k^2 / 2\ell^2)(\varepsilon^2 - 1)$, we obtain the eccentricity

$$\varepsilon = \sqrt{1 + \left(\frac{v_{\rm f}^2}{v_{\rm E}^2} - \frac{2a_{\rm E}}{a_{\rm J}} \right) \frac{d}{a_{\rm E}}} \ . \tag{9.121}$$

Note that the orbit is hyperbolic – the probe will escape the Sun – if $v_{\rm f} > v_{\rm E} \cdot \sqrt{2x}$. The condition that this orbit intersect Jupiter at $\phi = \phi_{\rm J}$ yields

$$\cos\alpha = \frac{1}{\varepsilon}\left(\frac{d}{a_{\rm J}} - 1\right) , \tag{9.122}$$

which determines the angle α. Interception of Neptune occurs at

$$\frac{d}{1 + \varepsilon\cos(\phi_{\rm N} - \phi_{\rm J} - \alpha)} = a_{\rm N} \qquad \Rightarrow \qquad \phi_{\rm N} = \phi_{\rm J} + \alpha + \cos^{-1}\frac{1}{\varepsilon}\left(\frac{d}{a_{\rm N}} - 1\right) . \tag{9.123}$$

We then have

$$\tau_{\rm JN} = \tau_{\rm E} \cdot \left(\frac{d}{a_{\rm E}}\right)^3 \int\limits_{\phi_{\rm J}}^{\phi_{\rm N}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 + \varepsilon\cos(\phi - \phi_{\rm J} - \alpha)\big]^2} . \tag{9.124}$$

The total time to Neptune is then the sum,

$$\tau_{\rm EN} = \tau_{\rm EJ} + \tau_{\rm JN} . \tag{9.125}$$

In Fig. 9.12, we plot the mission time $\tau_{\rm EN}$ *versus* the velocity at perihelion, $v_{\rm p} = \lambda v_{\rm E}$, for various values of κ. The value $\kappa = \infty$ corresponds to the case of no Jovian encounter at all.

9.6 Appendix II : Restricted Three-Body Problem

Problem : Consider the 'restricted three body problem' in which a light object of mass m (*e.g.* a satellite) moves in the presence of two celestial bodies of masses m_1 and m_2 (*e.g.* the sun and the earth, or the earth and the moon). Suppose m_1 and m_2 execute stable circular motion about their common center of mass. You may assume $m \ll m_2 \le m_1$.

(a) Show that the angular frequency for the motion of masses 1 and 2 is related to their (constant) relative separation, by

$$\omega_0^2 = \frac{GM}{r_0^3} , \tag{9.126}$$

where $M = m_1 + m_2$ is the total mass.

Solution : For a Kepler potential $U = -k/r$, the circular orbit lies at $r_0 = \ell^2/\mu k$, where $\ell = \mu r^2 \dot\phi$ is the angular momentum and $k = G m_1 m_2$. This gives

$$\omega_0^2 = \frac{\ell^2}{\mu^2\, r_0^4} = \frac{k}{\mu r_0^3} = \frac{GM}{r_0^3} , \tag{9.127}$$

with $\omega_0 = \dot\phi$.

(b) The satellite moves in the combined gravitational field of the two large bodies; the satellite itself is of course much too small to affect their motion. In deriving the motion for the satellilte, it is convenient to choose a reference frame whose origin is the CM and

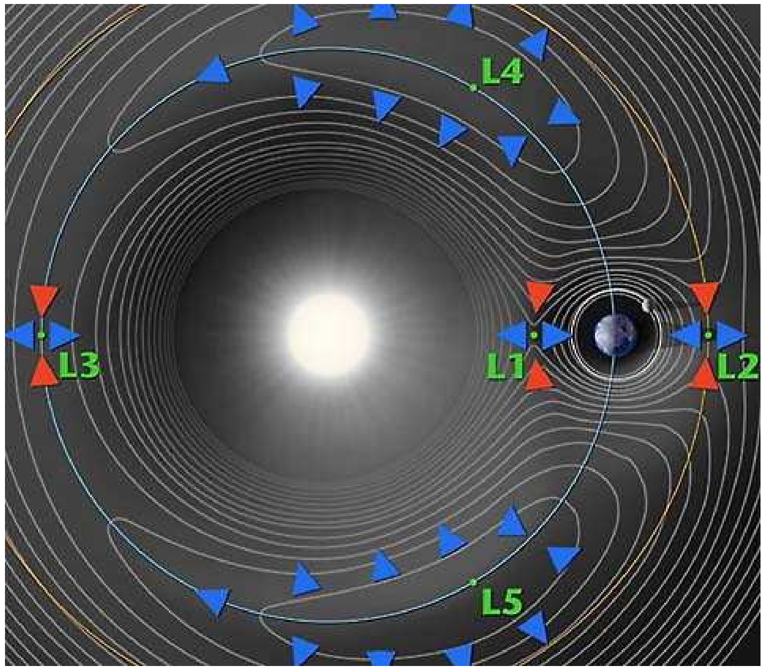


Figure 9.13: The Lagrange points for the earth-sun system. *Credit: WMAP project.*

which rotates with angular velocity ω_0. In the rotating frame the masses m_1 and m_2 lie, respectively, at $x_1 = -\alpha r_0$ and $x_2 = \beta r_0$, with

$$\alpha = \frac{m_2}{M} \qquad , \qquad \beta = \frac{m_1}{M} \tag{9.128}$$

and with $y_1 = y_2 = 0$. Note $\alpha + \beta = 1$.

Show that the Lagrangian for the satellite in this rotating frame may be written

$$L = \tfrac{1}{2}m\big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2}m\big(\dot{y} + \omega_0\, x\big)^2 + \frac{G\, m_1\, m}{\sqrt{(x + \alpha r_0)^2 + y^2}} + \frac{G\, m_2\, m}{\sqrt{(x - \beta r_0)^2 + y^2}} \ . \tag{9.129}$$

Solution : Let the original (inertial) coordinates be (x_0, y_0). Then let us define the rotated coordinates (x, y) as

$$x = \cos(\omega_0 t)\, x_0 + \sin(\omega_0 t)\, y_0 \tag{9.130}$$
$$y = -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 \ . \tag{9.131}$$

Therefore,

$$\dot{x} = \cos(\omega_0 t)\, \dot{x}_0 + \sin(\omega_0 t)\, \dot{y}_0 + \omega_0\, y \tag{9.132}$$
$$\dot{y} = -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 - \omega_0\, x \ . \tag{9.133}$$

Therefore

$$(\dot{x} - \omega_0\, y)^2 + (\dot{y} + \omega_0\, x)^2 = \dot{x}_0^2 + \dot{y}_0^2 \ , \tag{9.134}$$

The Lagrangian is then

$$L = \tfrac{1}{2}m\big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2}m\big(\dot{y} + \omega_0\, x\big)^2 + \frac{G\, m_1\, m}{\sqrt{(x - x_1)^2 + y^2}} + \frac{G\, m_2\, m}{\sqrt{(x - x_2)^2 + y^2}} \ , \qquad (9.135)$$

which, with $x_1 \equiv -\alpha r_0$ and $x_2 \equiv \beta r_0$, agrees with eqn. 9.129

(c) Lagrange discovered that there are five special points where the satellite remains fixed in the rotating frame. These are called the *Lagrange points* {L1, L2, L3, L4, L5}. A sketch of the Lagrange points for the earth-sun system is provided in Fig. 9.13. *Observation: In working out the rest of this problem, I found it convenient to measure all distances in units of r_0 and times in units of ω_0^{-1}, and to eliminate G by writing $Gm_1 = \beta\, \omega_0^2\, r_0^3$ and $Gm_2 = \alpha\, \omega_0^2\, r_0^3$.*

Assuming the satellite is stationary in the rotating frame, derive the equations for the positions of the Lagrange points.

Solution : At this stage it is convenient to measure all distances in units of r_0 and times in units of ω_0^{-1} to factor out a term $m\, r_0^2\, \omega_0^2$ from L, writing the dimensionless Lagrangian $\widetilde{L} \equiv L/(m\, r_0^2\, \omega_0^2)$. Using as well the definition of ω_0^2 to eliminate G, we have

$$\widetilde{L} = \tfrac{1}{2}\,(\dot{\xi} - \eta)^2 + \tfrac{1}{2}(\dot{\eta} + \xi)^2 + \frac{\beta}{\sqrt{(\xi + \alpha)^2 + \eta^2}} + \frac{\alpha}{\sqrt{(\xi - \beta)^2 + \eta^2}} \ , \qquad (9.136)$$

with

$$\xi \equiv \frac{x}{r_0} \quad , \quad \eta \equiv \frac{y}{r_0} \quad , \quad \dot{\xi} \equiv \frac{1}{\omega_0\, r_0}\frac{dx}{dt} \quad , \quad \dot{\eta} \equiv \frac{1}{\omega_0\, r_0}\frac{dy}{dt} \ . \qquad (9.137)$$

The equations of motion are then

$$\ddot{\xi} - 2\dot{\eta} = \xi - \frac{\beta(\xi + \alpha)}{d_1^3} - \frac{\alpha(\xi - \beta)}{d_2^3} \qquad (9.138)$$

$$\ddot{\eta} + 2\dot{\xi} = \eta - \frac{\beta\eta}{d_1^3} - \frac{\alpha\eta}{d_2^3} \ , \qquad (9.139)$$

where

$$d_1 = \sqrt{(\xi + \alpha)^2 + \eta^2} \qquad , \qquad d_2 = \sqrt{(\xi - \beta)^2 + \eta^2} \ . \qquad (9.140)$$

Here, $\xi \equiv x/r_0$, $\xi = y/r_0$, *etc.* Recall that $\alpha + \beta = 1$. Setting the time derivatives to zero yields the static equations for the Lagrange points:

$$\xi = \frac{\beta(\xi + \alpha)}{d_1^3} + \frac{\alpha(\xi - \beta)}{d_2^3} \qquad (9.141)$$

$$\eta = \frac{\beta\eta}{d_1^3} + \frac{\alpha\eta}{d_2^3} \ , \qquad (9.142)$$

(d) Show that the Lagrange points with $y = 0$ are determined by a single nonlinear equation. Show graphically that this equation always has three solutions, one with $x < x_1$, a second

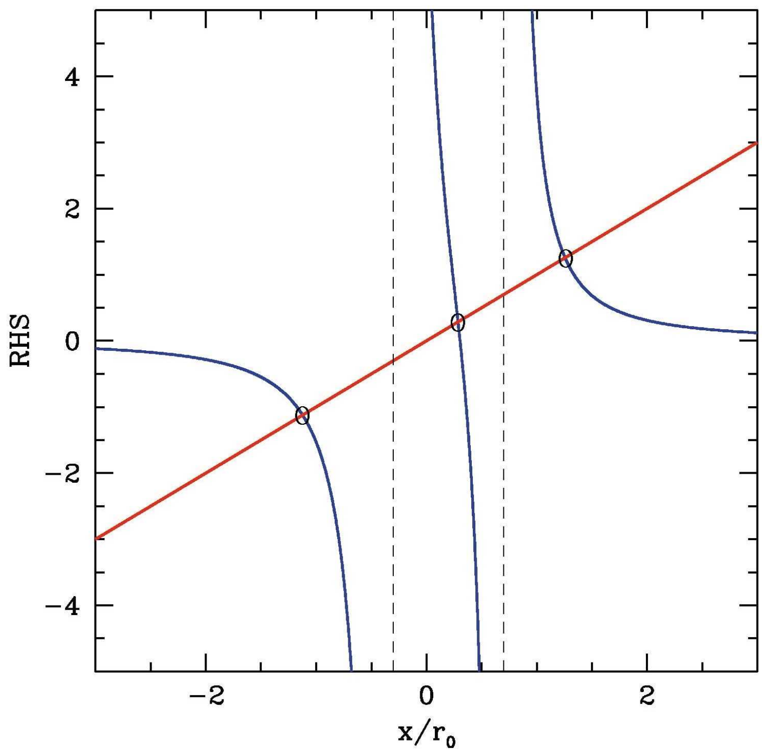


Figure 9.14: Graphical solution for the Lagrange points L1, L2, and L3.

with $x_1 < x < x_2$, and a third with $x > x_2$. These solutions correspond to the points L3, L1, and L2, respectively.

Solution : If $\eta = 0$ the second equation is automatically satisfied. The first equation then gives

$$\xi = \beta \cdot \frac{\xi + \alpha}{\left|\xi + \alpha\right|^3} + \alpha \cdot \frac{\xi - \beta}{\left|\xi - \beta\right|^3} \ . \tag{9.143}$$

The RHS of the above equation diverges to $+\infty$ for $\xi = -\alpha + 0^+$ and $\xi = \beta + 0^+$, and diverges to $-\infty$ for $\xi = -\alpha - 0^+$ and $\xi = \beta - 0^+$, where 0^+ is a positive infinitesimal. The situation is depicted in Fig. 9.14. Clearly there are three solutions, one with $\xi < -\alpha$, one with $-\alpha < \xi < \beta$, and one with $\xi > \beta$.

(e) Show that the remaining two Lagrange points, L4 and L5, lie along equilateral triangles with the two masses at the other vertices.

Solution : If $\eta \neq 0$, then dividing the second equation by η yields

$$1 = \frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3} \ . \tag{9.144}$$

Substituting this into the first equation,

$$\xi = \left(\frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3}\right)\xi + \left(\frac{1}{d_1^3} - \frac{1}{d_2^3}\right)\alpha\beta \ , \tag{9.145}$$

gives

$$d_1 = d_2 \ . \tag{9.146}$$

Reinserting this into the previous equation then gives the remarkable result,

$$d_1 = d_2 = 1 \ , \tag{9.147}$$

which says that each of L4 and L5 lies on an equilateral triangle whose two other vertices are the masses m_1 and m_2. The side length of this equilateral triangle is r_0. Thus, the dimensionless coordinates of L4 and L5 are

$$(\xi_{\rm L4},\ \eta_{\rm L4}) = \Big(\tfrac{1}{2} - \alpha,\ \tfrac{\sqrt{3}}{2}\Big) \qquad , \qquad (\xi_{\rm L5},\ \eta_{\rm L5}) = \Big(\tfrac{1}{2} - \alpha,\ -\tfrac{\sqrt{3}}{2}\Big) \ . \tag{9.148}$$

It turns out that L1, L2, and L3 are always unstable. Satellites placed in these positions must undergo periodic course corrections in order to remain approximately fixed. The SOlar and Heliopheric Observation satellite, *SOHO*, is located at L1, which affords a continuous unobstructed view of the Sun.

(f) Show that the Lagrange points L4 and L5 are stable (obviously you need only consider one of them) provided that the mass ratio m_1/m_2 is sufficiently large. Determine this critical ratio. Also find the frequency of small oscillations for motion in the vicinity of L4 and L5.

Solution : Now we write

$$\xi = \xi_{\rm L4} + \delta\xi \qquad , \qquad \eta = \eta_{\rm L4} + \delta\eta \ , \tag{9.149}$$

and derive the linearized dynamics. Expanding the equations of motion to lowest order in $\delta\xi$ and $\delta\eta$, we have

$$\begin{aligned} \delta\ddot\xi - 2\delta\dot\eta &= \bigg(1 - \beta + \tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} - \alpha - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3}{4}\,\delta\xi + \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\eta \end{aligned} \tag{9.150}$$

and

$$\begin{aligned} \delta\ddot\eta + 2\delta\dot\xi &= \bigg(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\xi + \tfrac{9}{4}\,\delta\eta \ , \end{aligned} \tag{9.151}$$

where we have defined

$$\varepsilon \equiv \beta - \alpha = \frac{m_1 - m_2}{m_1 + m_2} \ . \tag{9.152}$$

As defined, $\varepsilon \in [0,1]$.

Fourier transforming the differential equation, we replace each time derivative by $(-i\nu)$, and thereby obtain

$$\begin{pmatrix} \nu^2 + \frac{3}{4} & -2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon \\ 2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon & \nu^2 + \frac{9}{4} \end{pmatrix} \begin{pmatrix} \delta\hat{\xi} \\ \delta\hat{\eta} \end{pmatrix} = 0 \ . \tag{9.153}$$

Nontrivial solutions exist only when the determinant D vanishes. One easily finds

$$D(\nu^2) = \nu^4 - \nu^2 + \tfrac{27}{16}\left(1 - \varepsilon^2\right) \ , \tag{9.154}$$

which yields a quadratic equation in ν^2, with roots

$$\nu^2 = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{27\,\varepsilon^2 - 23} \ . \tag{9.155}$$

These frequencies are dimensionless. To convert to dimensionful units, we simply multiply the solutions for ν by ω_0, since we have rescaled time by ω_0^{-1}.

Note that the L4 and L5 points are stable only if $\varepsilon^2 > \frac{23}{27}$. If we define the mass ratio $\gamma \equiv m_1/m_2$, the stability condition is equivalent to

$$\gamma = \frac{m_1}{m_2} > \frac{\sqrt{27} + \sqrt{23}}{\sqrt{27} - \sqrt{23}} = 24.960 \ , \tag{9.156}$$

which is satisfied for both the Sun-Jupiter system ($\gamma = 1047$) – and hence for the Sun and any planet – and also for the Earth-Moon system ($\gamma = 81.2$).

Objects found at the L4 and L5 points are called *Trojans*, after the three large asteroids Agamemnon, Achilles, and Hector found orbiting in the L4 and L5 points of the Sun-Jupiter system. No large asteroids have been found in the L4 and L5 points of the Sun-Earth system.

Personal aside : David T. Wilkinson

The image in fig. 9.13 comes from the education and outreach program of the Wilkinson Microwave Anisotropy Probe (WMAP) project, a NASA mission, launched in 2001, which has produced some of the most important recent data in cosmology. The project is named in honor of David T. Wilkinson, who was a leading cosmologist at Princeton, and a founder of the Cosmic Background Explorer (COBE) satellite (launched in 1989). WMAP was sent to the L2 Lagrange point, on the night side of the earth, where it can constantly scan the cosmos with an ultra-sensitive microwave detector, shielded by the earth from interfering solar electromagnetic radiation. The L2 point is of course unstable, with a time scale of about 23 days. Satellites located at such points must undergo regular course and attitude corrections to remain situated.

During the summer of 1981, as an undergraduate at Princeton, I was a member of Wilkinson's "gravity group," working under Jeff Kuhn and Ken Libbrecht. It was a pretty big group and Dave – everyone would call him Dave – used to throw wonderful parties at his home, where we'd always play volleyball. I was very fortunate to get to know David Wilkinson a bit – after working in his group that summer I took a class from him the following year. He was a wonderful person, a superb teacher, and a world class physicist.

Chapter 10

Small Oscillations

10.1 Coupled Coordinates

We assume, for a set of n generalized coordinates $\{q_1, \ldots, q_n\}$, that the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} \ , \tag{10.1}$$

where the sum on σ and σ' from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix T_{ij} is

$$T_{\sigma\sigma'} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\!\theta \end{pmatrix} \qquad \Longrightarrow \qquad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2\big) \ . \tag{10.2}$$

The potential $U(q_1, \ldots, q_n)$ is assumed to be a function of the generalized coordinates alone: $U = U(q)$. A more general formulation of the problem of small oscillations is given in the appendix, section 10.8.

The generalized momenta are

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}\, \dot{q}_{\sigma'} \ , \tag{10.3}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} - \frac{\partial U}{\partial q_\sigma} \ . \tag{10.4}$$

The Euler-Lagrange equations are then $\dot{p}_\sigma = F_\sigma$, or

$$T_{\sigma\sigma'}\, \ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\, \frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma} \right) \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma} \tag{10.5}$$

which is a set of coupled nonlinear second order ODEs. Here we are using the Einstein 'summation convention', where we automatically sum over any and all repeated indices.

10.2 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar{q}_1, \ldots, \bar{q}_n\}$, which satisfies the n nonlinear equations

$$\frac{\partial U}{\partial q_\sigma}\bigg|_{q=\bar{q}} = 0 \ . \tag{10.6}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_\sigma \equiv \bar{q}_\sigma + \eta_\sigma \ . \tag{10.7}$$

The coordinates $\{\eta_1, \ldots, \eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2}\mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\mathrm{V}_{\sigma\sigma'}\,\eta_\sigma\eta_{\sigma'} \ , \tag{10.8}$$

where

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\dot{q}_\sigma\,\partial\dot{q}_{\sigma'}}\bigg|_{q=\bar{q}} \quad , \qquad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \ . \tag{10.9}$$

Writing $\boldsymbol{\eta}^{\mathrm{t}}$ for the row-vector $(\eta_1, \ldots, \eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathrm{t}}\,\mathrm{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathrm{t}}\,\mathrm{V}\,\boldsymbol{\eta} \ , \tag{10.10}$$

where T and V are the constant matrices of eqn. 10.9.

10.3 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_\sigma = \mathrm{A}_{\sigma i}\,\xi_i \ , \tag{10.11}$$

where both σ and i run from 1 to n. The $n \times n$ matrix $\mathrm{A}_{\sigma i}$ is known as the *modal matrix.* With the substitution $\boldsymbol{\eta} = \mathrm{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathrm{t}} = \boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}$, where $\mathrm{A}^{\mathrm{t}}_{i\sigma} = \mathrm{A}_{\sigma i}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\xi}}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A}\,\dot{\boldsymbol{\xi}} - \tfrac{1}{2}\,\boldsymbol{\xi}^{\mathrm{t}}\,\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A}\,\boldsymbol{\xi} \ . \tag{10.12}$$

We now choose the matrix A such that

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{T}\,\mathrm{A} = \mathbb{I} \tag{10.13}$$

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A} = \mathrm{diag}\left(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\right) \ . \tag{10.14}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2} \sum_{i=1}^{n} \Big(\dot{\xi}_i^2 - \omega_i^2 \, \xi_i^2 \Big) \ , \tag{10.15}$$

with the solution

$$\xi_i(t) = C_i \, \cos(\omega_i \, t) + D_i \sin(\omega_i \, t) \ , \tag{10.16}$$

where $\{C_1, \ldots, C_n\}$ and $\{D_1, \ldots, D_n\}$ are $2n$ constants of integration, and where no sum is implied on i. Note that

$$\boldsymbol{\xi} = \mathrm{A}^{-1} \boldsymbol{\eta} = \mathrm{A}^{\mathrm{t}} \mathrm{T} \, \boldsymbol{\eta} \ . \tag{10.17}$$

In terms of the original generalized displacements, the solution is

$$\eta_\sigma(t) = \sum_{i=1}^{n} \mathrm{A}_{\sigma i} \Big\{ C_i \, \cos(\omega_i t) + D_i \sin(\omega_i t) \Big\} \ , \tag{10.18}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$C_i = \mathrm{A}^{\mathrm{t}}_{i\sigma} \, \mathrm{T}_{\sigma\sigma'} \, \eta_{\sigma'}(0) \tag{10.19}$$

$$D_i = \omega_i^{-1} \mathrm{A}^{\mathrm{t}}_{i\sigma} \, \mathrm{T}_{\sigma\sigma'} \, \dot{\eta}_{\sigma'}(0) \ , \tag{10.20}$$

again with no implied sum on i on the RHS of the second equation, and where we have used $\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}}\,\mathrm{T}$, from eqn. 10.13. (The implied sums in eqn. 10.20 are over σ and σ'.)

Note that the normal coordinates have unusual dimensions: $[\boldsymbol{\xi}] = \sqrt{M} \cdot L$, where L is length and M is mass.

10.3.1 Can you really just choose an A so that both these wonderful things happen in 10.13 and 10.14?

Yes.

10.3.2 Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per 10.13 and 10.14. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $\mathcal{O}_1 \in \mathrm{O}(n)$ such that

  $$\mathcal{O}_1^{\mathrm{t}} \, \mathrm{T} \, \mathcal{O}_1 = \mathrm{T}_{\mathrm{d}} \ , \tag{10.21}$$

  where T_{d} is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $\mathrm{T}_{\mathrm{d}}^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of T_{d}. Consider the linear transformation $\mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}$. Its effect on T is
$$\mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2} = 1 \; . \tag{10.22}$$

- Since $\mathcal{O}_1$ and T_{d} are wholly derived from T, the only thing we know about
$$\widetilde{\mathrm{V}} \equiv \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2} \tag{10.23}$$
is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $\mathcal{O}_2 \in \mathrm{O}(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$ such that
$$\mathcal{O}_2^{\mathrm{t}}\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{T}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2 = 1 \tag{10.24}$$
$$\mathcal{O}_2^{\mathrm{t}}\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_1^{\mathrm{t}}\, \mathrm{V}\, \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2 = \mathrm{diag}\,(\omega_1^2, \ldots, \omega_n^2) \; . \tag{10.25}$$
All that remains is to identify the modal matrix $\mathrm{A} = \mathcal{O}_1\, \mathrm{T}_{\mathrm{d}}^{-1/2}\, \mathcal{O}_2$.

Note that it is *not possible* to simultaneously diagonalize *three* symmetric matrices in general.

10.3.3 Finding the modal matrix

While the above proof allows one to construct A by finding the two orthogonal matrices $\mathcal{O}_1$ and $\mathcal{O}_2$, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

We start with the equations of motion, $\mathrm{T}\, \ddot{\boldsymbol{\eta}} + \mathrm{V}\, \boldsymbol{\eta} = 0$. In component notation, we have
$$\mathrm{T}_{\sigma\sigma'}\, \ddot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\, \eta_{\sigma'} = 0 \; . \tag{10.26}$$

We now assume that $\boldsymbol{\eta}(t)$ oscillates with a single frequency ω, *i.e.* $\eta_\sigma(t) = \psi_\sigma\, e^{-i\omega t}$. This results in a set of linear algebraic equations for the components ψ_σ:
$$\big(\omega^2\, \mathrm{T}_{\sigma\sigma'} - \mathrm{V}_{\sigma\sigma'}\big)\, \psi_{\sigma'} = 0 \; . \tag{10.27}$$

These are n equations in n unknowns: one for each value of $\sigma = 1, \ldots, n$. Because the equations are homogeneous and linear, there is always a trivial solution $\boldsymbol{\psi} = 0$. In fact one might think this is the only solution, since
$$\big(\omega^2\, \mathrm{T} - \mathrm{V}\big)\, \boldsymbol{\psi} = 0 \qquad \stackrel{?}{\Longrightarrow} \qquad \boldsymbol{\psi} = \big(\omega^2\, \mathrm{T} - \mathrm{V}\big)^{-1}\, 0 = 0 \; . \tag{10.28}$$

However, this fails when the matrix $\omega^2\,\mathrm{T}-\mathrm{V}$ is defective[1], *i.e.* when

$$\det\big(\omega^2\,\mathrm{T}-\mathrm{V}\big)=0\ . \tag{10.29}$$

Since T and V are of rank n, the above determinant yields an n^{th} order polynomial in ω^2, whose n roots are the desired squared eigenfrequencies $\{\omega_1^2\,,\ldots,\,\omega_n^2\}$.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\sum_{\sigma'=1}^{n}\big(\omega_i^2\,\mathrm{T}_{\sigma\sigma'}-\mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'}=0 \tag{10.30}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^{(i)}$. That is, there are n equations ($\sigma=1,\ldots,n$), but one linear dependency since $\det\,(\omega_i^2\,\mathrm{T}-\mathrm{V})=0$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship,

$$\psi^{(i)}_{\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'}=\delta_{ij}\ . \tag{10.31}$$

To see this, let us duplicate eqn. 10.30, replacing i with j, and multiply both equations as follows:

$$\psi^{(j)}_{\sigma}\times\big(\omega_i^2\,\mathrm{T}_{\sigma\sigma'}-\mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'}=0 \tag{10.32}$$

$$\psi^{(i)}_{\sigma}\times\big(\omega_j^2\,\mathrm{T}_{\sigma\sigma'}-\mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(j)}_{\sigma'}=0\ . \tag{10.33}$$

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_i^2-\omega_j^2)\sum_{\sigma,\sigma'=1}^{n}\psi^{(i)}_{\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'}=0\ , \tag{10.34}$$

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$ corresponding to distinct eigenvalues $\omega_i^2\neq\omega_j^2$ are orthogonal: $(\boldsymbol{\psi}^{(i)})^{\mathrm{t}}\,\mathrm{T}\,\boldsymbol{\psi}^{(j)}=0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector – are fixed by imposing the condition of normalization:

$$\psi^{(i)}_{\sigma}\to\psi^{(i)}_{\sigma}\Big/\sqrt{\psi^{(i)}_{\mu}\,\mathrm{T}_{\mu\mu'}\,\psi^{(i)}_{\mu'}}\qquad\Longrightarrow\qquad\psi^{(i)}_{\sigma}\,\mathrm{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'}=\delta_{ij}\ . \tag{10.35}$$

The modal matrix is just the matrix of eigenvectors: $\mathrm{A}_{\sigma i}=\psi^{(i)}_{\sigma}$.

With the eigenvectors $\psi^{(i)}_{\sigma}$ thusly normalized, we have

$$\begin{aligned}0&=\psi^{(i)}_{\sigma}\,\big(\omega_j^2\,\mathrm{T}_{\sigma\sigma'}-\mathrm{V}_{\sigma\sigma'}\big)\,\psi^{(j)}_{\sigma'}\\&=\omega_j^2\,\delta_{ij}-\psi^{(i)}_{\sigma}\,\mathrm{V}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'}\ ,\end{aligned} \tag{10.36}$$

with no sum on j. This establishes the result

$$\mathrm{A}^{\mathrm{t}}\,\mathrm{V}\,\mathrm{A}=\mathrm{diag}\,\big(\omega_1^2\,,\ \ldots\ ,\,\omega_n^2\big)\ . \tag{10.37}$$

[1]The label *defective* has a distastefully negative connotation. In modern parlance, we should instead refer to such a matrix as *determinantally challenged*.

10.4 Example: Masses and Springs

Two blocks and three springs are configured as in Fig. 17.6. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

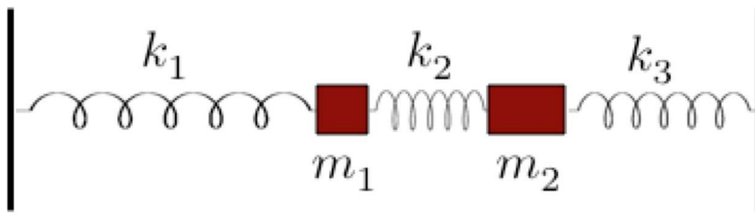


Figure 10.1: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.

(b) Find the T and V matrices.

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.

(d) Find the normal modes of oscillation.

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2} m_1\, \dot{x}_1^2 + \tfrac{1}{2} m_2\, \dot{x}_2^2 - \tfrac{1}{2} k_1\, x_1^2 - \tfrac{1}{2} k_2\, (x_2 - x_1)^2 - \tfrac{1}{2} k_3\, x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i\, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i\, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda\,\omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} .$$

The determinant is

$$\begin{aligned} \det\,(\omega^2 \mathrm{T} - \mathrm{V}) &= (2\lambda^2 - 11\lambda + 14)\,k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\,k^2 . \end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V})\,\vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)}\, \mathrm{T}_{ij}\, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \quad , \quad \mathcal{C}_+ = \frac{1}{\sqrt{6m}} . \tag{10.38}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) . \end{aligned}$$

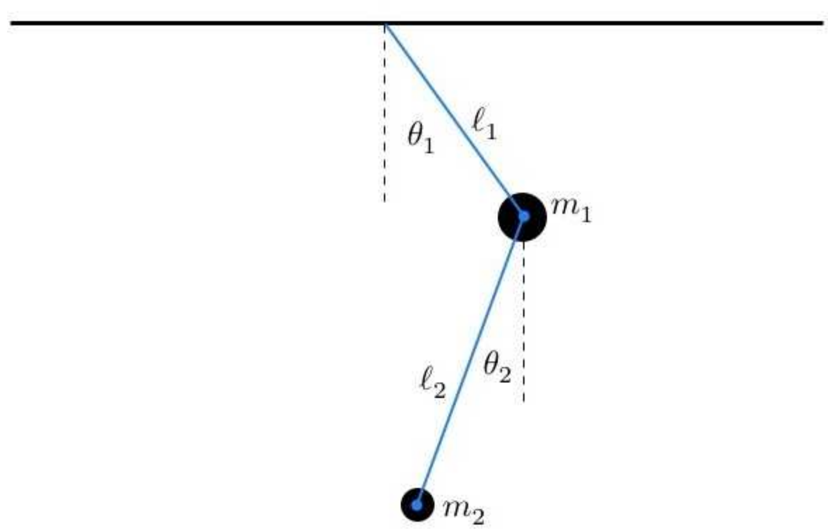


Figure 10.2: The double pendulum.

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega\_t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega\_t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

10.5 Example: Double Pendulum

As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The kinetic and potential energies are

$$T = m\ell^2 \dot\theta_1^2 + m\ell^2 \cos(\theta_1 - \theta_1)\, \dot\theta_1 \dot\theta_2 + \tfrac{1}{2} m\ell^2 \dot\theta_2^2 \tag{10.39}$$

$$V = -2mg\ell \cos\theta_1 - mg\ell \cos\theta_2 \ , \tag{10.40}$$

leading to

$$\mathrm{T} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \qquad , \qquad \mathrm{V} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \ . \tag{10.41}$$

Then

$$\omega^2 \mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \ , \tag{10.42}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\, \omega_0^2 \ . \tag{10.43}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathrm{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \qquad , \qquad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \ , \tag{10.44}$$

where $C_\pm$ are constants. One can check $\mathrm{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(j)}$ vanishes for $i \neq j$. We then normalize by demanding $\mathrm{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(i)} = 1$ (no sum on i), which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$\mathrm{A} = \begin{pmatrix} \psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^- \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}} \end{pmatrix} \ . \tag{10.45}$$

10.6 Zero Modes

Recall Noether's theorem, which says that for every continuous one-parameter family of coordinate transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q,\zeta) \qquad , \qquad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma \ , \tag{10.46}$$

which leaves the Lagrangian invariant, *i.e.* $dL/d\zeta = 0$, there is an associated conserved quantity,

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma} \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \qquad \text{satisfies} \qquad \frac{d\Lambda}{dt} = 0 \ . \tag{10.47}$$

For small oscillations, we write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$, hence

$$\Lambda_k = \sum_\sigma C_{k\sigma}\,\dot{\eta}_\sigma \ , \tag{10.48}$$

where k labels the one-parameter families (in the event there is more than one continuous symmetry), and where

$$C_{k\sigma} = \sum_{\sigma'} \mathrm{T}_{\sigma\sigma'} \frac{\partial \tilde{q}_{\sigma'}}{\partial \zeta_k}\bigg|_{\zeta=0} \ . \tag{10.49}$$

Therefore, we can define the (unnormalized) normal mode

$$\xi_k = \sum_\sigma C_{k\sigma}\,\eta_\sigma \ , \tag{10.50}$$

which satisfies $\ddot{\xi}_k = 0$. Thus, in systems with continuous symmetries, to each such continuous symmetry there is an associated zero mode of the small oscillations problem, *i.e.* a mode with $\omega_k^2 = 0$.

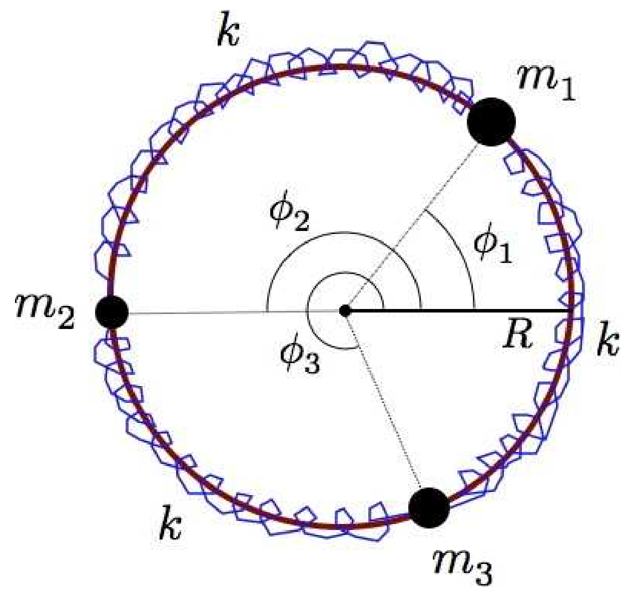


Figure 10.3: Coupled oscillations of three masses on a frictionless hoop of radius R. All three springs have the same force constant k, but the masses are all distinct.

10.6.1 Example of zero mode oscillations

The simplest example of a zero mode would be a pair of masses m_1 and m_2 moving frictionlessly along a line and connected by a spring of force constant k. We know from our study of central forces that the Lagrangian may be written

$$\begin{aligned} L &= \tfrac{1}{2}m_1\dot{x}_1^2 + \tfrac{1}{2}m_2\dot{x}_2^2 - \tfrac{1}{2}k(x_1-x_2)^2 \\ &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}\mu\dot{x}^2 - \tfrac{1}{2}kx^2 \ , \end{aligned} \tag{10.51}$$

where $X = (m_1x_1 + m_2x_2)/(m_1+m_2)$ is the center of mass position, $x = x_1 - x_2$ is the relative coordinate, $M = m_1 + m_2$ is the total mass, and $\mu = m_1m_2/(m_1+m_2)$ is the reduced mass. The relative coordinate obeys $\ddot{x} = -\omega_0^2\,x$, where the oscillation frequency is $\omega_0 = \sqrt{k/\mu}$. The center of mass coordinate obeys $\ddot{X} = 0$, *i.e.* its oscillation frequency is zero. The center of mass motion is a zero mode.

Another example is furnished by the system depicted in fig. 10.3, where three distinct masses m_1, m_2, and m_3 move around a frictionless hoop of radius R. The masses are connected to their neighbors by identical springs of force constant k. We choose as generalized coordinates the angles ϕ_σ ($\sigma = 1,2,3$), with the convention that

$$\phi_1 \le \phi_2 \le \phi_3 \le 2\pi + \phi_1 \ . \tag{10.52}$$

Let $R\chi$ be the equilibrium length for each of the springs. Then the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kR^2\Big\{(\phi_2-\phi_1-\chi)^2 + (\phi_3-\phi_2-\chi)^2 + (2\pi+\phi_1-\phi_3-\chi)^2\Big\} \\ &= \tfrac{1}{2}kR^2\Big\{(\phi_2-\phi_1)^2 + (\phi_3-\phi_2)^2 + (2\pi+\phi_1-\phi_3)^2 + 3\chi^2 - 4\pi\chi\Big\} \ . \end{aligned} \tag{10.53}$$

Note that the equilibrium angle χ enters only in an additive constant to the potential energy. Thus, for the calculation of the equations of motion, it is irrelevant. It doesn't matter whether or not the equilibrium configuration is unstretched ($\chi = 2\pi/3$) or not ($\chi \neq 2\pi/3$).

The kinetic energy is simple:

$$T = \tfrac{1}{2}R^2\Big(m_1\,\dot\phi_1^2 + m_2\,\dot\phi_2^2 + m_3\,\dot\phi_3^2\Big)\ . \tag{10.54}$$

The T and V matrices are then

$$\mathrm{T} = \begin{pmatrix} m_1R^2 & 0 & 0 \\ 0 & m_2R^2 & 0 \\ 0 & 0 & m_3R^2 \end{pmatrix} \quad , \quad \mathrm{V} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix}\ . \tag{10.55}$$

We then have

$$\omega^2\,\mathrm{T} - \mathrm{V} = kR^2 \begin{pmatrix} \frac{\omega^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega^2}{\Omega_3^2} - 2 \end{pmatrix}\ . \tag{10.56}$$

We compute the determinant to find the characteristic polynomial:

$$\begin{aligned} P(\omega) &= \det(\omega^2\,\mathrm{T} - \mathrm{V}) \qquad (10.57) \\ &= \frac{\omega^6}{\Omega_1^2\,\Omega_2^2\,\Omega_3^2} - 2\left(\frac{1}{\Omega_1^2\,\Omega_2^2} + \frac{1}{\Omega_2^2\,\Omega_3^2} + \frac{1}{\Omega_1^2\,\Omega_3^2}\right)\omega^4 + 3\left(\frac{1}{\Omega_1^2} + \frac{1}{\Omega_2^2} + \frac{1}{\Omega_3^2}\right)\omega^2\ , \end{aligned}$$

where $\Omega_i^2 \equiv k/m_i$. The equation $P(\omega) = 0$ yields a cubic equation in ω^2, but clearly ω^2 is a factor, and when we divide this out we obtain a quadratic equation. One root obviously is $\omega_1^2 = 0$. The other two roots are solutions to the quadratic equation:

$$\omega_{2,3}^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2 \pm \sqrt{\tfrac{1}{2}\big(\Omega_1^2 - \Omega_2^2\big)^2 + \tfrac{1}{2}\big(\Omega_2^2 - \Omega_3^2\big)^2 + \tfrac{1}{2}\big(\Omega_1^2 - \Omega_3^2\big)^2}\ . \tag{10.58}$$

To find the eigenvectors and the modal matrix, we set

$$\begin{pmatrix} \frac{\omega_j^2}{\Omega_1^2} - 2 & 1 & 1 \\ 1 & \frac{\omega_j^2}{\Omega_2^2} - 2 & 1 \\ 1 & 1 & \frac{\omega_j^2}{\Omega_3^2} - 2 \end{pmatrix} \begin{pmatrix} \psi_1^{(j)} \\ \psi_2^{(j)} \\ \psi_3^{(j)} \end{pmatrix} = 0\ , \tag{10.59}$$

Writing down the three coupled equations for the components of $\boldsymbol{\psi}^{(j)}$, we find

$$\left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)\psi_1^{(j)} = \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)\psi_2^{(j)} = \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)\psi_3^{(j)}\ . \tag{10.60}$$

We therefore conclude

$$\boldsymbol{\psi}^{(j)} = \mathcal{C}_j \begin{pmatrix} \left(\frac{\omega_j^2}{\Omega_1^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_2^2} - 3\right)^{-1} \\ \left(\frac{\omega_j^2}{\Omega_3^2} - 3\right)^{-1} \end{pmatrix}\ . \tag{10.61}$$

The normalization condition $\psi^{(i)}_\sigma \, \mathrm{T}_{\sigma\sigma'} \, \psi^{(j)}_{\sigma'} = \delta_{ij}$ then fixes the constants $\mathcal{C}_j$:

$$\left[m_1 \left(\frac{\omega_j^2}{\Omega_1^2} - 3 \right)^{-2} + \; m_2 \left(\frac{\omega_j^2}{\Omega_2^2} - 3 \right)^{-2} + \; m_3 \left(\frac{\omega_j^2}{\Omega_3^2} - 3 \right)^{-2} \right] \left| \mathcal{C}_j \right|^2 = 1 \; . \tag{10.62}$$

The Lagrangian is invariant under the one-parameter family of transformations

$$\phi_\sigma \longrightarrow \phi_\sigma + \zeta \tag{10.63}$$

for all $\sigma = 1, 2, 3$. The associated conserved quantity is

$$\begin{aligned} \Lambda &= \sum_\sigma \frac{\partial L}{\partial \dot\phi_\sigma} \, \frac{\partial \tilde\phi_\sigma}{\partial \zeta} \\ &= R^2 \left(m_1 \, \dot\phi_1 + m_2 \, \dot\phi_2 + m_3 \, \dot\phi_3 \right) \; , \end{aligned} \tag{10.64}$$

which is, of course, the total angular momentum relative to the center of the ring. Thus, from $\dot\Lambda = 0$ we identify the zero mode as ξ_1, where

$$\xi_1 = \mathcal{C} \left(m_1 \phi_1 + m_2 \, \phi_2 + m_3 \phi_3 \right) \; , \tag{10.65}$$

where $\mathcal{C}$ is a constant. Recall the relation $\eta_\sigma = \mathrm{A}_{\sigma i} \, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i. We can invert this relation to obtain

$$\xi_i = \mathrm{A}^{-1}_{i\sigma} \, \eta_\sigma = \mathrm{A}^{\mathrm{t}}_{i\sigma} \, \mathrm{T}_{\sigma\sigma'} \, \eta_{\sigma'} \; . \tag{10.66}$$

Here we have used the result $\mathrm{A}^{\mathrm{t}} \, \mathrm{T} \, \mathrm{A} = 1$ to write

$$\mathrm{A}^{-1} = \mathrm{A}^{\mathrm{t}} \, \mathrm{T} \; . \tag{10.67}$$

This is a convenient result, because it means that if we ever need to express the normal coordinates in terms of the generalized displacements, we don't have to invert any matrices – we just need to do one matrix multiplication. In our case here, the T matrix is diagonal, so the multiplication is trivial. From eqns. 10.65 and 10.66, we conclude that the matrix $\mathrm{A}^{\mathrm{t}} \, \mathrm{T}$ must have a first *row* which is proportional to (m_1, m_2, m_3). Since these are the very diagonal entries of T, we conclude that A^{t} itself must have a first row which is proportional to $(1, 1, 1)$, which means that the first *column* of A is proportional to $(1, 1, 1)$. But this is confirmed by eqn. 10.60 when we take $j = 1$, since $\omega^2_{j=1} = 0$: $\psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3$.

10.7 Chain of Mass Points

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$\begin{aligned} L &= \tfrac{1}{2} m \sum_n \dot x_n^2 - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\ &= \tfrac{1}{2} m \sum_n \dot u_n^2 - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \; , \end{aligned} \tag{10.68}$$

where $u_n \equiv x_n - na - b$ is the displacement from equilibrium of the $n^{\rm th}$ mass. The constant b is arbitrary. The Euler-Lagrange equations are

$$\begin{aligned}
\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot u_n}\bigg) = m\ddot u_n &= \frac{\partial L}{\partial u_n} \\
&= k(u_{n+1} - u_n) - k(u_n - u_{n-1}) \\
&= k(u_{n+1} + u_{n-1} - 2u_n) \ . \qquad (10.69)
\end{aligned}$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$u_n = \frac{1}{\sqrt{N}} \sum_q e^{iqan}\, \hat u_q \qquad (10.70)$$

$$\hat u_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, u_n \ , \qquad (10.71)$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat u_q\}$, the equations of motion become

$$\begin{aligned}
\ddot{\hat u}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot u_n \\
&= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (u_{n+1} + u_{n-1} - 2u_n) \\
&= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan}\, (e^{-iqa} + e^{+iqa} - 2)\, u_n \\
&= -\frac{2k}{m} \sin^2\!\big(\tfrac{1}{2}qa\big)\, \hat u_q \qquad (10.72)
\end{aligned}$$

Thus, the $\{\hat u_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$\omega_q = \frac{2k}{m} \big|\sin\big(\tfrac{1}{2}qa\big)\big| \ . \qquad (10.73)$$

This means that the modal matrix is

$$\mathrm{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan} \ , \qquad (10.74)$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. (The normal modes themselves are then $\xi_q = \mathrm{A}^\dagger_{qn} \mathrm{T}_{nn'} u_{n'} = \sqrt{m}\, \hat u_q$. For complex A, the normalizations are $\mathrm{A}^\dagger \mathrm{T} \mathrm{A} = \mathbb{I}$ and $\mathrm{A}^\dagger \mathrm{V} \mathrm{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$\mathrm{T}_{nn'} = m\, \delta_{n,n'} \qquad (10.75)$$

$$\mathrm{V}_{nn'} = 2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1} \qquad (10.76)$$

and that

$$
\begin{aligned}
(\mathrm{A}^\dagger \mathrm{T} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan}\, m\, \delta_{nn'}\, e^{iq'an'} \\
&= \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} = \delta_{qq'} \ ,
\end{aligned} \tag{10.77}
$$

and

$$
\begin{aligned}
(\mathrm{A}^\dagger \mathrm{V} \mathrm{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathrm{A}^*_{nq} \mathrm{T}_{nn'} \mathrm{A}_{n'q'} \\
&= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan} \Big(2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1} \Big)\, e^{iq'an'} \\
&= \frac{k}{m} \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} \left(2 - e^{-iq'a} - e^{iq'a} \right) \\
&= \frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \delta_{qq'} = \omega^2_q\, \delta_{qq'}
\end{aligned} \tag{10.78}
$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = \frac{2\pi}{a}$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbb{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[2]. One natural set to choose is $q \in \left[-\frac{\pi}{a}, \frac{\pi}{a} \right]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$
L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}^*_q\, \dot{\hat{u}}_q - k \sum_q (1 - \cos qa)\, \hat{u}^*_q\, \hat{u}_q \ , \tag{10.79}
$$

where the sum is over q in the first Brillouin zone. Note that

$$
\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}^*_q \ . \tag{10.80}
$$

This means that we can restrict the sum to half the Brillouin zone:

$$
L = \tfrac{1}{2} m \sum_{q \in [0, \frac{\pi}{a}]} \left\{ \dot{\hat{u}}^*_q\, \dot{\hat{u}}_q - \frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \hat{u}^*_q\, \hat{u}_q \right\} . \tag{10.81}
$$

[2]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

Now $\hat{u}_q$ and $\hat{u}^*_q$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}^*_q$ gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\hat{u}}^*_q}\right) = \frac{\partial L}{\partial \hat{u}^*_q} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2\, \hat{u}_q \ . \tag{10.82}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

10.7.1 Continuum limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write

$$u_n(t) \longrightarrow u(x = na, t) \tag{10.83}$$

in which case

$$T = \tfrac{1}{2} m \sum_n \dot{u}_n^2 \qquad \longrightarrow \qquad \tfrac{1}{2} m \int \frac{dx}{a} \left(\frac{\partial u}{\partial t}\right)^2 \tag{10.84}$$

$$V = \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \qquad \longrightarrow \qquad \tfrac{1}{2} k \int \frac{dx}{a} \left(\frac{u(x+a) - u(x)}{a}\right)^2 a^2 \tag{10.85}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int\! dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2}\, \mu \left(\frac{\partial u}{\partial t}\right)^2 - \tfrac{1}{2}\, \tau \left(\frac{\partial u}{\partial x}\right)^2 \ , \end{aligned} \tag{10.86}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[3]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[4]. The action is

$$S\big[u(x,t)\big] = \int_{t_a}^{t_b}\! dt \int_{x_a}^{x_b}\! dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \ , \tag{10.87}$$

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\,(\partial_t u)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\,(\partial_x u)}\right) = 0 \ . \tag{10.88}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \tag{10.89}$$

[3]For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.
[4]$\mathcal{L}$ may also depend explicitly on x and t.

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C\, e^{iqx}\, e^{-i\omega t} \ , \tag{10.90}$$

where

$$\omega = cq \ . \tag{10.91}$$

Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m} \sin^2\big(\tfrac{1}{2}qa\big) \longrightarrow \frac{ka^2}{m}\, q^2 = c^2\, q^2 \ , \tag{10.92}$$

and so the results agree.

10.8 Appendix I : General Formulation

In the development in section 10.1, we assumed that the kinetic energy T is a homogeneous function of degree 2, and the potential energy U a homogeneous function of degree 0, in the generalized velocities $\dot{q}_\sigma$. However, we've encountered situations where this is not so: problems with time-dependent holonomic constraints, such as the mass point on a rotating hoop, and problems involving charged particles moving in magnetic fields. The general Lagrangian is of the form

$$L = \tfrac{1}{2}\, T_{2\,\sigma\sigma'}(q)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} + T_{1\,\sigma}(q)\, \dot{q}_\sigma + T_0(q) - U_{1\,\sigma}(q)\, \dot{q}_\sigma - U_0(q) \ , \tag{10.93}$$

where the subscript 0, 1, or 2 labels the degree of homogeneity of each term in the generalized velocities. The generalized momenta are then

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{2\,\sigma\sigma'}\, \dot{q}_{\sigma'} + T_{1\,\sigma} - U_{1\,\sigma} \tag{10.94}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{\partial (T_0 - U_0)}{\partial q_\sigma} + \frac{\partial (T_{1\,\sigma'} - U_{1\,\sigma'})}{\partial q_\sigma}\, \dot{q}_{\sigma'} + \frac{1}{2}\, \frac{\partial T_{2\,\sigma'\sigma''}}{\partial q_\sigma}\, \dot{q}_{\sigma'}\, \dot{q}_{\sigma''} \ , \tag{10.95}$$

and the equations of motion are again $\dot{p}_\sigma = F_\sigma$. Once we solve

In equilibrium, we seek a time-independent solution of the form $q_\sigma(t) = \bar{q}_\sigma$. This entails

$$\frac{\partial}{\partial q_\sigma}\bigg|_{q=\bar{q}} \Big(U_0(q) - T_0(q) \Big) = 0 \ , \tag{10.96}$$

which give us n equations in the n unknowns $(q_1, \ldots, q_n)$. We then write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand in the notionally small quantities η_σ. It is important to understand that we assume η and all of its time derivatives as well are small. Thus, we can expand L to quadratic order in $(\eta, \dot{\eta})$ to obtain

$$L = \tfrac{1}{2}\, \mathrm{T}_{\sigma\sigma'}\, \dot{\eta}_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{B}_{\sigma\sigma'}\, \eta_\sigma\, \dot{\eta}_{\sigma'} - \tfrac{1}{2}\, \mathrm{V}_{\sigma\sigma'}\, \eta_\sigma\, \eta_{\sigma'} \ , \tag{10.97}$$

where

$$\mathrm{T}_{\sigma\sigma'} = T_{2\,\sigma\sigma'}(\bar{q}) \quad , \quad \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2\big(U_0 - T_0\big)}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \quad , \quad \mathrm{B}_{\sigma\sigma'} = 2\,\frac{\partial\big(U_{1\,\sigma'} - T_{1\,\sigma'}\big)}{\partial q_\sigma}\bigg|_{q=\bar{q}} \ . \tag{10.98}$$

Note that the T and V matrices are symmetric. The $\mathrm{B}_{\sigma\sigma'}$ term is new.

Now we can always write $\mathrm{B} = \frac{1}{2}(\mathrm{B}^{\mathrm{s}} + \mathrm{B}^{\mathrm{a}})$ as a sum over symmetric and antisymmetric parts, with $\mathrm{B}^{\mathrm{s}} = \mathrm{B} + \mathrm{B}^{\mathrm{t}}$ and $\mathrm{B}^{\mathrm{a}} = \mathrm{B} - \mathrm{B}^{\mathrm{t}}$. Since,

$$\mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\,\eta_\sigma\,\dot{\eta}_{\sigma'} = \frac{d}{dt}\Big(\tfrac{1}{2}\,\mathrm{B}^{\mathrm{s}}_{\sigma\sigma'}\,\eta_\sigma\,\eta_{\sigma'}\Big) \ , \tag{10.99}$$

any symmetric part to B contributes a total time derivative to L, and thus has no effect on the equations of motion. Therefore, we can project B onto its antisymmetric part, writing

$$\mathrm{B}_{\sigma\sigma'} = \left(\frac{\partial\big(U_{1\,\sigma'} - T_{1\,\sigma'}\big)}{\partial q_\sigma} - \frac{\partial\big(U_{1\,\sigma} - T_{1\,\sigma}\big)}{\partial q_{\sigma'}}\right)_{q=\bar{q}} \ . \tag{10.100}$$

We now have

$$p_\sigma = \frac{\partial L}{\partial\dot{\eta}_\sigma} = \mathrm{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \tfrac{1}{2}\,\mathrm{B}_{\sigma\sigma'}\,\eta_{\sigma'} \ , \tag{10.101}$$

and

$$F_\sigma = \frac{\partial L}{\partial\eta_\sigma} = -\tfrac{1}{2}\,\mathrm{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} - \mathrm{V}_{\sigma\sigma'}\,\eta_{\sigma'} \ . \tag{10.102}$$

The equations of motion, $\dot{p}_\sigma = F_\sigma$, then yield

$$\mathrm{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathrm{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \mathrm{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \ . \tag{10.103}$$

Let us write $\boldsymbol{\eta}(t) = \boldsymbol{\eta}\,e^{-i\omega t}$. We then have

$$\big(\omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V}\big)\,\boldsymbol{\eta} = 0 \ . \tag{10.104}$$

To solve eqn. 10.104, we set $P(\omega) = 0$, where $P(\omega) = \det\big[\mathrm{Q}(\omega)\big]$, with

$$\mathrm{Q}(\omega) \equiv \omega^2\,\mathrm{T} + i\omega\,\mathrm{B} - \mathrm{V} \ . \tag{10.105}$$

Since T, B, and V are real-valued matrices, and since $\det(M) = \det(M^{\mathrm{t}})$ for any matrix M, we can use $\mathrm{B}^{\mathrm{t}} = -\mathrm{B}$ to obtain $P(-\omega) = P(\omega)$ and $P(\omega^*) = \big[P(\omega)\big]^*$. This establishes that if $P(\omega) = 0$, *i.e.* if ω is an eigenfrequency, then $P(-\omega) = 0$ and $P(\omega^*) = 0$, *i.e.* $-\omega$ and ω^* are also eigenfrequencies (and hence $-\omega^*$ as well).

10.9 Appendix II : Additional Examples

10.9.1 Right Triatomic Molecule

A molecule consists of three identical atoms located at the vertices of a 45° right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Consider only planar motion of this molecule.

(a) Find three 'zero modes' for this system (*i.e.* normal modes whose associated eigenfrequencies vanish).

(b) Find the remaining three normal modes.

Solution

It is useful to choose the following coordinates:

$$(X_1, Y_1) = (x_1\, ,\, y_1) \tag{10.106}$$

$$(X_2, Y_2) = (a + x_2\, ,\, y_2) \tag{10.107}$$

$$(X_3, Y_3) = (x_3\, ,\, a + y_3)\ . \tag{10.108}$$

The three separations are then

$$\begin{aligned} d_{12} &= \sqrt{(a + x_2 - x_1)^2 + (y_2 - y_1)^2} \\ &= a + x_2 - x_1 + \ldots \end{aligned} \tag{10.109}$$

$$\begin{aligned} d_{23} &= \sqrt{(-a + x_3 - x_2)^2 + (a + y_3 - y_2)^2} \\ &= \sqrt{2}\, a - \tfrac{1}{\sqrt{2}}\big(x_3 - x_2\big) + \tfrac{1}{\sqrt{2}}\big(y_3 - y_2\big) + \ldots \end{aligned} \tag{10.110}$$

$$\begin{aligned} d_{13} &= \sqrt{(x_3 - x_1)^2 + (a + y_3 - y_1)^2} \\ &= a + y_3 - y_1 + \ldots\ . \end{aligned} \tag{10.111}$$

The potential is then

$$U = \tfrac{1}{2}k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}k\,\big(d_{23} - \sqrt{2}\,a\big)^2 + \tfrac{1}{2}k\,\big(d_{13} - a\big)^2 \tag{10.112}$$

$$\begin{aligned} &= \tfrac{1}{2}k\big(x_2 - x_1\big)^2 + \tfrac{1}{4}k\big(x_3 - x_2\big)^2 + \tfrac{1}{4}k\big(y_3 - y_2\big)^2 \\ &\qquad - \tfrac{1}{2}k\big(x_3 - x_2\big)\big(y_3 - y_2\big) + \tfrac{1}{2}k\big(y_3 - y_1\big)^2 \end{aligned} \tag{10.113}$$

Defining the row vector

$$\boldsymbol{\eta}^{\mathrm{t}} \equiv \big(x_1\ ,\ y_1\ ,\ x_2\ ,\ y_2\ ,\ x_3\ ,\ y_3\big)\ , \tag{10.114}$$

we have that U is a quadratic form:

$$U = \tfrac{1}{2}\eta_\sigma \mathrm{V}_{\sigma\sigma'}\eta_{\sigma'} = \tfrac{1}{2}\boldsymbol{\eta}^{\mathrm{t}}\,\mathrm{V}\,\boldsymbol{\eta}, \tag{10.115}$$

with

$$\mathrm{V} = \mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\mathrm{eq.}} = k \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{10.116}$$

The kinetic energy is simply

$$T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big)\ , \tag{10.117}$$

which entails

$$\mathrm{T}_{\sigma\sigma'} = m\,\delta_{\sigma\sigma'}\ . \tag{10.118}$$

(b) The three zero modes correspond to x-translation, y-translation, and rotation. Their eigenvectors, respectively, are

$$\boldsymbol{\psi}_1 = \frac{1}{\sqrt{3m}}\begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix}\quad,\quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{3m}}\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix}\quad,\quad \boldsymbol{\psi}_3 = \frac{1}{2\sqrt{3m}}\begin{pmatrix}1\\-1\\1\\2\\-2\\-1\end{pmatrix}\ . \tag{10.119}$$

To find the unnormalized rotation vector, we find the CM of the triangle, located at $\big(\frac{a}{3}\,,\,\frac{a}{3}\big)$, and sketch orthogonal displacements $\hat{\boldsymbol{z}} \times (\boldsymbol{R}_i - \boldsymbol{R}_{\mathrm{CM}})$ at the position of mass point i.

(c) The remaining modes may be determined by symmetry, and are given by

$$\boldsymbol{\psi}_4 = \frac{1}{2\sqrt{m}}\begin{pmatrix}-1\\-1\\0\\1\\1\\0\end{pmatrix}\quad,\quad \boldsymbol{\psi}_5 = \frac{1}{2\sqrt{m}}\begin{pmatrix}1\\-1\\-1\\0\\0\\1\end{pmatrix}\quad,\quad \boldsymbol{\psi}_6 = \frac{1}{2\sqrt{3m}}\begin{pmatrix}-1\\-1\\2\\-1\\-1\\2\end{pmatrix}\ , \tag{10.120}$$

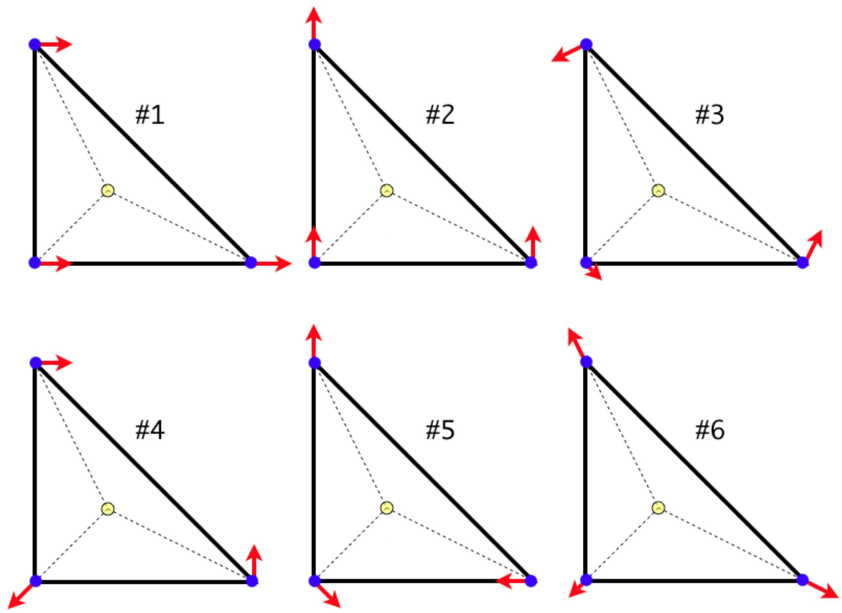


Figure 10.4: Normal modes of the 45° right triangle. The yellow circle is the location of the CM of the triangle.

with

$$\omega_1 = \sqrt{\frac{k}{m}} \quad , \quad \omega_2 = \sqrt{\frac{2k}{m}} \quad , \quad \omega_3 = \sqrt{\frac{3k}{m}} \ . \tag{10.121}$$

Since $\mathrm{T} = m \cdot 1$ is a multiple of the unit matrix, the orthogonormality relation $\psi_i^a \, \mathrm{T}_{ij} \, \psi_j^b = \delta^{ab}$ entails that the eigenvectors are mutually orthogonal in the usual dot product sense, with $\boldsymbol{\psi}_a \cdot \boldsymbol{\psi}_b = m^{-1} \, \delta_{ab}$. One can check that the eigenvectors listed here satisfy this condition.

The simplest of the set $\{\boldsymbol{\psi}_4, \boldsymbol{\psi}_5, \boldsymbol{\psi}_6\}$ to find is the uniform dilation $\boldsymbol{\psi}_6$, sometimes called the 'breathing' mode. This must keep the triangle in the same shape, which means that the deviations at each mass point are proportional to the distance to the CM. Next, it is simplest to find $\boldsymbol{\psi}_4$, in which the long and short sides of the triangle oscillate out of phase. Finally, the mode $\boldsymbol{\psi}_5$ must be orthogonal to all the remaining modes. No heavy lifting (*e.g.* `Mathematica`) is required!

10.9.2 Triple Pendulum

Consider a triple pendulum consisting of three identical masses m and three identical rigid massless rods of length ℓ, as depicted in Fig. 10.5.

(a) Find the T and V matrices.

(b) Find the equation for the eigenfrequencies.

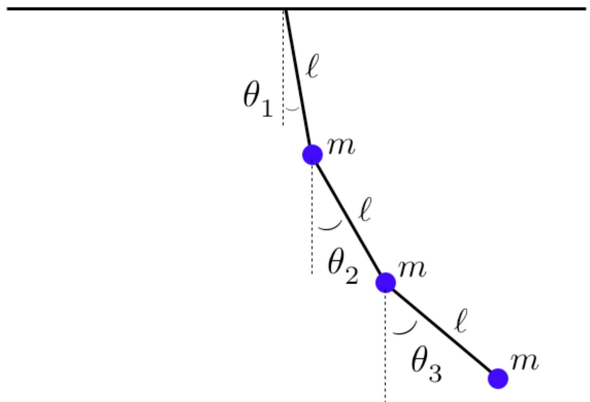


Figure 10.5: The triple pendulum.

(c) Numerically solve the eigenvalue equation for ratios ω_a^2/ω_0^2, where $\omega_0 = \sqrt{g/\ell}$. Find the three normal modes.

Solution

The Cartesian coordinates for the three masses are

$$\begin{aligned} x_1 &= \ell \sin\theta_1 & y_1 &= -\ell\cos\theta_1 \\ x_2 &= \ell\sin\theta_1 + \ell\sin\theta_2 & y_2 &= -\ell\cos\theta_1 - \ell\cos\theta_2 \\ x_3 &= \ell\sin\theta_1 + \ell\sin\theta_2 + \ell\sin\theta_3 & y_3 &= -\ell\cos\theta_1 - \ell\cos\theta_2 - \ell\cos\theta_3 \ . \end{aligned}$$

By inspection, we can write down the kinetic energy:

$$\begin{aligned} T &= \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^3 + \dot{y}_3^2\big) \\ &= \tfrac{1}{2} m\, \ell^2 \Big\{ 3\,\dot\theta_1^2 + 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\,\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 \\ &\qquad\qquad + 2\,\cos(\theta_1-\theta_3)\,\dot\theta_1\,\dot\theta_3 + 2\,\cos(\theta_2-\theta_3)\,\dot\theta_2\,\dot\theta_3 \Big\} \end{aligned}$$

The potential energy is

$$U = -mg\ell\Big\{3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3\Big\} \ ,$$

and the Lagrangian is $L = T - U$:

$$\begin{aligned} L = \tfrac{1}{2} m\,\ell^2 \Big\{ & 3\,\dot\theta_1^2 + 2\,\dot\theta_2^2 + \dot\theta_3^2 + 4\,\cos(\theta_1-\theta_2)\,\dot\theta_1\,\dot\theta_2 + 2\,\cos(\theta_1-\theta_3)\,\dot\theta_1\,\dot\theta_3 \\ & + 2\,\cos(\theta_2-\theta_3)\,\dot\theta_2\,\dot\theta_3 \Big\} + mg\ell\,\Big\{3\,\cos\theta_1 + 2\,\cos\theta_2 + \cos\theta_3\Big\} \ . \end{aligned}$$

The canonical momenta are given by

$$\begin{aligned}
\pi_1 &= \frac{\partial L}{\partial \dot\theta_1} = m\,\ell^2 \Big\{ 3\,\dot\theta_1 + 2\,\dot\theta_2\,\cos(\theta_1-\theta_2) + \dot\theta_3\,\cos(\theta_1-\theta_3) \Big\} \\
\pi_2 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ 2\,\dot\theta_2 + 2\,\dot\theta_1\,\cos(\theta_1-\theta_2) + \dot\theta_3\,\cos(\theta_2-\theta_3) \Big\} \\
\pi_3 &= \frac{\partial L}{\partial \dot\theta_2} = m\,\ell^2 \Big\{ \dot\theta_3 + \dot\theta_1\,\cos(\theta_1-\theta_3) + \dot\theta_2\,\cos(\theta_2-\theta_3) \Big\} \ .
\end{aligned}$$

The only conserved quantity is the total energy, $E = T + U$.

(a) As for the T and V matrices, we have

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = m\ell^2 \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = mg\ell \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ .$$

(b) The eigenfrequencies are roots of the equation $\det\,(\omega^2\,\mathrm{T} - \mathrm{V}) = 0$. Defining $\omega_0 \equiv \sqrt{g/\ell}$, we have

$$\omega^2\,\mathrm{T} - \mathrm{V} = m\ell^2 \begin{pmatrix} 3(\omega^2-\omega_0^2) & 2\,\omega^2 & \omega^2 \\ 2\,\omega^2 & 2(\omega^2-\omega_0^2) & \omega^2 \\ \omega^2 & \omega^2 & (\omega^2-\omega_0^2) \end{pmatrix}$$

and hence

$$\begin{aligned}
\det\,(\omega^2\mathrm{T} - \mathrm{V}) &= 3(\omega^2-\omega_0^2)\cdot\Big[2(\omega^2-\omega_0^2)^2 - \omega^4\Big] - 2\,\omega^2\cdot\Big[2\,\omega^2(\omega^2-\omega_0^2) - \omega^4\Big] \\
&\qquad + \omega^2\cdot\Big[2\,\omega^4 - 2\,\omega^2(\omega^2-\omega_0^2)\Big] \\
&= 6\,(\omega^2-\omega_0^2)^3 - 9\,\omega^4\,(\omega^2-\omega_0^2) + 4\omega^6 \\
&= \omega^6 - 9\,\omega_0^2\,\omega^4 + 18\,\omega_0^4\,\omega^2 - 6\,\omega_0^6 \ .
\end{aligned}$$

(c) The equation for the eigenfrequencies is

$$\lambda^3 - 9\,\lambda^2 + 18\,\lambda - 6 = 0 \ , \tag{10.122}$$

where $\omega^2 = \lambda\,\omega_0^2$. This is a cubic equation in λ. Numerically solving for the roots, one finds

$$\omega_1^2 = 0.415774\,\omega_0^2 \quad , \quad \omega_2^2 = 2.29428\,\omega_0^2 \quad , \quad \omega_3^2 = 6.28995\,\omega_0^2 \ . \tag{10.123}$$

I find the (unnormalized) eigenvectors to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} 1 \\ 1.2921 \\ 1.6312 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \begin{pmatrix} 1 \\ 0.35286 \\ -2.3981 \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \begin{pmatrix} 1 \\ -1.6450 \\ 0.76690 \end{pmatrix} \ . \tag{10.124}$$

10.9.3 Equilateral Linear Triatomic Molecule

Consider the vibrations of an equilateral triangle of mass points, depicted in figure 10.6 . The system is confined to the (x, y) plane, and in equilibrium all the strings are unstretched and of length a.

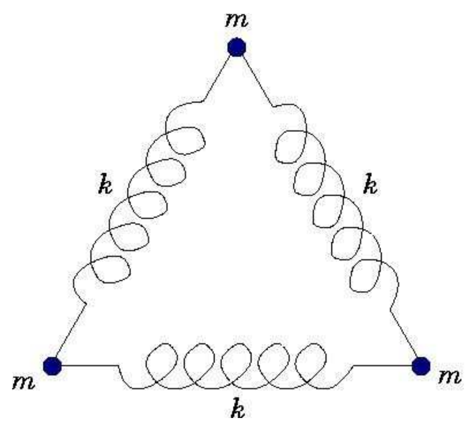


Figure 10.6: An equilateral triangle of identical mass points and springs.

(a) Choose as generalized coordinates the Cartesian displacements (x_i, y_i) with respect to equilibrium. Write down the exact potential energy.

(b) Find the T and V matrices.

(c) There are three normal modes of oscillation for which the corresponding eigenfrequencies all vanish: $\omega_a = 0$. Write down these modes explicitly, and provide a physical interpretation for why $\omega_a = 0$. Since this triplet is degenerate, there is no unique answer – any linear combination will also serve as a valid 'zero mode'. However, if you think physically, a natural set should emerge.

(d) The three remaining modes all have finite oscillation frequencies. They correspond to distortions of the triangular shape. One such mode is the "breathing mode" in which the triangle uniformly expands and contracts. Write down the eigenvector associated with this normal mode and compute its associated oscillation frequency.

(e) The fifth and sixth modes are degenerate. They must be orthogonal (with respect to the inner product defined by T) to all the other modes. See if you can figure out what these modes are, and compute their oscillation frequencies. As in (a), any linear combination of these modes will also be an eigenmode.

(f) Write down your full expression for the modal matrix A_{ai}, and check that it is correct by using `Mathematica`.

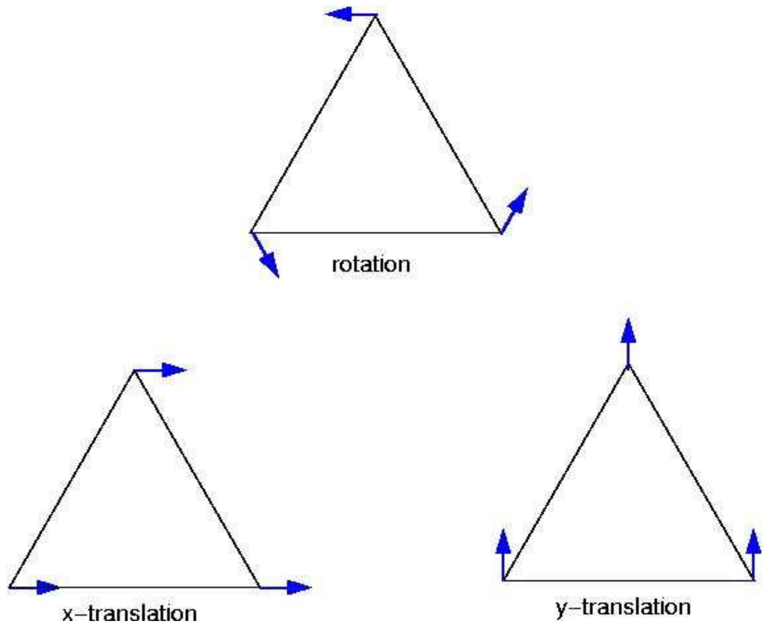


Figure 10.7: Zero modes of the mass-spring triangle.

Solution

Choosing as generalized coordinates the Cartesian displacements relative to equilibrium, we have the following:

$$\begin{aligned}
\#1 &: \big(x_1, y_1\big) \\
\#2 &: \big(a + x_2, y_2\big) \\
\#3 &: \big(\tfrac{1}{2}a + x_3, \tfrac{\sqrt{3}}{2}a + y_3\big) \ .
\end{aligned}$$

Let d_{ij} be the separation of particles i and j. The potential energy of the spring connecting them is then $\frac{1}{2}\,k\,(d_{ij} - a)^2$.

$$\begin{aligned}
d_{12}^2 &= \big(a + x_2 - x_1\big)^2 + \big(y_2 - y_1\big)^2 \\
d_{23}^2 &= \big(-\tfrac{1}{2}a + x_3 - x_2\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_2\big)^2 \\
d_{13}^2 &= \big(\tfrac{1}{2}a + x_3 - x_1\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_1\big)^2 \ .
\end{aligned}$$

The full potential energy is

$$U = \tfrac{1}{2}\,k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{23} - a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{13} - a\big)^2 \ . \tag{10.125}$$

This is a cumbersome expression, involving square roots.

To find T and V, we need to write T and V as quadratic forms, neglecting higher order terms. Therefore, we must expand $d_{ij} - a$ to linear order in the generalized coordinates.

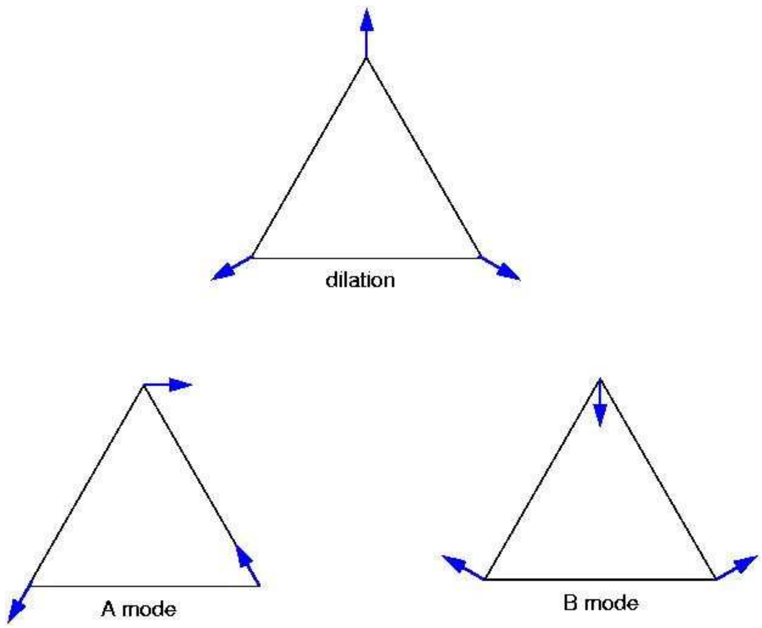


Figure 10.8: Finite oscillation frequency modes of the mass-spring triangle.

This results in the following:

$$\begin{aligned}
d_{12} &= a + \big(x_2 - x_1\big) + \ldots \\
d_{23} &= a - \tfrac{1}{2}\big(x_3 - x_2\big) + \tfrac{\sqrt{3}}{2}\big(y_3 - y_2\big) + \ldots \\
d_{13} &= a + \tfrac{1}{2}\big(x_3 - x_1\big) + \tfrac{\sqrt{3}}{2}\big(y_3 - y_1\big) + \ldots \ .
\end{aligned}$$

Thus,

$$\begin{aligned}
U = \tfrac{1}{2}\,k\,\big(x_2 - x_1\big)^2 &+ \tfrac{1}{8}\,k\,\big(x_2 - x_3 - \sqrt{3}\,y_2 + \sqrt{3}\,y_3\big)^2 \\
&+ \tfrac{1}{8}\,k\,\big(x_3 - x_1 + \sqrt{3}\,y_3 - \sqrt{3}\,y_1\big)^2 + \text{higher order terms} \ .
\end{aligned}$$

Defining

$$\big(q_1, q_2, q_3, q_4, q_5, q_6\big) = \big(x_1, y_1, x_2, y_2, x_3, y_3\big) \ ,$$

we may now read off

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{\bar{\mathrm{q}}} = k \begin{pmatrix} 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 \end{pmatrix}$$

The T matrix is trivial. From

$$T = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \ .$$

Figure 10.9: *John Henry*, statue by Charles O. Cooper (1972). "Now the man that invented the steam drill, he thought he was mighty fine. But John Henry drove fifteen feet, and the steam drill only made nine." - from *The Ballad of John Henry*.

we obtain

$$\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial \dot{q}_i \, \partial \dot{q}_j} = m \, \delta_{ij} \ ,$$

and $\mathrm{T} = m \cdot \mathbb{I}$ is a multiple of the unit matrix.

The zero modes are depicted graphically in figure 10.7. Explicitly, we have

$$\boldsymbol{\xi}_x = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_y = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\text{rot}} = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1/2 \\ -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ -1 \\ 0 \end{pmatrix} \ .$$

That these are indeed zero modes may be verified by direct multiplication:

$$\mathrm{V} \, \boldsymbol{\xi}_{x.y} = \mathrm{V} \, \boldsymbol{\xi}_{\text{rot}} = 0 \ . \tag{10.126}$$

The three modes with finite oscillation frequency are depicted graphically in figure 10.8.

Explicitly, we have

$$\boldsymbol{\xi}_{\rm A} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\rm B} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ 1/2 \\ \sqrt{3}/2 \\ 1/2 \\ 0 \\ -1 \end{pmatrix} \quad , \quad \boldsymbol{\xi}_{\rm dil} = \frac{1}{\sqrt{3m}} \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ 0 \\ 1 \end{pmatrix} .$$

The oscillation frequencies of these modes are easily checked by multiplying the eigenvectors by the matrix V. Since $\mathrm{T} = m \cdot \mathbb{I}$ is diagonal, we have $\mathrm{V}\,\boldsymbol{\xi}_a = m\omega_a^2\,\boldsymbol{\xi}_a$. One finds

$$\omega_{\rm A} = \omega_{\rm B} = \sqrt{\frac{3k}{2m}} \quad , \quad \omega_{\rm dil} = \sqrt{\frac{3k}{m}} .$$

`Mathematica`? I don't need no stinking `Mathematica`.

10.10 Aside : Christoffel Symbols

The coupled equations in eqn. 10.5 may be written in the form

$$\ddot{q}_\sigma + \Gamma^\sigma_{\mu\nu}\,\dot{q}_\mu\,\dot{q}_\nu = F_\sigma \ , \tag{10.127}$$

with

$$\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2}\,T^{-1}_{\sigma\alpha}\bigg(\frac{\partial T_{\alpha\mu}}{\partial q_\nu} + \frac{\partial T_{\alpha\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\alpha}\bigg) \tag{10.128}$$

and

$$F_\sigma = -T^{-1}_{\sigma\alpha}\,\frac{\partial U}{\partial q_\alpha} \ . \tag{10.129}$$

The components of the rank-three tensor $\Gamma^\sigma_{\alpha\beta}$ are known as *Christoffel symbols*, in the case where $T_{\mu\nu}(q)$ defines a *metric* on the space of generalized coordinates.

Chapter 11

Elastic Collisions

11.1 Center of Mass Frame

A collision or 'scattering event' is said to be *elastic* if it results in no change in the internal state of any of the particles involved. Thus, no internal energy is liberated or captured in an elastic process.

Consider the elastic scattering of two particles. Recall the relation between laboratory coordinates $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ and the CM and relative coordinates $\{\boldsymbol{R}, \boldsymbol{r}\}$:

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad \boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\,\boldsymbol{r} \tag{11.1}$$

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\,\boldsymbol{r} \tag{11.2}$$

If external forces are negligible, the CM momentum $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ is constant, and therefore the frame of reference whose origin is tied to the CM position is an inertial frame of reference. In this frame,

$$\boldsymbol{v}_1^{\text{CM}} = \frac{m_2\,\boldsymbol{v}}{m_1 + m_2} \quad , \quad \boldsymbol{v}_2^{\text{CM}} = -\frac{m_1\,\boldsymbol{v}}{m_1 + m_2} \ , \tag{11.3}$$

where $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_1^{\text{CM}} - \boldsymbol{v}_2^{\text{CM}}$ is the relative velocity, which is the same in both L and CM frames. Note that the CM momenta satisfy

$$\boldsymbol{p}_1^{\text{CM}} = m_1 \boldsymbol{v}_1^{\text{CM}} = \mu \boldsymbol{v} \tag{11.4}$$

$$\boldsymbol{p}_2^{\text{CM}} = m_2 \boldsymbol{v}_2^{\text{CM}} = -\mu \boldsymbol{v} \ , \tag{11.5}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Thus, $\boldsymbol{p}_1^{\text{CM}} + \boldsymbol{p}_2^{\text{CM}} = 0$ and the total momentum in the CM frame is zero. We may then write

$$\boldsymbol{p}_1^{\text{CM}} \equiv p_0 \hat{\boldsymbol{n}} \quad , \quad \boldsymbol{p}_2^{\text{CM}} \equiv -p_0 \hat{\boldsymbol{n}} \quad \Rightarrow \quad E^{\text{CM}} = \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} = \frac{p_0^2}{2\mu} \ . \tag{11.6}$$

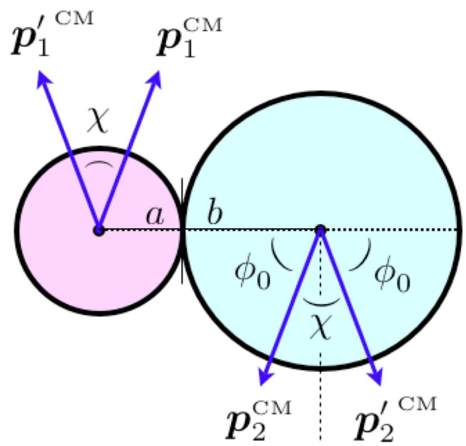


Figure 11.1: The scattering of two hard spheres of radii a and b The scattering angle is χ.

The energy is evaluated when the particles are asymptotically far from each other, in which case the potential energy is assumed to be negligible. After the collision, energy and momentum conservation require

$$\boldsymbol{p}_1'^{\,\rm CM} \equiv p_0\hat{\boldsymbol{n}}' \qquad , \qquad \boldsymbol{p}_2'^{\,\rm CM} \equiv -p_0\hat{\boldsymbol{n}}' \qquad \Rightarrow \qquad E'^{\rm CM} = E^{\rm CM} = \frac{p_0^2}{2\mu} \ . \tag{11.7}$$

The angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is the *scattering angle* χ:

$$\boldsymbol{n}\cdot\boldsymbol{n}' \equiv \cos\chi \ . \tag{11.8}$$

The value of χ depends on the details of the scattering process, *i.e.* on the interaction potential $U(r)$. As an example, consider the scattering of two hard spheres, depicted in Fig. 11.1. The potential is

$$U(r) = \begin{cases} \infty & \text{if } r \le a+b \\ 0 & \text{if } r > a+b \ . \end{cases} \tag{11.9}$$

Clearly the scattering angle is $\chi = \pi - 2\phi_0$, where ϕ_0 is the angle between the initial momentum of either sphere and a line containing their two centers at the moment of contact.

There is a simple geometric interpretation of these results, depicted in Fig. 11.2. We have

$$\boldsymbol{p}_1 = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_1' = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}}' \tag{11.10}$$

$$\boldsymbol{p}_2 = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_2' = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}}' \ . \tag{11.11}$$

So draw a circle of radius p_0 whose center is the origin. The vectors $p_0\hat{\boldsymbol{n}}$ and $p_0\hat{\boldsymbol{n}}'$ must both lie along this circle. We define the angle ψ between $\boldsymbol{V}$ and $\boldsymbol{n}$:

$$\hat{\boldsymbol{V}}\cdot\boldsymbol{n} = \cos\psi \ . \tag{11.12}$$

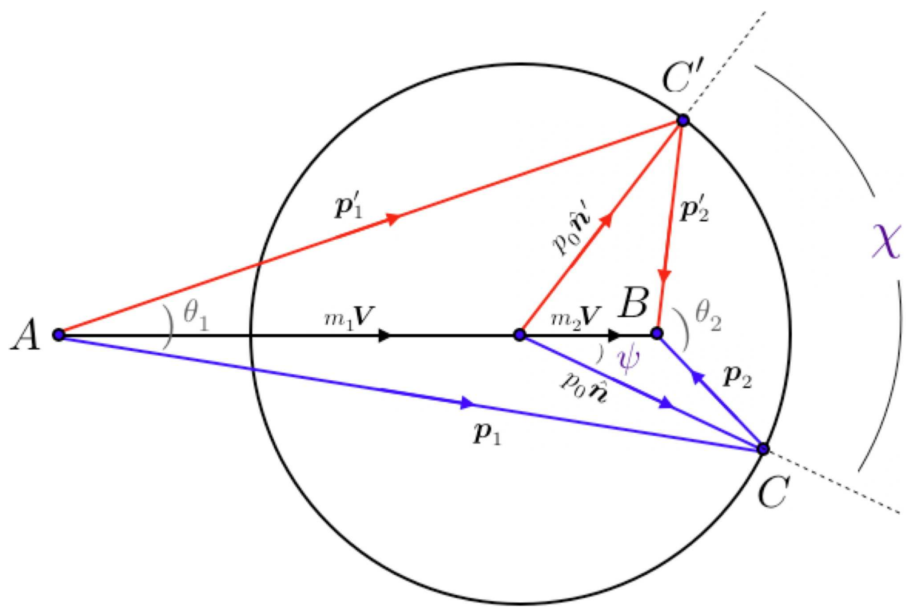


Figure 11.2: Scattering of two particles of masses m_1 and m_2. The scattering angle χ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

It is now an exercise in geometry, using the law of cosines, to determine everything of interest in terms of the quantities V, v, ψ, and χ. For example, the momenta are

$$p_1 = \sqrt{m_1^2\,V^2 + \mu^2 v^2 + 2m_1\mu V v\cos\psi} \tag{11.13}$$

$$p_1' = \sqrt{m_1^2\,V^2 + \mu^2 v^2 + 2m_1\mu V v\cos(\chi-\psi)} \tag{11.14}$$

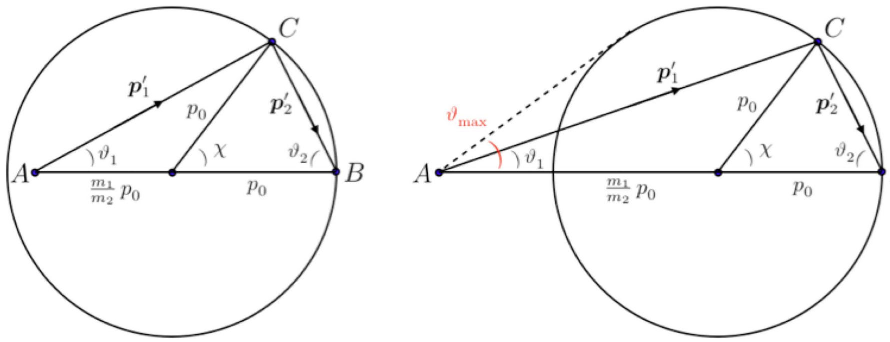
$$p_2 = \sqrt{m_2^2\,V^2 + \mu^2 v^2 - 2m_2\mu V v\cos\psi} \tag{11.15}$$

$$p_2' = \sqrt{m_2^2\,V^2 + \mu^2 v^2 - 2m_2\mu V v\cos(\chi-\psi)}\ , \tag{11.16}$$

and the scattering angles are

$$\theta_1 = \tan^{-1}\!\left(\frac{\mu v\sin\psi}{\mu v\cos\psi + m_1 V}\right) + \tan^{-1}\!\left(\frac{\mu v\sin(\chi-\psi)}{\mu v\cos(\chi-\psi) + m_1 V}\right) \tag{11.17}$$

$$\theta_2 = \tan^{-1}\!\left(\frac{\mu v\sin\psi}{\mu v\cos\psi - m_2 V}\right) + \tan^{-1}\!\left(\frac{\mu v\sin(\chi-\psi)}{\mu v\cos(\chi-\psi) - m_2 V}\right)\ . \tag{11.18}$$



(a) $m_1 < m_2$ (b) $m_1 > m_2$

Figure 11.3: Scattering when particle 2 is initially at rest.

If particle 2, say, is initially at rest, the situation is somewhat simpler. In this case, $\boldsymbol{V} = m_1\boldsymbol{V}/(m_1+m_2)$ and $m_2\boldsymbol{V} = \mu\boldsymbol{v}$, which means the point B lies on the circle in Fig. 11.3 ($m_1 \neq m_2$) and Fig. 11.4 ($m_1 = m_2$). Let $\vartheta_{1,2}$ be the angles between the directions of motion after the collision and the direction $\boldsymbol{V}$ of impact. The scattering angle χ is the angle through which particle 1 turns in the CM frame. Clearly

$$\tan\vartheta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \qquad , \qquad \vartheta_2 = \tfrac{1}{2}(\pi - \chi) \ . \tag{11.19}$$

We can also find the speeds v_1' and v_2' in terms of v and χ, from

$$p_1'^{\,2} = p_0^2 + \big(\tfrac{m_1}{m_2}\,p_0\big)^2 - 2\,\tfrac{m_1}{m_2}\,p_0^2\,\cos(\pi-\chi) \tag{11.20}$$

and

$$p_2^2 = 2\,p_0^2\,(1-\cos\chi) \ . \tag{11.21}$$

These equations yield

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2\cos\chi}}{m_1+m_2}\,v \qquad , \qquad v_2' = \frac{2m_1 v}{m_1+m_2}\,\sin(\tfrac{1}{2}\chi) \ . \tag{11.22}$$

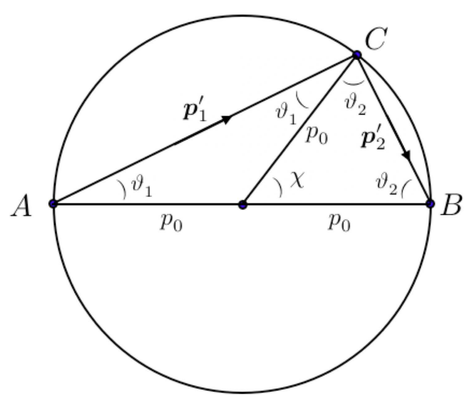


Figure 11.4: Scattering of identical mass particles when particle 2 is initially at rest.

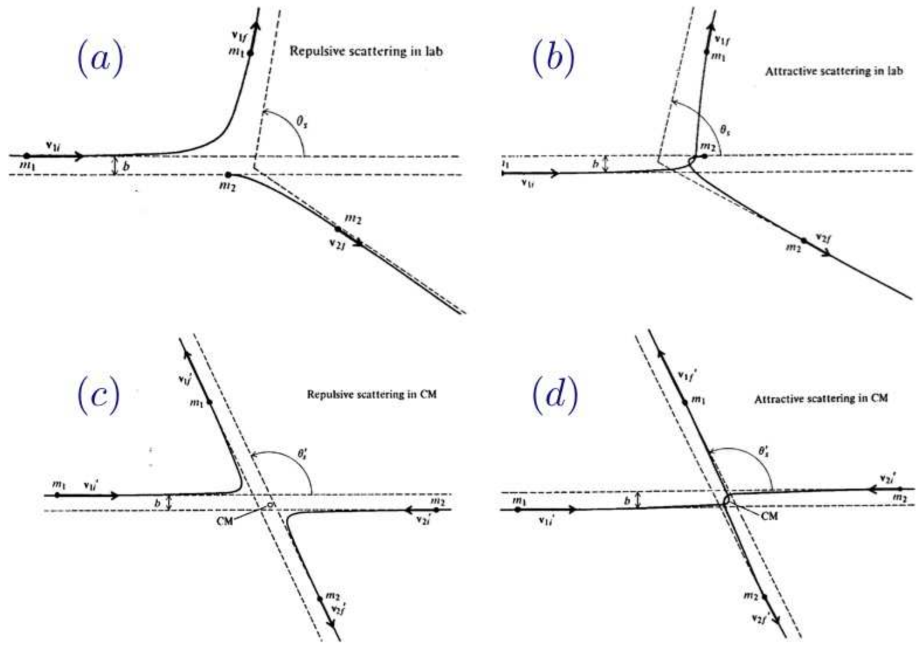


Figure 11.5: Repulsive (A,C) and attractive (B,D) scattering in the lab (A,B) and CM (C,D) frames, assuming particle 2 starts from rest in the lab frame. (From Barger and Olsson.)

The angle $\vartheta_{\rm max}$ from Fig. 11.3(b) is given by $\sin\vartheta_{\rm max} = \frac{m_2}{m_1}$. Note that when $m_1 = m_2$ we have $\vartheta_1 + \vartheta_2 = \pi$. A sketch of the orbits in the cases of both repulsive and attractive scattering, in both the laboratory and CM frames, in shown in Fig. 11.5.

11.2 Central Force Scattering

Consider a single particle of mass μ movng in a central potential $U(r)$, or a two body central force problem in which μ is the reduced mass. Recall that

$$\frac{dr}{dt} = \frac{d\phi}{dt} \cdot \frac{dr}{d\phi} = \frac{\ell}{\mu r^2} \cdot \frac{dr}{d\phi} \ , \tag{11.23}$$

and therefore

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r) \ . \end{aligned} \tag{11.24}$$

Solving for $\frac{dr}{d\phi}$, we obtain

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2\mu r^4}{\ell^2}\big(E - U(r)\big) - r^2} \ , \tag{11.25}$$

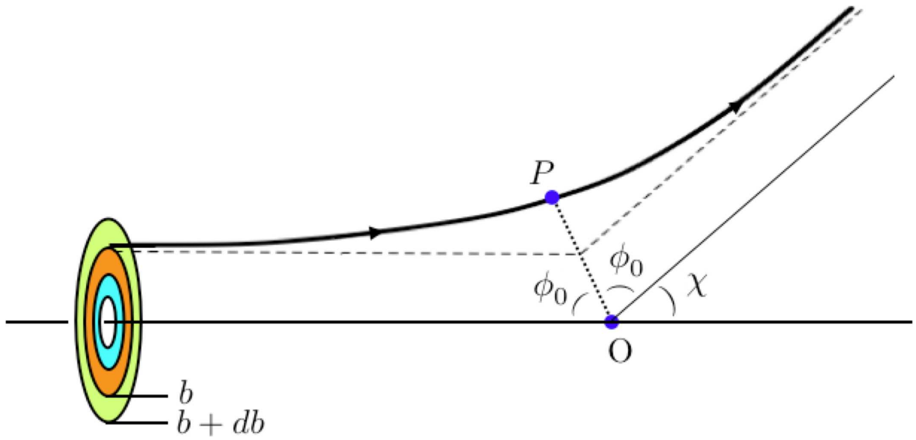


Figure 11.6: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

Consulting Fig. 11.6, we have that

$$\phi_0 = \frac{\ell}{\sqrt{2\mu}} \int_{r_{\rm p}}^{\infty} \frac{dr}{r^2 \sqrt{E - U_{\rm eff}(r)}} \ , \tag{11.26}$$

where $r_{\rm p}$ is the radial distance at periapsis, and where

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \tag{11.27}$$

is the effective potential, as before. From Fig. 11.6, we conclude that the scattering angle is

$$\chi = \big|\pi - 2\phi_0\big| \ . \tag{11.28}$$

It is convenient to define the *impact parameter* b as the distance of the asymptotic trajectory from a parallel line containing the force center. The geometry is shown again in Fig. 11.6. Note that the energy and angular momentum, which are conserved, can be evaluated at infinity using the impact parameter:

$$E = \tfrac{1}{2}\mu v_\infty^2 \qquad , \qquad \ell = \mu v_\infty b \ . \tag{11.29}$$

Substituting for $\ell(b)$, we have

$$\phi_0(E, b) = \int_{r_{\rm p}}^{\infty} \frac{dr}{r^2} \, \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U(r)}{E}}} \ , \tag{11.30}$$

In physical applications, we are often interested in the deflection of a beam of incident particles by a scattering center. We define the *differential scattering cross section* $d\sigma$ by

$$d\sigma = \frac{\#\text{ of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \ . \tag{11.31}$$

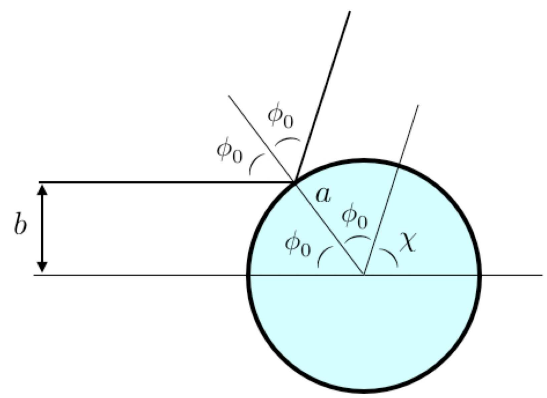


Figure 11.7: Geometry of hard sphere scattering.

Now for particles of a given energy E there is a unique relationship between the scattering angle χ and the impact parameter b, as we have just derived in eqn. 11.30. The differential solid angle is given by $d\Omega = 2\pi \sin\chi \, d\chi$, hence

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi} \left| \frac{db}{d\chi} \right| = \left| \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} \right| . \tag{11.32}$$

Note that $\frac{d\sigma}{d\Omega}$ has dimensions of area. The integral of $\frac{d\sigma}{d\Omega}$ over all solid angle is the *total scattering cross section*,

$$\sigma_{\rm T} = 2\pi \int\limits_0^\pi d\chi \, \sin\chi \, \frac{d\sigma}{d\Omega} . \tag{11.33}$$

11.2.1 Hard sphere scattering

Consider a point particle scattering off a hard sphere of radius a, or two hard spheres of radii a_1 and a_2 scattering off each other, with $a \equiv a_1 + a_2$. From the geometry of Fig. 11.7, we have $b = a \sin\phi_0$ and $\phi_0 = \frac{1}{2}(\pi - \chi)$, so

$$b^2 = a^2 \sin^2\left(\tfrac{1}{2}\pi - \tfrac{1}{2}\chi\right) = \tfrac{1}{2}a^2\,(1 + \cos\chi) . \tag{11.34}$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} = \tfrac{1}{4}\,a^2 \tag{11.35}$$

and $\sigma_{\rm T} = \pi a^2$. The total scattering cross section is simply the area of a sphere of radius a projected onto a plane perpendicular to the incident flux.

11.2.2 Rutherford scattering

Consider scattering by the Kepler potential $U(r) = -\frac{k}{r}$. We assume that the orbits are unbound, *i.e.* they are Keplerian hyperbolae with $E > 0$, described by the equation

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon\,\cos\phi} \qquad \Rightarrow \qquad \cos\phi_0 = \pm\,\frac{1}{\varepsilon}\ . \tag{11.36}$$

Recall that the eccentricity is given by

$$\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} = 1 + \left(\frac{\mu b v_\infty}{k}\right)^2 . \tag{11.37}$$

We then have

$$\begin{aligned}\left(\frac{\mu b v_\infty}{k}\right)^2 &= \varepsilon^2 - 1 \\ &= \sec^2\!\phi_0 - 1 = \tan^2\!\phi_0 = \text{ctn}^2\big(\tfrac{1}{2}\chi\big)\ .\end{aligned} \tag{11.38}$$

Therefore

$$b(\chi) = \frac{k}{\mu v_\infty^2}\,\text{ctn}\big(\tfrac{1}{2}\chi\big) \tag{11.39}$$

We finally obtain

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} = \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d\,\text{ctn}^2\big(\frac{1}{2}\chi\big)}{d\cos\chi} \\ &= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d}{d\cos\chi}\left(\frac{1+\cos\chi}{1-\cos\chi}\right) \\ &= \left(\frac{k}{2\mu v_\infty^2}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big)\ ,\end{aligned} \tag{11.40}$$

which is the same as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big)\ . \tag{11.41}$$

Since $\frac{d\sigma}{d\Omega} \propto \chi^{-4}$ as $\chi \to 0$, the total cross section $\sigma_{\rm T}$ diverges! This is a consequence of the long-ranged nature of the Kepler/Coulomb potential. In electron-atom scattering, the Coulomb potential of the nucleus is *screened* by the electrons of the atom, and the $1/r$ behavior is cut off at large distances.

11.2.3 Transformation to laboratory coordinates

We previously derived the relation

$$\tan\vartheta = \frac{\sin\chi}{\gamma + \cos\chi}\ , \tag{11.42}$$

where $\vartheta \equiv \vartheta_1$ is the scattering angle for particle 1 in the laboratory frame, and $\gamma = \frac{m_1}{m_2}$ is the ratio of the masses. We now derive the differential scattering cross section in the laboratory frame. To do so, we note that particle conservation requires

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} \cdot\ 2\pi \sin\vartheta\, d\vartheta = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \cdot\ 2\pi \sin\chi\, d\chi\ , \tag{11.43}$$

which says

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \cdot\ \frac{d\cos\chi}{d\cos\vartheta}\ . \tag{11.44}$$

From

$$\begin{aligned}\cos\vartheta &= \frac{1}{\sqrt{1+\tan^2\!\vartheta}} \\ &= \frac{\gamma+\cos\chi}{\sqrt{1+\gamma^2+2\gamma\cos\chi}}\ ,\end{aligned} \tag{11.45}$$

we derive

$$\frac{d\cos\vartheta}{d\cos\chi} = \frac{1+\gamma\cos\chi}{\big(1+\gamma^2+2\gamma\cos\chi\big)^{3/2}} \tag{11.46}$$

and, accordingly,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} = \frac{\big(1+\gamma^2+2\gamma\cos\chi\big)^{3/2}}{1+\gamma\cos\chi} \cdot \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM}\ . \tag{11.47}$$

Chapter 12

Noninertial Reference Frames

12.1 Accelerated Coordinate Systems

A reference frame which is fixed with respect to a rotating rigid body is not inertial. The parade example of this is an observer fixed on the surface of the earth. Due to the rotation of the earth, such an observer is in a noninertial frame, and there are corresponding corrections to Newton's laws of motion which must be accounted for in order to correctly describe mechanical motion in the observer's frame. As is well known, these corrections involve fictitious centrifugal and Coriolis forces.

Consider an inertial frame with a fixed set of coordinate axes $\hat{\boldsymbol{e}}_\mu$, where μ runs from 1 to d, the dimension of space. Any vector $\boldsymbol{A}$ may be written in either basis:

$$\boldsymbol{A} = \sum_\mu A_\mu\, \hat{\boldsymbol{e}}_\mu = \sum_\mu A'_\mu\, \hat{\boldsymbol{e}}'_\mu \ , \tag{12.1}$$

where $A_\mu = \boldsymbol{A} \cdot \hat{\boldsymbol{e}}_\mu$ and $A'_\mu = \boldsymbol{A} \cdot \hat{\boldsymbol{e}}'_\mu$ are projections onto the different coordinate axes. We may now write

$$\begin{aligned} \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} &= \sum_\mu \frac{dA_\mu}{dt}\, \hat{\boldsymbol{e}}_\mu \\ &= \sum_i \frac{dA'_\mu}{dt}\, \hat{\boldsymbol{e}}'_\mu + \sum_i A'_\mu\, \frac{d\hat{\boldsymbol{e}}'_\mu}{dt} \ . \end{aligned} \tag{12.2}$$

The first term on the RHS is $(d\boldsymbol{A}/dt)_{\text{body}}$, the time derivative of $\boldsymbol{A}$ along body-fixed axes, *i.e.* as seen by an observer rotating with the body. But what is $d\hat{\boldsymbol{e}}'_i/dt$? Well, we can always expand it in the $\{\hat{\boldsymbol{e}}'_i\}$ basis:

$$d\hat{\boldsymbol{e}}'_\mu = \sum_j d\Omega_{\mu\nu}\, \hat{\boldsymbol{e}}'_\nu \quad \Longleftrightarrow \quad d\Omega_{\mu\nu} \equiv d\hat{\boldsymbol{e}}'_\mu \cdot \hat{\boldsymbol{e}}'_\nu \ . \tag{12.3}$$

Note that $d\Omega_{\mu\nu} = -d\Omega_{\nu\mu}$ is antisymmetric, because

$$0 = d\big(\hat{\boldsymbol{e}}'_\mu \cdot \hat{\boldsymbol{e}}'_\nu\big) = d\Omega_{\nu\mu} + d\Omega_{\mu\nu} \ , \tag{12.4}$$

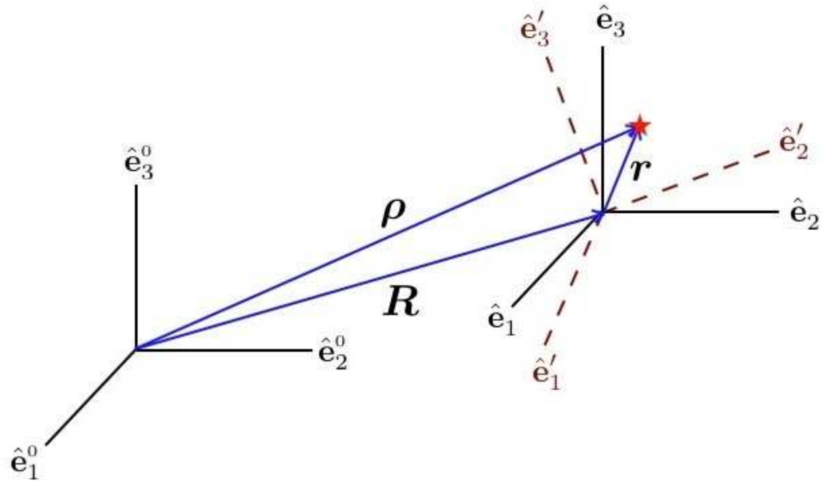


Figure 12.1: Reference frames related by both translation and rotation.

because $\hat{\boldsymbol{e}}'_\mu \cdot \hat{\boldsymbol{e}}'_\nu = \delta_{\mu\nu}$ is a constant. Now we may define $d\Omega_{12} \equiv d\Omega_3$, *et cyc.*, so that

$$d\Omega_{\mu\nu} = \sum_\sigma \epsilon_{\mu\nu\sigma}\, d\Omega_\sigma \quad , \quad \omega_\sigma \equiv \frac{d\Omega_\sigma}{dt} \ , \tag{12.5}$$

which yields

$$\frac{d\hat{\boldsymbol{e}}'_\mu}{dt} = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}'_\mu \ . \tag{12.6}$$

Finally, we obtain the important result

$$\boxed{\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{A}} \tag{12.7}$$

which is valid for any vector $\boldsymbol{A}$.

Applying this result to the position vector $\boldsymbol{r}$, we have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \ . \tag{12.8}$$

Applying it twice,

$$\begin{aligned}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega}\times\right)\left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega}\times\right)\boldsymbol{r} \\ &= \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt}\times\boldsymbol{r} + 2\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \ .\end{aligned} \tag{12.9}$$

Note that $d\boldsymbol{\omega}/dt$ appears with no "inertial" or "body" label. This is because, upon invoking eq. 12.7,

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{body}} + \boldsymbol{\omega}\times\boldsymbol{\omega} \ , \tag{12.10}$$

and since $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$, inertial and body-fixed observers will agree on the value of $\dot{\boldsymbol{\omega}}_{\text{inertial}} = \dot{\boldsymbol{\omega}}_{\text{body}} \equiv \dot{\boldsymbol{\omega}}$.

12.1.1 Translations

Suppose that frame K moves with respect to an inertial frame K^0, such that the origin of K lies at $\boldsymbol{R}(t)$. Suppose further that frame K' rotates with respect to K, but shares the same origin (see Fig. 12.1). Consider the motion of an object lying at position $\boldsymbol{\rho}$ relative to the origin of K^0, and $\boldsymbol{r}$ relative to the origin of K/K'. Thus,

$$\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r} \ , \tag{12.11}$$

and

$$\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{R}}{dt}\right)_{\text{inertial}} + \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \tag{12.12}$$

$$\begin{aligned}\left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} \\ &\quad + 2\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \ .\end{aligned} \tag{12.13}$$

Here, $\boldsymbol{\omega}$ is the angular velocity in the frame K or K'.

12.1.2 Motion on the surface of the earth

The earth both rotates about its axis and orbits the Sun. If we add the infinitesimal effects of the two rotations,

$$\begin{aligned} d\boldsymbol{r}_1 &= \boldsymbol{\omega}_1 \times \boldsymbol{r}\, dt \\ d\boldsymbol{r}_2 &= \boldsymbol{\omega}_2 \times (\boldsymbol{r} + d\boldsymbol{r}_1)\, dt \\ d\boldsymbol{r} &= d\boldsymbol{r}_1 + d\boldsymbol{r}_2 \\ &= (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\, dt \times \boldsymbol{r} + \mathcal{O}\big((dt)^2\big) \ . \end{aligned} \tag{12.14}$$

Thus, *infinitesimal rotations add.* Dividing by dt, this means that

$$\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}_i \ , \tag{12.15}$$

where the sum is over all the rotations. For the earth, $\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rot}} + \boldsymbol{\omega}_{\text{orb}}$.

- The rotation about earth's axis, $\boldsymbol{\omega}_{\text{rot}}$ has magnitude $\omega_{\text{rot}} = 2\pi/(1\,\text{day}) = 7.29 \times 10^{-5}\,\text{s}^{-1}$. The radius of the earth is $R_{\text{e}} = 6.37 \times 10^3$ km.

- The orbital rotation about the Sun, $\boldsymbol{\omega}_{\text{orb}}$ has magnitude $\omega_{\text{orb}} = 2\pi/(1\,\text{yr}) = 1.99 \times 10^{-7}\,\text{s}^{-1}$. The radius of the earth is $a_{\text{e}} = 1.50 \times 10^{8}\,\text{km}$.

Thus, $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} = 365.25$, which is of course the number of days (*i.e.* rotational periods) in a year (*i.e.* orbital period). There is also a very slow precession of the earth's axis of rotation, the period of which is about 25,000 years, which we will ignore. Note $\dot{\boldsymbol{\omega}} = 0$ for the earth. Thus, applying Newton's second law and then invoking eq. 12.14, we arrive at

$$m\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{earth}} = \boldsymbol{F}^{(\text{tot})} - m\left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{Sun}} - \ 2m\,\boldsymbol{\omega}\times\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{earth}} - \ m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) \ , \qquad (12.16)$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rot}} + \boldsymbol{\omega}_{\text{orb}}$, and where $\ddot{\boldsymbol{R}}_{\text{Sun}}$ is the acceleration of the center of the earth around the Sun, assuming the Sun-fixed frame to be inertial. The force $\boldsymbol{F}_{(\text{tot})}$ is the total force on the object, and arises from three parts: (i) gravitational pull of the Sun, (ii) gravitational pull of the earth, and (iii) other earthly forces, such as springs, rods, surfaces, electric fields, *etc.*

On the earth's surface, the ratio of the Sun's gravity to the earth's is

$$\frac{F_{\odot}}{F_{\text{e}}} = \frac{GM_{\odot}m}{a_{\text{e}}^2}\bigg/\frac{GM_{\text{e}}m}{R_{\text{e}}^2} = \frac{M_{\odot}}{M_{\text{e}}}\left(\frac{R_{\text{e}}}{a_{\text{e}}}\right)^2 \approx 6.02\times 10^{-4} \ . \qquad (12.17)$$

In fact, it is clear that the Sun's field precisely cancels with the term $m\,\ddot{\boldsymbol{R}}_{\text{Sun}}$ at the earth's center, leaving only gradient contributions of even lower order, *i.e.* multiplied by $R_{\text{e}}/a_{\text{e}} \approx 4.25 \times 10^{-5}$. Thus, to an excellent approximation, we may neglect the Sun entirely and write

$$\boxed{\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}'}{m} + \boldsymbol{g} - 2\,\boldsymbol{\omega}\times\frac{d\boldsymbol{r}}{dt} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})} \qquad (12.18)$$

Note that we've dropped the 'earth' label here and henceforth. We define $\boldsymbol{g} = -GM_{\text{e}}\,\hat{\boldsymbol{r}}/r^2$, the acceleration due to gravity; $\boldsymbol{F}'$ is the sum of all earthly forces other than the earth's gravity. The last two terms on the RHS are corrections to $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ due to the noninertial frame of the earth, and are recognized as the Coriolis and centrifugal acceleration terms, respectively.

12.2 Spherical Polar Coordinates

The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ varies with position. In terms of the body-fixed triad $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we have

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \qquad (12.19)$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \qquad (12.20)$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \ . \qquad (12.21)$$

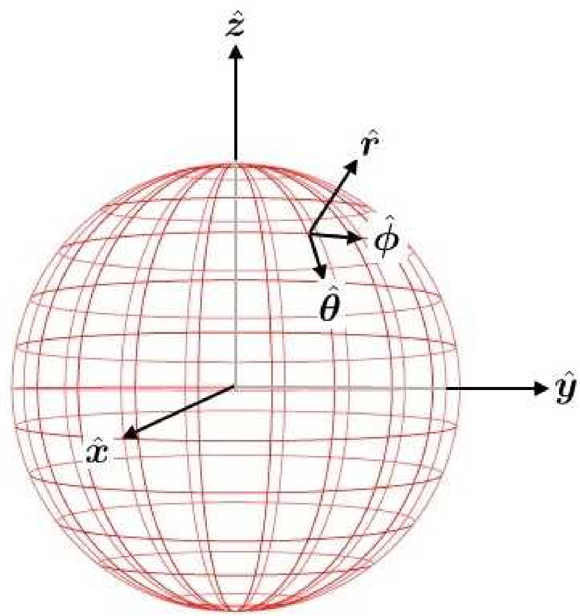


Figure 12.2: The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$.

Inverting the relation between the triads $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we obtain

$$\hat{\boldsymbol{x}} = \sin\theta\,\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \tag{12.22}$$

$$\hat{\boldsymbol{y}} = \sin\theta\,\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \tag{12.23}$$

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ . \tag{12.24}$$

The differentials of these unit vectors are

$$d\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}}\,d\theta + \sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi \tag{12.25}$$

$$d\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{r}}\,d\theta + \cos\theta\,\hat{\boldsymbol{\phi}}\,d\phi \tag{12.26}$$

$$d\hat{\boldsymbol{\phi}} = -\sin\theta\,\hat{\boldsymbol{r}}\,d\phi - \cos\theta\,\hat{\boldsymbol{\theta}}\,d\phi\ . \tag{12.27}$$

Thus,

$$\begin{aligned}\dot{\boldsymbol{r}} &= \frac{d}{dt}\big(r\,\hat{\boldsymbol{r}}\big) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}} \\ &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}\ .\end{aligned} \tag{12.28}$$

If we differentiate a second time, we find, after some tedious accounting,

$$\begin{aligned}\ddot{\boldsymbol{r}} = &\big(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\!\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} + \big(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{\theta}} \\ &+ \big(2\,\dot{r}\,\dot{\phi}\,\sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi}\,\cos\theta + r\,\sin\theta\,\ddot{\phi}\,\big)\,\hat{\boldsymbol{\phi}}\ .\end{aligned} \tag{12.29}$$

12.3 Centrifugal Force

One major distinction between the Coriolis and centrifugal forces is that the Coriolis force acts only on moving particles, whereas the centrifugal force is present even when $\dot{\boldsymbol{r}} = 0$. Thus, the equation for stationary equilibrium on the earth's surface is

$$m\boldsymbol{g} + \boldsymbol{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = 0 \ , \tag{12.30}$$

involves the centrifugal term. We can write this as $\boldsymbol{F}' + m\widetilde{\boldsymbol{g}} = 0$, where

$$\widetilde{\boldsymbol{g}} = -\frac{GM_\text{e}\,\hat{\boldsymbol{r}}}{r^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{12.31}$$

$$= -\big(g_0 - \omega^2 R_\text{e}\, \sin^2\theta\big)\,\hat{\boldsymbol{r}} + \omega^2\ R_\text{e}\,\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} \ , \tag{12.32}$$

where $g_0 = GM_\text{e}/R_\text{e}^2 = 980\,\text{cm/s}^2$. Thus, on the equator, $\tilde{\boldsymbol{g}} = -\big(g_0 - \omega^2 R_\text{e}\big)\,\hat{\boldsymbol{r}}$, with $\omega^2 R_\text{e} \approx 3.39\,\text{cm/s}^2$, a small but significant correction. Thus, you weigh less on the equator. Note also the term in $\widetilde{\boldsymbol{g}}$ along $\hat{\boldsymbol{\theta}}$. This means that a plumb bob suspended from a general point above the earth's surface won't point exactly toward the earth's center. Moreover, if the earth were replaced by an equivalent mass of fluid, the fluid would rearrange itself so as to make its surface locally perpendicular to $\widetilde{\boldsymbol{g}}$. Indeed, the earth (and Sun) do exhibit quadrupolar distortions in their mass distributions – both are oblate spheroids. In fact, the observed difference $\tilde{g}(\theta = \frac{\pi}{2}) - \tilde{g}(\theta = 0) \approx 5.2\,\text{cm/s}^2$, which is 53% greater than the naïvely expected value of $3.39\,\text{cm/s}^2$. The earth's oblateness enhances the effect.

12.3.1 Rotating tube of fluid

Consider a cylinder filled with a liquid, rotating with angular frequency ω about its symmetry axis $\hat{\boldsymbol{z}}$. In steady state, the fluid is stationary in the rotating frame, and we may write, for any given element of fluid

$$0 = \boldsymbol{f}' + \boldsymbol{g} - \omega^2\,\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{r}) \ , \tag{12.33}$$

where $\boldsymbol{f}'$ is the force per unit mass on the fluid element. Now consider a fluid element on the surface. Since there is no static friction to the fluid, any component of $\boldsymbol{f}'$ parallel to the fluid's surface will cause the fluid to flow in that direction. This contradicts the steady state assumption. Therefore, we must have $\boldsymbol{f}' = f'\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the local unit normal to the fluid surface. We write the equation for the fluid's surface as $z = z(\rho)$. Thus, with $\boldsymbol{r} = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$, Newton's second law yields

$$f'\,\hat{\boldsymbol{n}} = g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}} \ , \tag{12.34}$$

where $\boldsymbol{g} = -g\,\hat{\boldsymbol{z}}$ is assumed. From this, we conclude that the unit normal to the fluid surface and the force per unit mass are given by

$$\hat{\boldsymbol{n}}(\rho) = \frac{g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}}}{\sqrt{g^2 + \omega^4\,\rho^2}} \qquad , \qquad f'(\rho) = \sqrt{g^2 + \omega^4\rho^2} \ . \tag{12.35}$$

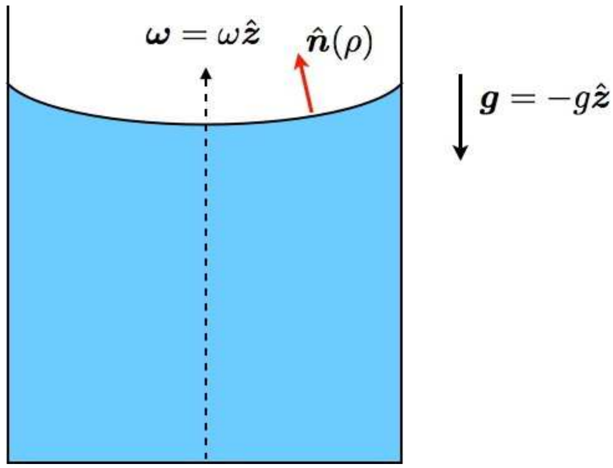


Figure 12.3: A rotating cylinder of fluid.

Now suppose $\boldsymbol{r}(\rho,\phi) = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$ is a point on the surface of the fluid. We have that

$$d\boldsymbol{r} = \hat{\boldsymbol{\rho}}\,d\rho + z'(\rho)\,\hat{\boldsymbol{z}}\,d\rho + \rho\,\hat{\boldsymbol{\phi}}\,d\phi\ , \tag{12.36}$$

where $z' = \frac{dz}{d\rho}$, and where we have used $d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\,d\phi$, which follows from eqn. 12.25 after setting $\theta = \frac{\pi}{2}$. Now $d\boldsymbol{r}$ must lie along the surface, therefore $\hat{\boldsymbol{n}} \cdot d\boldsymbol{r} = 0$, which says

$$g\,\frac{dz}{d\rho} = \omega^2\,\rho\ . \tag{12.37}$$

Integrating this equation, we obtain the shape of the surface:

$$z(\rho) = z_0 + \frac{\omega^2\rho^2}{2g}\ . \tag{12.38}$$

12.4 The Coriolis Force

The Coriolis force is given by $\boldsymbol{F}_{\rm Cor} = -2m\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}}$. According to (12.18), the acceleration of a free particle ($\boldsymbol{F}' = 0$) isn't along $\widetilde{\boldsymbol{g}}$ – an orthogonal component is generated by the Coriolis force. To actually solve the coupled equations of motion is difficult because the unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ change with position, and hence with time. The following standard problem highlights some of the effects of the Coriolis and centrifugal forces.

PROBLEM: A cannonball is dropped from the top of a tower of height h located at a northerly latitude of λ. Assuming the cannonball is initially at rest with respect to the tower, and neglecting air resistance, calculate its deflection (magnitude and direction) due to (a) centrifugal and (b) Coriolis forces by the time it hits the ground. Evaluate for the case $h = 100$ m, $\lambda = 45°$. The radius of the earth is $R_{\rm e} = 6.4 \times 10^6$ m.

SOLUTION: The equation of motion for a particle near the earth's surface is

$$\ddot{\boldsymbol{r}} = -2\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}} - g_0\,\hat{\boldsymbol{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r})\ , \tag{12.39}$$

where $\omega = \omega\,\hat{\boldsymbol{z}}$, with $\omega = 2\pi/(24\,\text{hrs}) = 7.3\times 10^{-5}\,\text{rad/s}$. Here, $g_0 = GM_\text{e}/R_\text{e}^2 = 980\,\text{cm/s}^2$. We use a locally orthonormal coordinate system $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + (R_\text{e} + z)\,\hat{\boldsymbol{r}}\ , \tag{12.40}$$

where $R_\text{e} = 6.4\times 10^6\,\text{m}$ is the radius of the earth. Expressing $\hat{\boldsymbol{z}}$ in terms of our chosen orthonormal triad,

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ , \tag{12.41}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the polar angle, or 'colatitude'. Since the height of the tower and the deflections are all very small on the scale of R_e, we may regard the orthonormal triad as fixed and time-independent. (In general, these unit vectors change as a function of $\boldsymbol{r}$.) Thus, we have $\dot{\boldsymbol{r}} \simeq \dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}$, and we find

$$\hat{\boldsymbol{z}}\times\dot{\boldsymbol{r}} = -\dot{y}\,\cos\theta\,\hat{\boldsymbol{\theta}} + (\dot{x}\,\cos\theta + \dot{z}\,\sin\theta)\,\hat{\boldsymbol{\phi}} - \dot{y}\sin\theta\,\hat{\boldsymbol{r}} \tag{12.42}$$

$$\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = -\omega^2 R_\text{e}\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} - \omega^2 R_\text{e}\sin^2\!\theta\,\hat{\boldsymbol{r}}\ , \tag{12.43}$$

where we neglect the $\mathcal{O}(z)$ term in the second equation, since $z \ll R_\text{e}$.

The equation of motion, written in components, is then

$$\ddot{x} = 2\omega\cos\theta\,\dot{y} + \omega^2 R_\text{e}\sin\theta\cos\theta \tag{12.44}$$

$$\ddot{y} = -2\omega\cos\theta\,\dot{x} - 2\omega\sin\theta\,\dot{z} \tag{12.45}$$

$$\ddot{z} = -g_0 + 2\omega\sin\theta\,\dot{y} + \omega^2 R_\text{e}\sin^2\theta\ . \tag{12.46}$$

While these (inhomogeneous) equations are linear, they also are coupled, so an exact analytical solution is not trivial to obtain (but see below). Fortunately, the deflections are small, so we can solve this perturbatively. We write $x = x^{(0)} + \delta x$, *etc.*, and solve to lowest order by including only the g_0 term on the RHS. This gives $z^{(0)}(t) = z_0 - \frac{1}{2}g_0\,t^2$, along with $x^{(0)}(t) = y^{(0)}(t) = 0$. We then substitute this solution on the RHS and solve for the deflections, obtaining

$$\delta x(t) = \tfrac{1}{2}\omega^2 R_\text{e}\,\sin\theta\cos\theta\,t^2 \tag{12.47}$$

$$\delta y(t) = \tfrac{1}{3}\omega g_0\,\sin\theta\,t^3 \tag{12.48}$$

$$\delta z(t) = \tfrac{1}{2}\omega^2 R_\text{e}\,\sin^2\!\theta\,t^2\ . \tag{12.49}$$

The deflection along $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{r}}$ is due to the centrifugal term, while that along $\hat{\boldsymbol{\phi}}$ is due to the Coriolis term. (At higher order, the two terms interact and the deflection in any given direction can't uniquely be associated to a single fictitious force.) To find the deflection of an object dropped from a height h, solve $z^{(0)}(t^*) = 0$ to obtain $t^* = \sqrt{2h/g_0}$ for the drop time, and substitute. For $h = 100\,\text{m}$ and $\lambda = \frac{\pi}{2}$, find $\delta x(t^*) = 17\,\text{cm}$ south (centrifugal) and $\delta y(t^*) = 1.6\,\text{cm}$ east (Coriolis).

In fact, an exact solution to (12.46) is readily obtained, via the following analysis. The equations of motion may be written $\dot{\boldsymbol{v}} = 2i\omega\mathcal{J}\boldsymbol{v} + \boldsymbol{b}$, or

$$\begin{pmatrix}\dot{v}_x\\ \dot{v}_y\\ \dot{v}_x\end{pmatrix} = 2i\,\omega\,\overbrace{\begin{pmatrix}0 & -i\cos\theta & 0\\ i\cos\theta & 0 & i\sin\theta\\ 0 & -i\sin\theta & 0\end{pmatrix}}^{\mathcal{J}}\begin{pmatrix}v_x\\ v_y\\ v_x\end{pmatrix} + \overbrace{\begin{pmatrix}g_1\sin\theta\cos\theta\\ 0\\ -g_0 + g_1\sin^2\theta\end{pmatrix}}^{\boldsymbol{b}} \tag{12.50}$$

with $g_1 \equiv \omega^2 R_e$. Note that $\mathcal{J}^\dagger = \mathcal{J}$, *i.e.* $\mathcal{J}$ is a Hermitian matrix. The formal solution is

$$\boldsymbol{v}(t) = e^{2i\omega\mathcal{J}t}\,\boldsymbol{v}(0) + \left(\frac{e^{2i\omega\mathcal{J}t}-1}{2i\omega}\right)\mathcal{J}^{-1}\,\boldsymbol{b}\ . \tag{12.51}$$

When working with matrices, it is convenient to work in an eigenbasis. The characteristic polynomial for $\mathcal{J}$ is $P(\lambda) = \det(\lambda \cdot 1 - \mathcal{J}) = \lambda\,(\lambda^2 - 1)$, hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = +1$, and $\lambda_3 = -1$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta \\ i \\ \sin\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta \\ -i \\ \sin\theta \end{pmatrix}\ . \tag{12.52}$$

Note that $\boldsymbol{\psi}_a^\dagger \cdot \boldsymbol{\psi}_{a'} = \delta_{aa'}$.

Expanding $\boldsymbol{v}$ and $\boldsymbol{b}$ in this eigenbasis, we have $\dot v_a = 2i\omega\lambda_a v_a + b_a$, where $v_a = \psi^*_{ia}\, v_i$ and $b_a = \psi^*_{ia}\, b_i$. The solution is

$$v_a(t) = v_a(0)\, e^{2i\lambda_a\omega t} + \left(\frac{e^{2i\lambda_a\omega t}-1}{2i\lambda_a\omega}\right) b_a\ , \tag{12.53}$$

which entails

$$v_i(t) = \left(\sum_a \psi_{ia}\left(\frac{e^{2i\lambda_a\omega t}-1}{2i\lambda_a\omega}\right)\psi^*_{ja}\right) b_j\ , \tag{12.54}$$

where we have taken $\boldsymbol{v}(0) = 0$, *i.e.* the object is released from rest. Doing the requisite matrix multiplications,

$$\begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} t\sin^2\!\theta + \frac{\sin 2\omega t}{2\omega}\cos^2\!\theta & \frac{\sin^2\!\omega t}{\omega}\cos\theta & -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta \\ -\frac{\sin^2\!\omega t}{\omega}\cos\theta & \frac{\sin 2\omega t}{2\omega} & -\frac{\sin^2\!\omega t}{\omega}\sin\theta \\ -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta & \frac{\sin^2\!\omega t}{\omega}\sin\theta & t\cos^2\!\theta + \frac{\sin 2\omega t}{2\omega}\sin^2\!\theta \end{pmatrix}\begin{pmatrix} g_1\sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\!\theta \end{pmatrix}, \tag{12.55}$$

which says

$$\begin{aligned} v_x(t) &= \left(\tfrac{1}{2}\sin 2\theta + \tfrac{\sin 2\omega t}{4\omega t}\sin 2\theta\right)\cdot g_0 t + \tfrac{\sin 2\omega t}{4\omega t}\sin 2\theta \cdot g_1 t \\ v_y(t) &= \tfrac{\sin^2\!\omega t}{\omega t}\cdot g_0 t - \tfrac{\sin^2\!\omega t}{\omega t}\sin\theta\cdot g_1 t \\ v_z(t) &= -\left(\cos^2\!\theta + \tfrac{\sin 2\omega t}{2\omega t}\sin^2\!\theta\right)\cdot g_0 t + \tfrac{\sin^2\!\omega t}{2\omega t}\cdot g_1 t\ . \end{aligned} \tag{12.56}$$

Why is the deflection always to the east? The earth rotates eastward, and an object starting from rest in the earth's frame has initial angular velocity equal to that of the earth. To conserve angular momentum, the object must speed up as it falls.

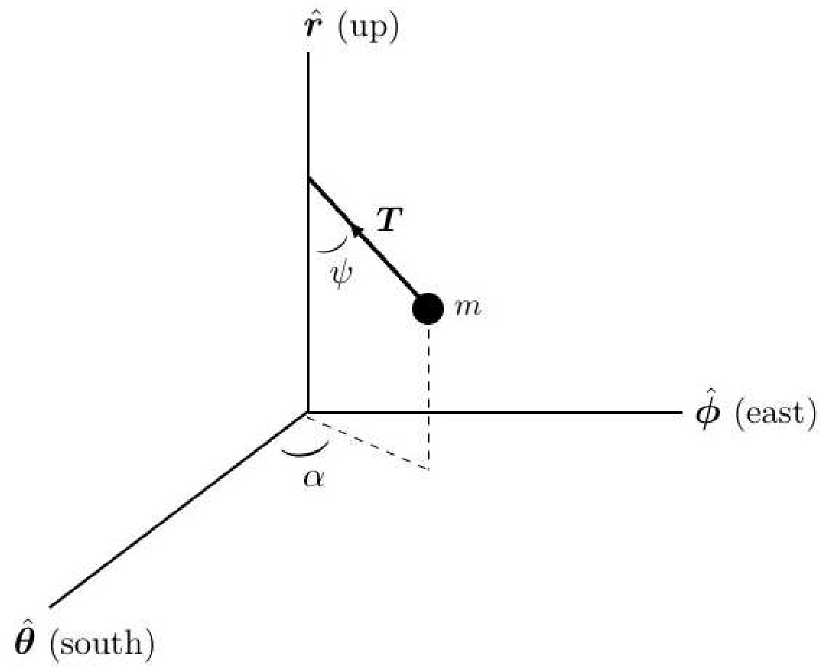


Figure 12.4: Foucault's pendulum.

12.4.1 Foucault's pendulum

A pendulum swinging over one of the poles moves in a fixed inertial plane while the earth rotates underneath. Relative to the earth, the plane of motion of the pendulum makes one revolution every day. What happens at a general latitude? Assume the pendulum is located at colatitude θ and longitude ϕ. Assuming the length scale of the pendulum is small compared to $R_{\rm e}$, we can regard the local triad $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{r}}\}$ as fixed. The situation is depicted in Fig. 12.4. We write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}} \ , \tag{12.57}$$

with

$$x = \ell\,\sin\psi\,\cos\alpha \quad , \quad y = \ell\,\sin\psi\,\sin\alpha \quad , \quad z = \ell\,(1-\cos\psi) \ . \tag{12.58}$$

In our analysis we will ignore centrifugal effects, which are of higher order in ω, and we take $\boldsymbol{g} = -g\,\hat{\boldsymbol{r}}$. We also idealize the pendulum, and consider the suspension rod to be of negligible mass.

The total force on the mass m is due to gravity and tension:

$$\begin{aligned}\boldsymbol{F} &= m\boldsymbol{g} + \boldsymbol{T} \\ &= \big(-T\sin\psi\,\cos\alpha,\ -T\sin\psi\,\sin\alpha,\ T\cos\psi - mg\big) \\ &= \big(-Tx/\ell,\ -Ty/\ell, T - Mg - Tz/\ell\big) \ .\end{aligned} \tag{12.59}$$

The Coriolis term is

$$\begin{aligned}\boldsymbol{F}_{\rm Cor} &= -2m\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}} \qquad (12.60)\\ &= -2m\,\omega\big(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\big)\times\big(\dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}\big)\\ &= 2m\omega\,\big(\dot{y}\cos\theta,\ -\dot{x}\cos\theta - \dot{z}\sin\theta,\ \dot{y}\sin\theta\big)\ . \qquad (12.61)\end{aligned}$$

The equations of motion are $m\,\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{F}_{\rm Cor}$:

$$m\ddot{x} = -Tx/\ell + 2m\omega\cos\theta\,\dot{y} \qquad (12.62)$$

$$m\ddot{y} = -Ty/\ell - 2m\omega\cos\theta\,\dot{x} - 2m\omega\sin\theta\,\dot{z} \qquad (12.63)$$

$$m\ddot{z} = T - mg - Tz/\ell + 2m\omega\sin\theta\,\dot{y}\ . \qquad (12.64)$$

These three equations are to be solved for the three unknowns x, y, and T. Note that

$$x^2 + y^2 + (\ell - z)^2 = \ell^2\ , \qquad (12.65)$$

so $z = z(x,y)$ is not an independent degree of freedom. This equation may be recast in the form $z = (x^2 + y^2 + z^2)/2\ell$ which shows that if x and y are both small, then z is at least of second order in smallness. Therefore, we will approximate $z \simeq 0$, in which case $\dot{z}$ may be neglected from the second equation of motion. The third equation is used to solve for T:

$$T \simeq mg - 2m\omega\sin\theta\,\dot{y}\ . \qquad (12.66)$$

Adding the first plus i times the second then gives the complexified equation

$$\begin{aligned}\ddot{\xi} &= -\frac{T}{m\ell}\,\xi - 2i\omega\cos\theta\,\dot{\xi}\\ &\approx -\omega_0^2\,\xi - 2i\omega\cos\theta\,\dot{\xi} \qquad (12.67)\end{aligned}$$

where $\xi \equiv x + iy$, and where $\omega_0 = \sqrt{g/\ell}$. Note that we have approximated $T \approx mg$ in deriving the second line.

It is now a trivial matter to solve the homogeneous linear ODE of eq. 12.67. Writing

$$\xi = \xi_0\,e^{-i\Omega t} \qquad (12.68)$$

and plugging in to find Ω, we obtain

$$\Omega^2 - 2\omega_\perp\Omega - \omega_0^2 = 0\ , \qquad (12.69)$$

with $\omega_\perp \equiv \omega\cos\theta$. The roots are

$$\Omega_\pm = \omega_\perp \pm \sqrt{\omega_0^2 + \omega_\perp^2}\ , \qquad (12.70)$$

hence the most general solution is

$$\xi(t) = A_+\,e^{-i\Omega_+ t} + A_-\,e^{-i\Omega_- t}\ . \qquad (12.71)$$

Finally, if we take as initial conditions $x(0) = a$, $y(0) = 0$, $\dot{x}(0) = 0$, and $\dot{y}(0) = 0$, we obtain

$$x(t) = \left(\frac{a}{\nu}\right) \cdot \Big\{\omega_\perp \sin(\omega_\perp t)\, \sin(\nu t) + \nu \cos(\omega_\perp t)\, \cos(\nu t)\Big\} \tag{12.72}$$

$$y(t) = \left(\frac{a}{\nu}\right) \cdot \Big\{\omega_\perp \cos(\omega_\perp t)\, \sin(\nu t) - \nu \sin(\omega_\perp t)\, \cos(\nu t)\Big\}\ , \tag{12.73}$$

with $\nu = \sqrt{\omega_0^2 + \omega_\perp^2}$. Typically $\omega_0 \gg \omega_\perp$, since $\omega = 7.3 \times 10^{-5}\,\text{s}^{-1}$. In the limit $\omega_\perp \ll \omega_0$, then, we have $\nu \approx \omega_0$ and

$$x(t) \simeq a \cos(\omega_\perp t)\, \cos(\omega_0 t) \quad , \quad y(t) \simeq -a \sin(\omega_\perp t)\, \cos(\omega_0 t)\ , \tag{12.74}$$

and the plane of motion rotates with angular frequency $-\omega_\perp$, *i.e.* the period is $|\sec\theta|$ days. Viewed from above, the rotation is clockwise in the northern hemisphere, where $\cos\theta > 0$ and counterclockwise in the southern hemisphere, where $\cos\theta < 0$.

Chapter 13

Rigid Body Motion and Rotational Dynamics

13.1 Rigid Bodies

A rigid body consists of a group of particles whose separations are all fixed in magnitude. Six independent coordinates are required to completely specify the position and orientation of a rigid body. For example, the location of the first particle is specified by three coordinates. A second particle requires only two coordinates since the distance to the first is fixed. Finally, a third particle requires only one coordinate, since its distance to the first two particles is fixed (think about the intersection of two spheres). The positions of all the remaining particles are then determined by their distances from the first three. Usually, one takes these six coordinates to be the center-of-mass position $\boldsymbol{R} = (X, Y, Z)$ and three angles specifying the orientation of the body (*e.g.* the Euler angles).

As derived previously, the equations of motion are

$$\boldsymbol{P} = \sum_i m_i\, \dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{P}} = \boldsymbol{F}^{(\text{ext})} \tag{13.1}$$

$$\boldsymbol{L} = \sum_i m_i\, \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{L}} = \boldsymbol{N}^{(\text{ext})} \; . \tag{13.2}$$

These equations determine the motion of a rigid body.

13.1.1 Examples of rigid bodies

Our first example of a rigid body is of a wheel rolling with constant angular velocity $\dot{\phi} = \omega$, and without slipping, This is shown in Fig. 13.1. The no-slip condition is $dx = R\, d\phi$, so $\dot{x} = V_{\text{CM}} = R\omega$. The velocity of a point within the wheel is

$$\boldsymbol{v} = \boldsymbol{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{r} \; , \tag{13.3}$$

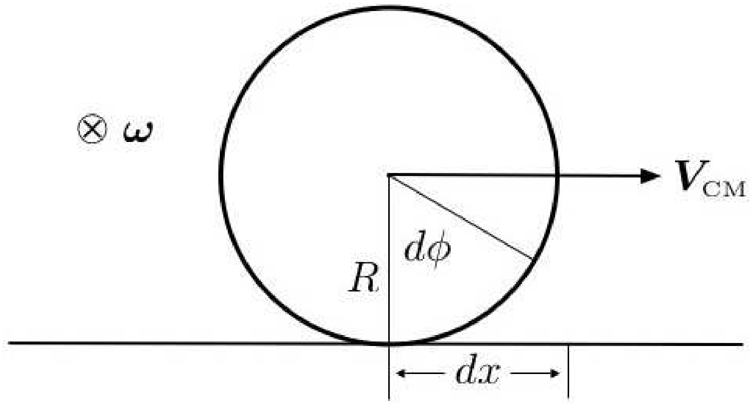


Figure 13.1: A wheel rolling to the right without slipping.

where $\boldsymbol{r}$ is measured from the center of the disk. The velocity of a point on the surface is then given by $\boldsymbol{v} = \omega R\big(\hat{\boldsymbol{x}} + \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}}\big)$.

As a second example, consider a bicycle wheel of mass M and radius R affixed to a light, firm rod of length d, as shown in Fig. 13.2. Assuming $\boldsymbol{L}$ lies in the (x, y) plane, one computes the gravitational torque $\boldsymbol{N} = \boldsymbol{r} \times (M\boldsymbol{g}) = Mgd\,\hat{\boldsymbol{\phi}}$. The angular momentum vector then rotates with angular frequency $\dot{\phi}$. Thus,

$$d\phi = \frac{dL}{L} \quad \Longrightarrow \quad \dot{\phi} = \frac{Mgd}{L} \ . \tag{13.4}$$

But $L = MR^2\omega$, so the precession frequency is

$$\omega_{\rm p} = \dot{\phi} = \frac{gd}{\omega R^2} \ . \tag{13.5}$$

For $R = d = 30\,{\rm cm}$ and $\omega/2\pi = 200\,{\rm rpm}$, find $\omega_{\rm p}/2\pi \approx 15\,{\rm rpm}$. Note that we have here ignored the contribution to $\boldsymbol{L}$ from the precession itself, which lies along $\hat{\boldsymbol{z}}$, resulting in the *nutation* of the wheel. This is justified if $L_{\rm p}/L = (d^2/R^2) \cdot (\omega_{\rm p}/\omega) \ll 1$.

13.2 The Inertia Tensor

Suppose first that a point within the body itself is fixed. This eliminates the translational degrees of freedom from consideration. We now have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\rm inertial} = \boldsymbol{\omega} \times \boldsymbol{r} \ , \tag{13.6}$$

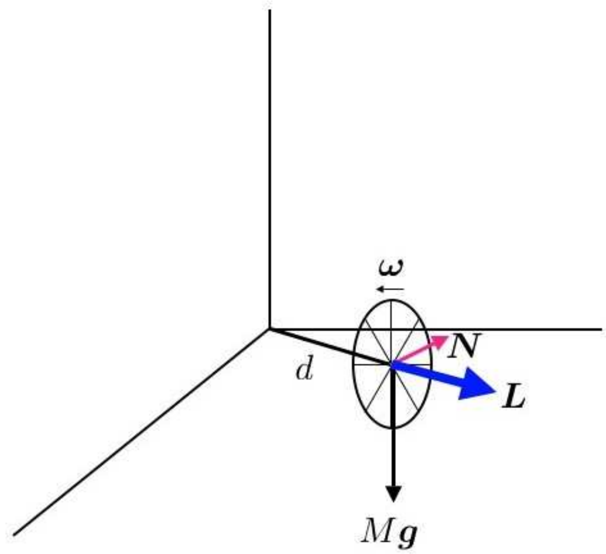


Figure 13.2: Precession of a spinning bicycle wheel.

since $\dot{\boldsymbol{r}}_{\text{body}} = 0$. The kinetic energy is then

$$
\begin{aligned}
T &= \tfrac{1}{2}\sum_i m_i \bigg(\frac{d\boldsymbol{r}_i}{dt}\bigg)^2_{\text{inertial}} = \tfrac{1}{2}\sum_i m_i\,(\boldsymbol{\omega}\times\boldsymbol{r}_i)\cdot(\boldsymbol{\omega}\times\boldsymbol{r}_i) \\
&= \tfrac{1}{2}\sum_i m_i\Big[\omega^2\,\boldsymbol{r}_i^2 - (\boldsymbol{\omega}\cdot\boldsymbol{r}_i)^2\Big] \equiv \tfrac{1}{2} I_{\alpha\beta}\,\omega_\alpha\,\omega_\beta\ , \qquad (13.7)
\end{aligned}
$$

where ω_α is the component of $\boldsymbol{\omega}$ along the body-fixed axis $\boldsymbol{e}_\alpha$. The quantity $I_{\alpha\beta}$ is the *inertia tensor,*

$$
\begin{aligned}
I_{\alpha\beta} &= \sum_i m_i\Big(\boldsymbol{r}_i^2\,\delta_{\alpha\beta} - r_{i,\alpha}\,r_{i,\beta}\Big) \qquad (13.8) \\
&= \int\! d^d r\,\varrho(\boldsymbol{r})\,\Big(\boldsymbol{r}^2\,\delta_{\alpha\beta} - r_\alpha\,r_\beta\Big) \quad \text{(continuous media)}\ . \qquad (13.9)
\end{aligned}
$$

The angular momentum is

$$
\begin{aligned}
\boldsymbol{L} &= \sum_i m_i\,\boldsymbol{r}_i \times \bigg(\frac{d\boldsymbol{r}_i}{dt}\bigg)_{\text{inertial}} \\
&= \sum_i m_i\,\boldsymbol{r}_i \times (\boldsymbol{\omega}\times\boldsymbol{r}_i) = I_{\alpha\beta}\,\omega_\beta\ . \qquad (13.10)
\end{aligned}
$$

The diagonal elements of $I_{\alpha\beta}$ are called the *moments of inertia*, while the off-diagonal elements are called the *products of inertia.*

13.2.1 Coordinate transformations

Consider the basis transformation

$$\hat{\boldsymbol{e}}'_\alpha = \mathcal{R}_{\alpha\beta}\,\hat{\boldsymbol{e}}_\beta \ . \tag{13.11}$$

We demand $\hat{\boldsymbol{e}}'_\alpha \cdot \hat{\boldsymbol{e}}'_\beta = \delta_{\alpha\beta}$, which means $\mathcal{R} \in \mathrm{O}(d)$ is an orthogonal matrix, *i.e.* $\mathcal{R}^{\mathrm{t}} = \mathcal{R}^{-1}$. Thus the inverse transformation is $\boldsymbol{e}_\alpha = \mathcal{R}^{\mathrm{t}}_{\alpha\beta}\boldsymbol{e}'_\beta$. Consider next a general vector $\boldsymbol{A} = A_\beta\,\hat{\boldsymbol{e}}_\beta$. Expressed in terms of the new basis $\{\hat{\boldsymbol{e}}'_\alpha\}$, we have

$$\boldsymbol{A} = A_\beta\,\overbrace{\mathcal{R}^{\mathrm{t}}_{\beta\alpha}\,\hat{\boldsymbol{e}}'_\alpha}^{\hat{\mathbf{e}}_\beta} = \overbrace{\mathcal{R}_{\alpha\beta}A_\beta}^{A'_\alpha}\,\hat{\boldsymbol{e}}'_\alpha \tag{13.12}$$

Thus, the components of $\boldsymbol{A}$ transform as $A'_\alpha = \mathcal{R}_{\alpha\beta}\,A_\beta$. This is true for any vector.

Under a rotation, the density $\rho(\boldsymbol{r})$ must satisfy $\rho'(\boldsymbol{r}') = \rho(\boldsymbol{r})$. This is the transformation rule for scalars. The inertia tensor therefore obeys

$$\begin{aligned} I'_{\alpha\beta} &= \int\! d^3r'\,\rho'(\boldsymbol{r}')\left[\boldsymbol{r}'^2\,\delta_{\alpha\beta} - r'_\alpha\,r'_\beta\right] \\ &= \int\! d^3r\,\rho(\boldsymbol{r})\left[\boldsymbol{r}^2\,\delta_{\alpha\beta} - \big(\mathcal{R}_{\alpha\mu}r_\mu\big)\big(\mathcal{R}_{\beta\nu}r_\nu\big)\right] \\ &= \mathcal{R}_{\alpha\mu}\,I_{\mu\nu}\,\mathcal{R}^{\mathrm{t}}_{\nu\beta} \ . \end{aligned} \tag{13.13}$$

I.e. $I' = \mathcal{R}I\mathcal{R}^{\mathrm{t}}$, the transformation rule for tensors. The angular frequency $\boldsymbol{\omega}$ is a vector, so $\omega'_\alpha = \mathcal{R}_{\alpha\mu}\,\omega_\mu$. The angular momentum $\boldsymbol{L}$ also transforms as a vector. The kinetic energy is $T = \frac{1}{2}\,\boldsymbol{\omega}^{\mathrm{t}} \cdot I \cdot \boldsymbol{\omega}$, which transforms as a scalar.

13.2.2 The case of no fixed point

If there is no fixed point, we can let $\boldsymbol{r}'$ denote the distance from the center-of-mass (CM), which will serve as the instantaneous origin in the body-fixed frame. We then adopt the notation where $\boldsymbol{R}$ is the CM position of the rotating body, as observed in an inertial frame, and is computed from the expression

$$\boldsymbol{R} = \frac{1}{M}\sum_i m_i\,\boldsymbol{\rho}_i = \frac{1}{M}\int\! d^3r\,\rho(\boldsymbol{r})\,\boldsymbol{r} \ , \tag{13.14}$$

where the total mass is of course

$$M = \sum_i m_i = \int\! d^3r\,\rho(\boldsymbol{r}) \ . \tag{13.15}$$

The kinetic energy and angular momentum are then

$$T = \tfrac{1}{2}M\dot{\boldsymbol{R}}^2 + \tfrac{1}{2}I_{\alpha\beta}\,\omega_\alpha\,\omega_\beta \tag{13.16}$$

$$L_\alpha = \epsilon_{\alpha\beta\gamma}MR_\beta\dot{R}_\gamma + I_{\alpha\beta}\,\omega_\beta \ , \tag{13.17}$$

where $I_{\alpha\beta}$ is given in eqs. 13.8 and 13.9, where the origin is the CM.

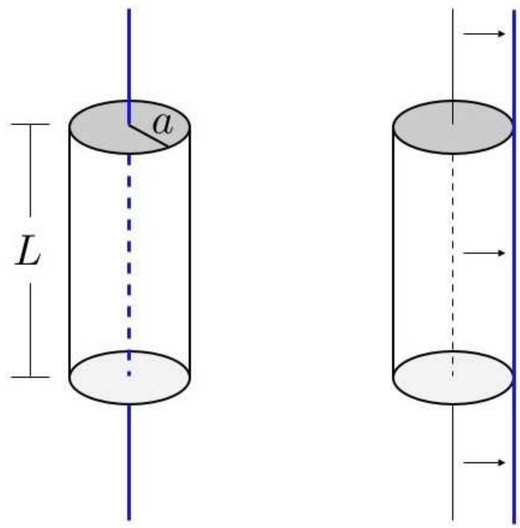


Figure 13.3: Application of the parallel axis theorem to a cylindrically symmetric mass distribution.

13.3 Parallel Axis Theorem

Suppose $I_{\alpha\beta}$ is given in a body-fixed frame. If we displace the origin in the body-fixed frame by $\boldsymbol{d}$, then let $I_{\alpha\beta}(\boldsymbol{d})$ be the inertial tensor with respect to the new origin. If, relative to the origin at $\boldsymbol{0}$ a mass element lies at position $\boldsymbol{r}$, then relative to an origin at $\boldsymbol{d}$ it will lie at $\boldsymbol{r}-\boldsymbol{d}$. We then have

$$I_{\alpha\beta}(\boldsymbol{d}) = \sum_i m_i \Big\{ (\boldsymbol{r}_i^2 - 2\boldsymbol{d}\cdot\boldsymbol{r}_i + \boldsymbol{d}^2)\,\delta_{\alpha\beta} - (r_{i,\alpha} - d_\alpha)(r_{i,\beta} - d_\beta) \Big\} \ . \tag{13.18}$$

If $\boldsymbol{r}_i$ is measured with respect to the CM, then

$$\sum_i m_i\,\boldsymbol{r}_i = 0 \tag{13.19}$$

and

$$I_{\alpha\beta}(\boldsymbol{d}) = I_{\alpha\beta}(0) + M\big(\boldsymbol{d}^2 \delta_{\alpha\beta} - d_\alpha d_\beta\big) \ , \tag{13.20}$$

a result known as the *parallel axis theorem.*

As an example of the theorem, consider the situation depicted in Fig. 13.3, where a cylindrically symmetric mass distribution is rotated about is symmetry axis, and about an axis tangent to its side. The component I_{zz} of the inertia tensor is easily computed when the origin lies along the symmetry axis:

$$\begin{aligned} I_{zz} &= \int d^3\!r\,\rho(\boldsymbol{r})\,(\boldsymbol{r}^2 - z^2) = \rho L \cdot 2\pi \int_0^a dr_\perp\, r_\perp^3 \\ &= \tfrac{\pi}{2}\rho L a^4 = \tfrac{1}{2} M a^2 \ , \end{aligned} \tag{13.21}$$

where $M = \pi a^2 L\rho$ is the total mass. If we compute I_{zz} about a vertical axis which is tangent to the cylinder, the parallel axis theorem tells us that

$$I'_{zz} = I_{zz} + Ma^2 = \tfrac{3}{2}Ma^2 \ . \tag{13.22}$$

Doing this calculation by explicit integration of $\int dm\, r_\perp^2$ would be tedious!

13.3.1 Example

Problem: Compute the CM and the inertia tensor for the planar right triangle of Fig. 13.4, assuming it to be of uniform two-dimensional mass density ρ.

Solution: The total mass is $M = \frac{1}{2}\rho\, ab$. The x-coordinate of the CM is then

$$\begin{aligned} X &= \frac{1}{M}\int_0^a \!dx \int_0^{b(1-\frac{x}{a})} \!dy\, \rho\, x = \frac{\rho}{M}\int_0^a \!dx\, b\big(1-\tfrac{x}{a}\big)\, x \\ &= \frac{\rho\, a^2 b}{M}\int_0^1 \!du\, u(1-u) = \frac{\rho\, a^2 b}{6M} = \tfrac{1}{3}\, a \ . \end{aligned} \tag{13.23}$$

Clearly we must then have $Y = \frac{1}{3}\, b$, which may be verified by explicit integration.

We now compute the inertia tensor, with the origin at $(0,0,0)$. Since the figure is planar, $z = 0$ everywhere, hence $I_{xz} = I_{zx} = 0$, $I_{yz} = I_{zy} = 0$, and also $I_{zz} = I_{xx} + I_{yy}$. We now compute the remaining independent elements:

$$\begin{aligned} I_{xx} &= \rho \int_0^a \!dx \int_0^{b(1-\frac{x}{a})} \!dy\, y^2 = \rho \int_0^a dx\, \tfrac{1}{3}\, b^3 \left(1 - \tfrac{x}{a}\right)^3 \\ &= \tfrac{1}{3}\rho\, ab^3 \int_0^1 \!du\, (1-u)^3 = \tfrac{1}{12}\rho\, ab^3 = \tfrac{1}{6} M b^2 \end{aligned} \tag{13.24}$$

and

$$\begin{aligned} I_{xy} &= -\rho \int_0^a \!dx \int_0^{b(1-\frac{x}{a})} \!dy\, x\, y = -\tfrac{1}{2}\rho\, b^2 \int_0^a dx\, x \left(1 - \tfrac{x}{a}\right)^2 \\ &= -\tfrac{1}{2}\rho\, a^2 b^2 \int_0^1 du\, u\, (1-u)^2 = -\tfrac{1}{24}\rho\, a^2 b^2 = -\tfrac{1}{12} M ab \ . \end{aligned} \tag{13.25}$$

Thus,

$$I = \frac{M}{6}\begin{pmatrix} b^2 & -\frac{1}{2}ab & 0 \\ -\frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \ . \tag{13.26}$$

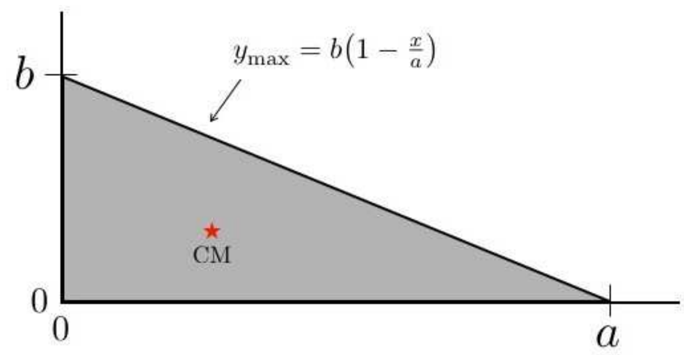


Figure 13.4: A planar mass distribution in the shape of a triangle.

Suppose we wanted the inertia tensor relative in a coordinate system where the CM lies at the origin. What we computed in eqn. 13.26 is $I(\boldsymbol{d})$, with $\boldsymbol{d} = -\frac{a}{3}\hat{\boldsymbol{x}} - \frac{b}{3}\,\hat{\boldsymbol{y}}$. Thus,

$$\boldsymbol{d}^2\delta_{\alpha\beta} - d_\alpha\, d_\beta = \frac{1}{9}\begin{pmatrix} b^2 & -ab & 0 \\ -ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \ . \tag{13.27}$$

Since

$$I(\boldsymbol{d}) = I^{\rm CM} + M\Big(\boldsymbol{d}^2\delta_{\alpha\beta} - d_\alpha\, d_\beta\Big) \ , \tag{13.28}$$

we have that

$$I^{\rm CM} = I(\boldsymbol{d}) - M\Big(\boldsymbol{d}^2\delta_{\alpha\beta} - d_\alpha\, d_\beta\Big) \tag{13.29}$$

$$= \frac{M}{18}\begin{pmatrix} b^2 & \frac{1}{2}ab & 0 \\ \frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \ . \tag{13.30}$$

13.3.2 General planar mass distribution

For a general planar mass distribution,

$$\rho(x,y,z) = \sigma(x,y)\,\delta(z) \ , \tag{13.31}$$

which is confined to the plane $z = 0$, we have

$$I_{xx} = \int\! dx \int\! dy\, \sigma(x,y)\, y^2 \tag{13.32}$$

$$I_{yy} = \int\! dx \int\! dy\, \sigma(x,y)\, x^2 \tag{13.33}$$

$$I_{xy} = -\int\! dx \int\! dy\, \sigma(x,y)\, xy \tag{13.34}$$

and $I_{zz} = I_{xx} + I_{yy}$, regardless of the two-dimensional mass distribution $\sigma(x,y)$.

13.4 Principal Axes of Inertia

We found that an orthogonal transformation to a new set of axes $\hat{\boldsymbol{e}}'_\alpha = \mathcal{R}_{\alpha\beta}\hat{\boldsymbol{e}}_\beta$ entails $I' = \mathcal{R}I\mathcal{R}^{\rm t}$ for the inertia tensor. Since $I = I^{\rm t}$ is manifestly a symmetric matrix, it can be brought to diagonal form by such an orthogonal transformation. To find $\mathcal{R}$, follow this recipe:

1. Find the diagonal elements of I' by setting $P(\lambda) = 0$, where
$$P(\lambda) = \det\big(\lambda \cdot 1 - I\big) \ , \tag{13.35}$$
is the characteristic polynomial for I, and 1 is the unit matrix.

2. For each eigenvalue λ_a, solve the d equations
$$\sum_\nu I_{\mu\nu}\,\psi^a_\nu = \lambda_a\,\psi^a_\mu \ . \tag{13.36}$$
Here, ψ^a_μ is the $\mu^{\rm th}$ component of the $a^{\rm th}$ eigenvector. Since $(\lambda \cdot 1 - I)$ is degenerate, these equations are linearly dependent, which means that the first $d-1$ components may be determined in terms of the $d^{\rm th}$ component.

3. Because $I = I^{\rm t}$, eigenvectors corresponding to different eigenvalues are orthogonal. In cases of degeneracy, the eigenvectors may be chosen to be orthogonal, *e.g.* via the Gram-Schmidt procedure.

4. Due to the underdetermined aspect to step 2, we may choose an arbitrary normalization for each eigenvector. It is conventional to choose the eigenvectors to be orthonormal: $\sum_\mu \psi^a_\mu\,\psi^b_\mu = \delta^{ab}$.

5. The matrix $\mathcal{R}$ is explicitly given by $\mathcal{R}_{a\mu} = \psi^a_\mu$, the matrix whose row vectors are the eigenvectors $\boldsymbol{\psi}^a$. Of course $\mathcal{R}^{\rm t}$ is then the corresponding matrix of column vectors.

6. The eigenvectors form a complete basis. The resolution of unity may be expressed as
$$\sum_a \psi^a_\mu\,\psi^a_\nu = \delta_{\mu\nu} \ . \tag{13.37}$$

As an example, consider the inertia tensor for a general planar mass distribution, which is of the form
$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \ , \tag{13.38}$$
where $I_{yx} = I_{xy}$ and $I_{zz} = I_{xx} + I_{yy}$. Define
$$A = \tfrac{1}{2}\big(I_{xx} + I_{yy}\big) \tag{13.39}$$
$$B = \sqrt{\tfrac{1}{4}\big(I_{xx} - I_{yy}\big)^2 + I^2_{xy}} \tag{13.40}$$
$$\vartheta = \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right) \ , \tag{13.41}$$

so that

$$I = \begin{pmatrix} A + B\cos\vartheta & B\sin\vartheta & 0 \\ B\sin\vartheta & A - B\cos\vartheta & 0 \\ 0 & 0 & 2A \end{pmatrix} \ , \tag{13.42}$$

The characteristic polynomial is found to be

$$P(\lambda) = (\lambda - 2A)\Big[(\lambda - A)^2 - B^2\Big] \ , \tag{13.43}$$

which gives $\lambda_1 = A$, $\lambda_{2,3} = A \pm B$. The corresponding normalized eigenvectors are

$$\boldsymbol{\psi}^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^2 = \begin{pmatrix} \cos\frac{1}{2}\vartheta \\ \sin\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^3 = \begin{pmatrix} -\sin\frac{1}{2}\vartheta \\ \cos\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \tag{13.44}$$

and therefore

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\frac{1}{2}\vartheta & \sin\frac{1}{2}\vartheta & 0 \\ -\sin\frac{1}{2}\vartheta & \cos\frac{1}{2}\vartheta & 0 \end{pmatrix} \ . \tag{13.45}$$

13.5 Euler's Equations

Let us now choose our coordinate axes to be the principal axes of inertia, with the CM at the origin. We may then write

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad , \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{L} = \begin{pmatrix} I_1\,\omega_1 \\ I_2\,\omega_2 \\ I_3\,\omega_3 \end{pmatrix} \ . \tag{13.46}$$

The equations of motion are

$$\begin{aligned} \boldsymbol{N}^{\rm ext} &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\rm inertial} \\ &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\rm body} + \boldsymbol{\omega} \times \boldsymbol{L} \\ &= I\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\,\boldsymbol{\omega}) \ . \end{aligned}$$

Thus, we arrive at *Euler's equations:*

$$I_1\,\dot\omega_1 = (I_2 - I_3)\,\omega_2\,\omega_3 + N_1^{\rm ext} \tag{13.47}$$
$$I_2\,\dot\omega_2 = (I_3 - I_1)\,\omega_3\,\omega_1 + N_2^{\rm ext} \tag{13.48}$$
$$I_3\,\dot\omega_3 = (I_1 - I_2)\,\omega_1\,\omega_2 + N_3^{\rm ext} \ . \tag{13.49}$$

These are coupled and nonlinear. Also note the fact that the external torque must be evaluated along body-fixed principal axes. We can however make progress in the case

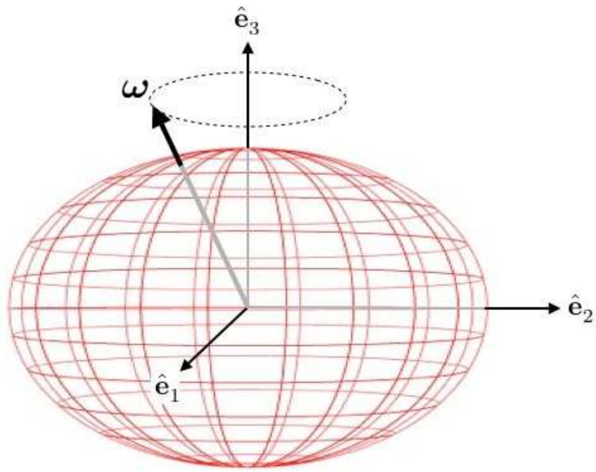


Figure 13.5: Wobbling of a torque-free symmetric top.

where $\boldsymbol{N}^{\text{ext}} = 0$, *i.e.* when there are no external torques. This is true for a body in free space, or in a uniform gravitational field. In the latter case,

$$\boldsymbol{N}^{\text{ext}} = \sum_i \boldsymbol{r}_i \times (m_i\, \boldsymbol{g}) = \Big(\sum_i m_i \boldsymbol{r}_i\Big) \times \boldsymbol{g} \ , \tag{13.50}$$

where $\boldsymbol{g}$ is the uniform gravitational acceleration. In a body-fixed frame whose origin is the CM, we have $\sum_i m_i \boldsymbol{r}_i = 0$, and the external torque vanishes!

Precession of torque-free symmetric tops: Consider a body which has a symmetry axis $\hat{\boldsymbol{e}}_3$. This guarantees $I_1 = I_2$, but in general we still have $I_1 \neq I_3$. In the absence of external torques, the last of Euler's equations says $\dot{\omega}_3 = 0$, so ω_3 is a constant. The remaining two equations are then

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right) \omega_3\, \omega_2 \quad , \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right) \omega_3\, \omega_1 \ . \tag{13.51}$$

I.e.$\dot{\omega}_1 = -\Omega\, \omega_2$ and $\dot{\omega}_2 = +\Omega\, \omega_1$, with

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right) \omega_3 \ , \tag{13.52}$$

which are the equations of a harmonic oscillator. The solution is easily obtained:

$$\omega_1(t) = \omega_\perp \cos\big(\Omega t + \delta\big) \quad , \quad \omega_2(t) = \omega_\perp \sin\big(\Omega t + \delta\big) \quad , \quad \omega_3(t) = \omega_3 \ , \tag{13.53}$$

where $\omega_\perp$ and δ are constants of integration, and where $|\boldsymbol{\omega}| = (\omega_\perp^2 + \omega_3^2)^{1/2}$. This motion is sketched in Fig. 13.5. Note that the perpendicular components of $\boldsymbol{\omega}$ oscillate harmonically, and that the angle $\boldsymbol{\omega}$ makes with respect to $\hat{\boldsymbol{e}}_3$ is $\lambda = \tan^{-1}(\omega_\perp/\omega_3)$.

For the earth, $(I_3 - I_1)/I_1 \approx \frac{1}{305}$, so $\omega_3 \approx \omega$, and $\Omega \approx \omega/305$, yielding a precession period of 305 days, or roughly 10 months. Astronomical observations reveal such a precession, known as the *Chandler wobble.* For the earth, the precession angle is $\lambda_{\text{Chandler}} \simeq 6 \times 10^{-7}$ rad, which means that the North Pole moves by about 4 meters during the wobble. The Chandler wobble has a period of about 14 months, so the naïve prediction of 305 days is off by a substantial amount. This discrepancy is attributed to the mechanical properties of the earth: elasticity and fluidity. The earth is not solid![1]

Asymmetric tops: Next, consider the torque-free motion of an asymmetric top, where $I_1 \neq I_2 \neq I_3 \neq I_1$. Unlike the symmetric case, there is no conserved component of $\boldsymbol{\omega}$. True, we can invoke conservation of energy and angular momentum,

$$E = \tfrac{1}{2} I_1\, \omega_1^2 + \tfrac{1}{2} I_2\, \omega_2^2 + \tfrac{1}{2} I_3\, \omega_3^2 \tag{13.54}$$

$$\boldsymbol{L}^2 = I_1^2\, \omega_1^2 + I_2^2\, \omega_2^2 + I_3^2\, \omega_3^2 \ , \tag{13.55}$$

and, in principle, solve for ω_1 and ω_2 in terms of ω_3, and then invoke Euler's equations (which must honor these conservation laws). However, the nonlinearity greatly complicates matters and in general this approach is a dead end.

We can, however, find a *particular* solution quite easily – one in which the rotation is about a single axis. Thus, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ is indeed a solution for all time, according to Euler's equations. Let us now perturb about this solution, to explore its stability. We write

$$\boldsymbol{\omega} = \omega_0\, \hat{\boldsymbol{e}}_3 + \delta\boldsymbol{\omega} \ , \tag{13.56}$$

and we invoke Euler's equations, linearizing by dropping terms quadratic in $\delta\boldsymbol{\omega}$. This yield

$$I_1\, \delta\dot{\omega}_1 = (I_2 - I_3)\, \omega_0\, \delta\omega_2 + \mathcal{O}(\delta\omega_2\, \delta\omega_3) \tag{13.57}$$

$$I_2\, \delta\dot{\omega}_2 = (I_3 - I_1)\, \omega_0\, \delta\omega_1 + \mathcal{O}(\delta\omega_3\, \delta\omega_1) \tag{13.58}$$

$$I_3\, \delta\dot{\omega}_3 = 0 + \mathcal{O}(\delta\omega_1\, \delta\omega_2) \ . \tag{13.59}$$

Taking the time derivative of the first equation and invoking the second, and *vice versa,* yields

$$\delta\ddot{\omega}_1 = -\Omega^2\, \delta\omega_1 \quad , \quad \delta\ddot{\omega}_2 = -\Omega^2\, \delta\omega_2 \ , \tag{13.60}$$

with

$$\Omega^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1\, I_2} \cdot \omega_0^2 \ . \tag{13.61}$$

The solution is then $\delta\omega_1(t) = C\, \cos(\Omega t + \delta)$.

If $\Omega^2 > 0$, then Ω is real, and the deviation results in a harmonic precession. This occurs if I_3 is either the largest or the smallest of the moments of inertia. If, however, I_3 is the middle moment, then $\Omega^2 < 0$, and Ω is purely imaginary. The perturbation will in general increase exponentially with time, which means that the initial solution to Euler's equations is *unstable* with respect to small perturbations. This result can be vividly realized using a tennis racket, and sometimes goes by the name of the "tennis racket theorem."

[1]The earth is a layered like a *Mozartkugel,* with a solid outer shell, an inner fluid shell, and a solid (iron) core.

13.5.1 Example

PROBLEM: A unsuspecting solid spherical planet of mass M_0 rotates with angular velocity ω_0. Suddenly, a giant asteroid of mass αM_0 smashes into and sticks to the planet at a location which is at polar angle θ relative to the initial rotational axis. The new mass distribution is no longer spherically symmetric, and the rotational axis will precess. Recall Euler's equation

$$\frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} = \boldsymbol{N}^{\text{ext}} \tag{13.62}$$

for rotations in a body-fixed frame.

(a) What is the new inertia tensor $I_{\alpha\beta}$ along principle center-of-mass frame axes? Don't forget that the CM is no longer at the center of the sphere! Recall $I = \frac{2}{5}MR^2$ for a solid sphere.

(b) What is the period of precession of the rotational axis in terms of the original length of the day $2\pi/\omega_0$?

SOLUTION: Let's choose body-fixed axes with $\hat{\boldsymbol{z}}$ pointing from the center of the planet to the smoldering asteroid. The CM lies a distance

$$d = \frac{\alpha M_0 \cdot R + M_0 \cdot 0}{(1+\alpha)M_0} = \frac{\alpha}{1+\alpha}\,R \tag{13.63}$$

from the center of the sphere. Thus, relative to the center of the sphere, we have

$$I = \tfrac{2}{5}M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{13.64}$$

Now we shift to a frame with the CM at the origin, using the parallel axis theorem,

$$I_{\alpha\beta}(\boldsymbol{d}) = I^{\text{CM}}_{\alpha\beta} + M\big(\boldsymbol{d}^2\,\delta_{\alpha\beta} - d_\alpha d_\beta\big) \ . \tag{13.65}$$

Thus, with $\boldsymbol{d} = d\hat{\boldsymbol{z}}$,

$$I^{\text{CM}}_{\alpha\beta} = \tfrac{2}{5}M_0R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - (1+\alpha)M_0 d^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{13.66}$$

$$= M_0R^2 \begin{pmatrix} \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 & 0 \\ 0 & \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} \quad . \tag{13.67}$$

In the absence of external torques, Euler's equations along principal axes read

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\,\omega_2\,\omega_3 \\ I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\,\omega_3\,\omega_1 \\ I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\,\omega_1\,\omega_2 \end{aligned} \tag{13.68}$$

Since $I_1 = I_2$, $\omega_3(t) = \omega_3(0) = \omega_0 \cos\theta$ is a constant. We then obtain $\dot{\omega}_1 = \Omega\omega_2$, and $\dot{\omega}_2 = -\Omega\omega_1$, with

$$\Omega = \frac{I_2 - I_3}{I_1}\,\omega_3 = \frac{5\alpha}{7\alpha + 2}\,\omega_3 \ . \tag{13.69}$$

The period of precession τ in units of the pre-cataclysmic day is

$$\frac{\tau}{T} = \frac{\omega}{\Omega} = \frac{7\alpha + 2}{5\alpha\cos\theta} \ . \tag{13.70}$$

13.6 Euler's Angles

In d dimensions, an orthogonal matrix $\mathcal{R} \in \mathrm{O}(d)$ has $\frac{1}{2}d(d-1)$ independent parameters. To see this, consider the constraint $\mathcal{R}^{\mathrm{t}}\mathcal{R} = 1$. The matrix $\mathcal{R}^{\mathrm{t}}\mathcal{R}$ is manifestly symmetric, so it has $\frac{1}{2}d(d+1)$ independent entries (*e.g.* on the diagonal and above the diagonal). This amounts to $\frac{1}{2}d(d+1)$ constraints on the d^2 components of $\mathcal{R}$, resulting in $\frac{1}{2}d(d-1)$ freedoms. Thus, in $d = 3$ rotations are specified by three parameters. The *Euler angles* $\{\phi, \theta, \psi\}$ provide one such convenient parameterization.

A general rotation $\mathcal{R}(\phi, \theta, \psi)$ is built up in three steps. We start with an orthonormal triad $\hat{\boldsymbol{e}}^0_\mu$ of body-fixed axes. The first step is a rotation by an angle ϕ about $\hat{\boldsymbol{e}}^0_3$:

$$\hat{\boldsymbol{e}}'_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\boldsymbol{e}}^0_3, \phi\big)\,\hat{\boldsymbol{e}}^0_\nu \quad , \quad \mathcal{R}\big(\hat{\boldsymbol{e}}^0_3, \phi\big) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.71}$$

This step is shown in panel (a) of Fig. 13.6. The second step is a rotation by θ about the new axis $\hat{\boldsymbol{e}}'_1$:

$$\hat{\boldsymbol{e}}''_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\boldsymbol{e}}'_1, \theta\big)\,\hat{\boldsymbol{e}}'_\nu \quad , \quad \mathcal{R}\big(\hat{\boldsymbol{e}}'_1, \theta\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{13.72}$$

This step is shown in panel (b) of Fig. 13.6. The third and final step is a rotation by ψ about the new axis $\hat{\boldsymbol{e}}''_3$:

$$\hat{\boldsymbol{e}}'''_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\boldsymbol{e}}''_3, \psi\big)\,\hat{\boldsymbol{e}}''_\nu \quad , \quad \mathcal{R}\big(\hat{\boldsymbol{e}}''_3, \psi\big) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{13.73}$$

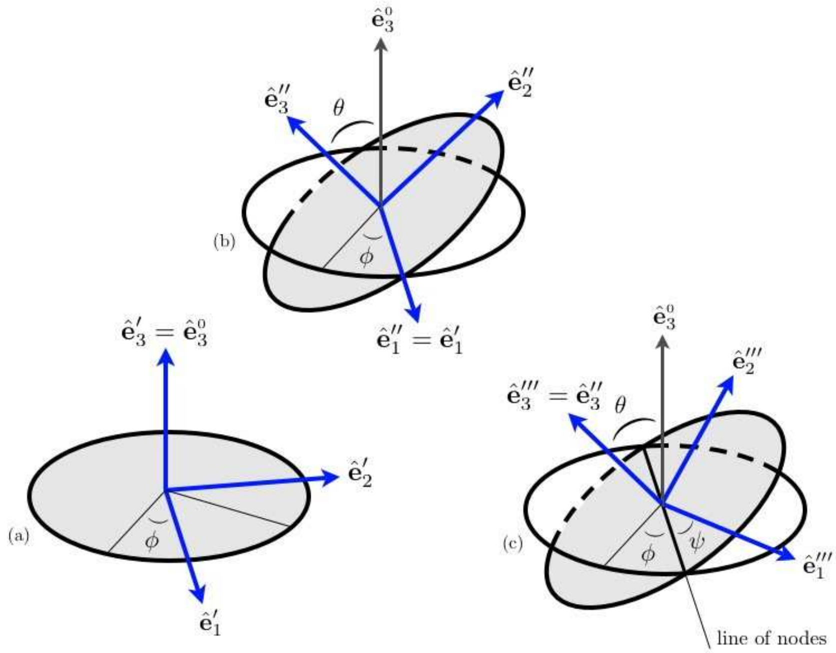


Figure 13.6: A general rotation, defined in terms of the Euler angles $\{\phi, \theta, \psi\}$. Three successive steps of the transformation are shown.

This step is shown in panel (c) of Fig. 13.6. Putting this all together,

$$\mathcal{R}(\phi,\theta,\psi) = \mathcal{R}\big(\hat{\boldsymbol{e}}_3'',\phi\big)\,\mathcal{R}\big(\hat{\boldsymbol{e}}_1',\theta\big)\,\mathcal{R}\big(\hat{\boldsymbol{e}}_3^0,\psi\big) \tag{13.74}$$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} .$$

Next, we'd like to relate the components $\omega_\mu = \boldsymbol{\omega}\cdot\hat{\boldsymbol{e}}_\mu$ (with $\hat{\boldsymbol{e}}_\mu \equiv \hat{\boldsymbol{e}}_\mu'''$) of the rotation in the body-fixed frame to the derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$. To do this, we write

$$\boldsymbol{\omega} = \dot{\phi}\,\hat{\boldsymbol{e}}_\phi + \dot{\theta}\,\hat{\boldsymbol{e}}_\theta + \dot{\psi}\,\hat{\boldsymbol{e}}_\psi \ , \tag{13.75}$$

where

$$\hat{\boldsymbol{e}}_3^0 = \hat{\boldsymbol{e}}_\phi = \sin\theta\sin\psi\,\hat{\boldsymbol{e}}_1 + \sin\theta\cos\psi\,\hat{\boldsymbol{e}}_2 + \cos\theta\,\hat{\boldsymbol{e}}_3 \tag{13.76}$$

$$\hat{\boldsymbol{e}}_\theta = \cos\psi\,\hat{\boldsymbol{e}}_1 - \sin\psi\,\hat{\boldsymbol{e}}_2 \qquad (\text{“line of nodes”}) \tag{13.77}$$

$$\hat{\boldsymbol{e}}_\psi = \hat{\boldsymbol{e}}_3 \ . \tag{13.78}$$

This gives

$$\omega_1 = \boldsymbol{\omega} \cdot \hat{\boldsymbol{e}}_1 = \dot{\phi}\,\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{13.79}$$
$$\omega_2 = \boldsymbol{\omega} \cdot \hat{\boldsymbol{e}}_2 = \dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi \tag{13.80}$$
$$\omega_3 = \boldsymbol{\omega} \cdot \hat{\boldsymbol{e}}_3 = \dot{\phi}\cos\theta + \dot{\psi}\ . \tag{13.81}$$

Note that

$$\dot{\phi} \leftrightarrow \text{precession} \quad , \quad \dot{\theta} \leftrightarrow \text{nutation} \quad , \quad \dot{\psi} \leftrightarrow \text{axial rotation}\ . \tag{13.82}$$

The general form of the kinetic energy is then

$$\begin{aligned} T = {\textstyle\frac{1}{2}} I_1 \big(\dot{\phi}\,\sin\theta\sin\psi + \dot{\theta}\cos\psi\big)^2 \\ + {\textstyle\frac{1}{2}} I_2 \left(\dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi\right)^2 + {\textstyle\frac{1}{2}} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2\ . \end{aligned} \tag{13.83}$$

Note that

$$\boldsymbol{L} = p_\phi\,\hat{\boldsymbol{e}}_\phi + p_\theta\,\hat{\boldsymbol{e}}_\theta + p_\psi\,\hat{\boldsymbol{e}}_\psi\ , \tag{13.84}$$

which may be verified by explicit computation.

13.6.1 Torque-free symmetric top

A body falling in a gravitational field experiences no net torque about its CM:

$$\boldsymbol{N}^{\text{ext}} = \sum_i \boldsymbol{r}_i \times (-m_i\,\boldsymbol{g}) = \boldsymbol{g} \times \sum_i m_i\,\boldsymbol{r}_i = 0\ . \tag{13.85}$$

For a symmetric top with $I_1 = I_2$, we have

$$T = {\textstyle\frac{1}{2}} I_1 \big(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\!\theta\big) + {\textstyle\frac{1}{2}} I_3 \big(\dot{\phi}\cos\theta + \dot{\psi}\big)^2\ . \tag{13.86}$$

The potential is cyclic in the Euler angles, hence the equations of motion are

$$\frac{d}{dt}\,\frac{\partial T}{\partial(\dot{\phi},\dot{\theta},\dot{\psi})} = \frac{\partial T}{\partial(\phi,\theta,\psi)}\ . \tag{13.87}$$

Since ϕ and ψ are cyclic in T, their conjugate momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = I_1\,\dot{\phi}\,\sin^2\!\theta + I_3\,(\dot{\phi}\cos\theta + \dot{\psi})\,\cos\theta \tag{13.88}$$
$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = I_3\,(\dot{\phi}\cos\theta + \dot{\psi})\ . \tag{13.89}$$

Note that $p_\psi = I_3\,\omega_3$, hence ω_3 is constant, as we have already seen.

To solve for the motion, we first note that $\boldsymbol{L}$ is conserved in the inertial frame. We are therefore permitted to define $\hat{\boldsymbol{L}} = \hat{\boldsymbol{e}}^0_3 = \hat{\boldsymbol{e}}_\phi$. Thus, $p_\phi = L$. Since $\hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_\psi = \cos\theta$, we have

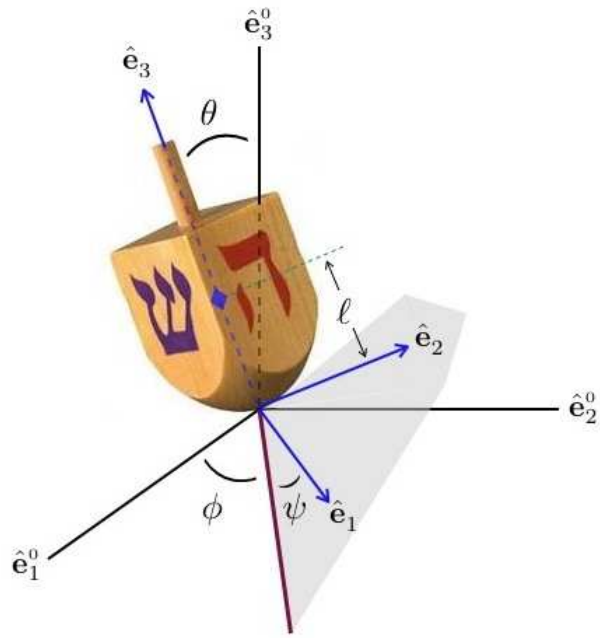


Figure 13.7: A *dreidl* is a symmetric top. The four-fold symmetry axis guarantees $I_1 = I_2$. The blue diamond represents the center-of-mass.

that $p_\psi = \boldsymbol{L} \cdot \hat{\boldsymbol{e}}_\psi = L\cos\theta$. Finally, $\hat{\boldsymbol{e}}_\phi \cdot \hat{\boldsymbol{e}}_\theta = 0$, which means $p_\theta = \boldsymbol{L} \cdot \hat{\boldsymbol{e}}_\theta = 0$. From the equations of motion,

$$\dot{p}_\theta = I_1\,\ddot{\theta} = \big(I_1\,\dot{\phi}\cos\theta - p_\psi\big)\,\dot{\phi}\sin\theta\ , \tag{13.90}$$

hence we must have

$$\dot{\theta} = 0 \quad , \quad \dot{\phi} = \frac{p_\psi}{I_1\cos\theta}\ . \tag{13.91}$$

Note that $\dot{\theta} = 0$ follows from conservation of $p_\psi = L\cos\theta$. From the equation for p_ψ, we may now conclude

$$\dot{\psi} = \frac{p_\psi}{I_3} - \frac{p_\psi}{I_1} = \left(\frac{I_3 - I_1}{I_3}\right)\omega_3\ , \tag{13.92}$$

which recapitulates (13.52), with $\dot{\psi} = \varOmega$.

13.6.2 Symmetric top with one point fixed

Consider the case of a symmetric top with one point fixed, as depicted in Fig. 13.7. The Lagrangian is

$$L = \tfrac{1}{2}I_1\big(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\!\theta\big) + \tfrac{1}{2}I_3\big(\dot{\phi}\cos\theta + \dot{\psi}\big)^2 - Mg\ell\cos\theta\ . \tag{13.93}$$

Here, ℓ is the distance from the fixed point to the CM, and the inertia tensor is defined along principal axes whose origin lies at the fixed point (not the CM!). Gravity now supplies a

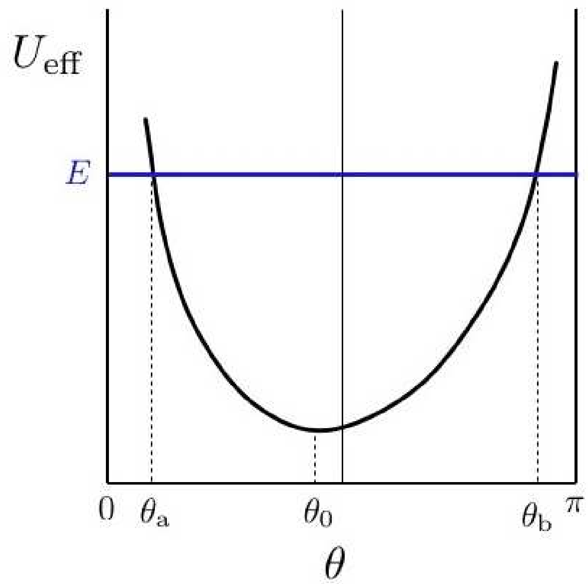


Figure 13.8: The effective potential of eq. 13.102.

torque, but as in the torque-free case, the Lagrangian is still cyclic in ϕ and ψ, so

$$p_\phi = (I_1 \sin^2\theta + I_3 \cos^2\theta)\,\dot\phi + I_3 \cos\theta\,\dot\psi \tag{13.94}$$

$$p_\psi = I_3 \cos\theta\,\dot\phi + I_3\,\dot\psi \tag{13.95}$$

are each conserved. We can invert these relations to obtain $\dot\phi$ and $\dot\psi$ in terms of $\{p_\phi, p_\psi, \theta\}$:

$$\dot\phi = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta} \quad , \quad \dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos\theta)\cos\theta}{I_1 \sin^2\theta} \ . \tag{13.96}$$

In addition, since $\partial L/\partial t = 0$, the total energy is conserved:

$$E = T + U = \tfrac{1}{2} I_1\,\dot\theta^2 + \overbrace{\frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta}^{U_{\rm eff}(\theta)} \ , \tag{13.97}$$

where the term under the brace is the effective potential $U_{\rm eff}(\theta)$.

The problem thus reduces to the one-dimensional dynamics of $\theta(t)$, *i.e.*

$$I_1\,\ddot\theta = -\frac{\partial U_{\rm eff}}{\partial\theta} \ , \tag{13.98}$$

with

$$U_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \ . \tag{13.99}$$

Using energy conservation, we may write

$$dt = \pm\sqrt{\frac{I_1}{2}}\,\frac{d\theta}{\sqrt{E - U_{\rm eff}(\theta)}} \ . \tag{13.100}$$

and thus the problem is reduced to quadratures:

$$t(\theta) = t(\theta_0) \pm \sqrt{\frac{I_1}{2}} \int_{\theta_0}^{\theta} d\vartheta \, \frac{1}{\sqrt{E - U_{\rm eff}(\vartheta)}} \ . \tag{13.101}$$

We can gain physical insight into the motion by examining the shape of the effective potential,

$$U_{\rm eff}(\theta) = \frac{(p_\phi - p_\psi \, \cos\theta)^2}{2I_1 \, \sin^2\!\theta} + Mg\ell \cos\theta + \frac{p_\psi^2}{2I_3} \ , \tag{13.102}$$

over the interval $\theta \in [0, \pi]$. Clearly $U_{\rm eff}(0) = U_{\rm eff}(\pi) = \infty$, so the motion must be bounded. What is not yet clear, but what is nonetheless revealed by some additional analysis, is that $U_{\rm eff}(\theta)$ has a single minimum on this interval, at $\theta = \theta_0$. The turning points for the θ motion are at $\theta = \theta_{\rm a}$ and $\theta = \theta_{\rm b}$, where $U_{\rm eff}(\theta_{\rm a}) = U_{\rm eff}(\theta_{\rm b}) = E$. Clearly if we expand about θ_0 and write $\theta = \theta_0 + \eta$, the η motion will be harmonic, with

$$\eta(t) = \eta_0 \, \cos(\Omega t + \delta) \quad , \quad \Omega = \sqrt{\frac{U''_{\rm eff}(\theta_0)}{I_1}} \ . \tag{13.103}$$

To prove that $U_{\rm eff}(\theta)$ has these features, let us define $u \equiv \cos\theta$. Then $\dot u = -\,\dot\theta \sin\theta$, and from $E = \frac{1}{2} I_1 \, \dot\theta^2 + U_{\rm eff}(\theta)$ we derive

$$\dot u^2 = \left(\frac{2E}{I_1} - \frac{p_\psi^2}{I_1 I_3}\right)(1 - u^2) - \frac{2Mg\ell}{I_1}\,(1 - u^2)\, u - \left(\frac{p_\phi - p_\psi \, u}{I_1}\right)^2 \equiv f(u) \ . \tag{13.104}$$

The turning points occur at $f(u) = 0$. The function $f(u)$ is cubic, and the coefficient of the cubic term is $2Mg\ell/I_1$, which is positive. Clearly $f(u = \pm 1) = -(p_\phi \mp p_\psi)^2/I_1^2$ is negative, so there must be at least one solution to $f(u) = 0$ on the interval $u \in (1, \infty)$. Clearly there can be at most three real roots for $f(u)$, since the function is cubic in u, hence there are at most two turning points on the interval $u \in [-1, 1]$. Thus, $U_{\rm eff}(\theta)$ has the form depicted in fig. 13.8.

To apprehend the full motion of the top in an inertial frame, let us follow the symmetry axis $\hat{\boldsymbol{e}}_3$:

$$\hat{\boldsymbol{e}}_3 = \sin\theta \sin\phi \, \hat{\boldsymbol{e}}_1^0 - \sin\theta \cos\phi \, \hat{\boldsymbol{e}}_2^0 + \cos\theta \, \hat{\boldsymbol{e}}_3^0 \ . \tag{13.105}$$

Once we know $\theta(t)$ and $\phi(t)$ we're done. The motion $\theta(t)$ is described above: θ oscillates between turning points at $\theta_{\rm a}$ and $\theta_{\rm b}$. As for $\phi(t)$, we have already derived the result

$$\dot\phi = \frac{p_\phi - p_\psi \, \cos\theta}{I_1 \, \sin^2\!\theta} \ . \tag{13.106}$$

Thus, if $p_\phi > p_\psi \cos\theta_{\rm a}$, then $\dot\phi$ will remain positive throughout the motion. If, on the other hand, we have

$$p_\psi \cos\theta_{\rm b} < p_\phi < p_\psi \cos\theta_{\rm a} \ , \tag{13.107}$$

then $\dot\phi$ changes sign at an angle $\theta^* = \cos^{-1}\big(p_\phi/p_\psi\big)$. The motion is depicted in Fig. 13.9. An extensive discussion of this problem is given in H. Goldstein, *Classical Mechanics*.

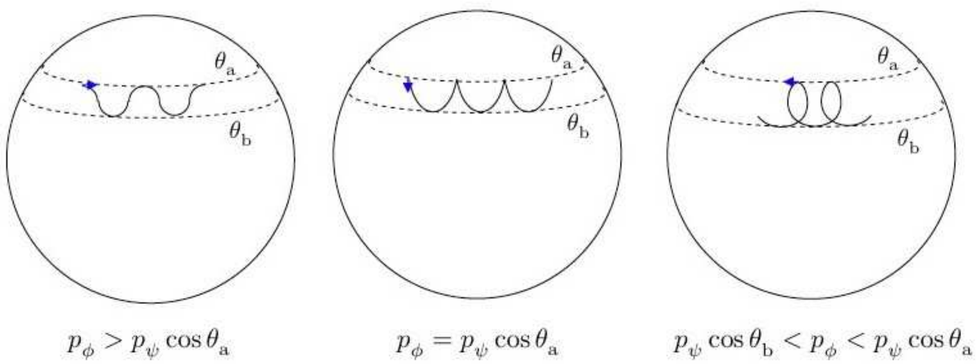


Figure 13.9: Precession and nutation of the symmetry axis of a symmetric top.

13.7 Rolling and Skidding Motion of Real Tops

The material in this section is based on the corresponding sections from V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective*. This is an excellent book which contains many interesting applications and examples.

13.7.1 Rolling tops

In most tops, the point of contact rolls or skids along the surface. Consider the peg end top of Fig. 13.10, executing a circular rolling motion, as sketched in Fig. 13.11. There are three components to the force acting on the top: gravity, the normal force from the surface, and friction. The frictional force is perpendicular to the CM velocity, and results in centripetal acceleration of the top:

$$f = M\Omega^2\rho \leq \mu Mg \ , \tag{13.108}$$

where Ω is the frequency of the CM motion and μ is the coefficient of friction. If the above inequality is violated, the top starts to slip.

The frictional and normal forces combine to produce a torque $N = Mg\ell\sin\theta - f\ell\cos\theta$ about the CM[2]. This torque is tangent to the circular path of the CM, and causes $\boldsymbol{L}$ to precess. We assume that the top is spinning rapidly, so that $\boldsymbol{L}$ very nearly points along the symmetry axis of the top itself. (As we'll see, this is true for slow precession but not for fast precession, where the precession frequency is proportional to ω_3.) The precession is then governed by the equation

$$\begin{aligned} N &= Mg\ell\sin\theta - f\ell\cos\theta \\ &= \big|\dot{\boldsymbol{L}}\big| = \big|\boldsymbol{\Omega}\times\boldsymbol{L}\big| \approx \Omega\, I_3\,\omega_3 \sin\theta \ , \end{aligned} \tag{13.109}$$

[2]Gravity of course produces no net torque about the CM.

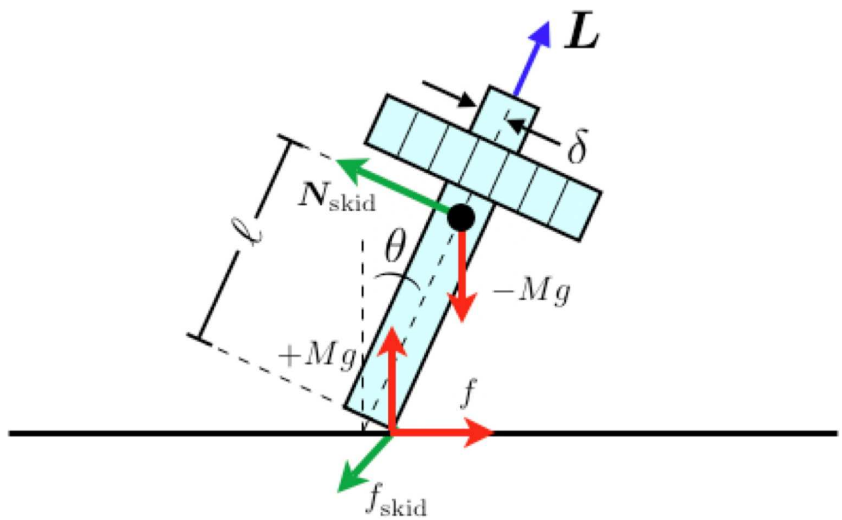


Figure 13.10: A top with a peg end. The frictional forces $\boldsymbol{f}$ and $\boldsymbol{f}_{\rm skid}$ are shown. When the top rolls without skidding, $\boldsymbol{f}_{\rm skid}=0$.

where $\hat{\mathbf{e}}_3$ is the instantaneous symmetry axis of the top. Substituting $f=M\Omega^2\rho$,

$$\frac{Mg\ell}{I_3\,\omega_3}\left(1-\frac{\Omega^2\rho}{g}\,{\rm ctn}\,\theta\right)=\Omega\ , \tag{13.110}$$

which is a quadratic equation for Ω. We supplement this with the 'no slip' condition,

$$\omega_3\,\delta=\Omega\left(\rho+\ell\sin\theta\right)\ , \tag{13.111}$$

resulting in two equations for the two unknowns Ω and ρ.

Substituting for $\rho(\Omega)$ and solving for Ω, we obtain

$$\Omega=\frac{I_3\,\omega_3}{2M\ell^2\cos\theta}\left\{1+\frac{Mg\ell\delta}{I_3}\,{\rm ctn}\,\theta\pm\sqrt{\left(1+\frac{Mg\ell\delta}{I_3}\,{\rm ctn}\,\theta\right)^2-\frac{4M\ell^2}{I_3}\cdot\frac{Mg\ell}{I_3\,\omega_3^2}}\right\}\ . \tag{13.112}$$

This in order to have a real solution we must have

$$\omega_3\geq\frac{2M\ell^2\sin\theta}{I_3\sin\theta+Mg\ell\delta\cos\theta}\sqrt{\frac{g}{\ell}}\ . \tag{13.113}$$

If the inequality is satisfied, there are two possible solutions for Ω, corresponding to fast and slow precession. Usually one observes slow precession. Note that it is possible that $\rho<0$, in which case the CM and the peg end lie on opposite sides of a circle from each other.

13.7.2 Skidding tops

A skidding top experiences a frictional force which opposes the skidding velocity, until $v_{\rm skid}=0$ and a pure rolling motion sets in. This force provides a torque which makes the

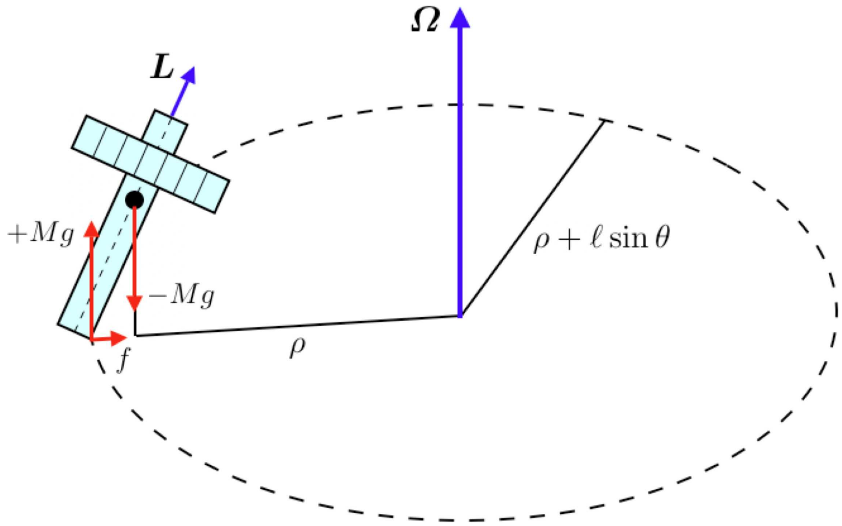


Figure 13.11: Circular rolling motion of the peg top.

top *rise:*

$$\dot{\theta} = -\frac{N_{\text{skid}}}{L} = -\frac{\mu M g \ell}{I_3\,\omega_3} \,. \tag{13.114}$$

Suppose $\delta \approx 0$, in which case $\rho + \ell \sin\theta = 0$, from eqn. 13.111, and the point of contact remains fixed. Now recall the effective potential for a symmetric top with one point fixed:

$$U_{\text{eff}}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \,. \tag{13.115}$$

We demand $U'_{\text{eff}}(\theta_0) = 0$, which yields

$$\cos\theta_0 \cdot \beta^2 - p_\psi \sin^2\!\theta_0 \cdot \beta + Mg\ell I_1 \sin^4\!\theta_0 = 0 \,, \tag{13.116}$$

where

$$\beta \equiv p_\phi - p_\psi \cos\theta_0 = I_1 \sin^2\!\theta_0\,\dot{\phi} \,. \tag{13.117}$$

Solving the quadratic equation for β, we find

$$\dot{\phi} = \frac{I_3\,\omega_3}{2I_1\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4Mg\ell I_1\cos\theta_0}{I_3^2\,\omega_3^2}}\,\right) . \tag{13.118}$$

This is simply a recapitulation of eqn. 13.112, with $\delta = 0$ and with $M\ell^2$ replaced by I_1. Note $I_1 = M\ell^2$ by the parallel axis theorem if $I_1^{\text{CM}} = 0$. But to the extent that $I_1^{\text{CM}} \neq 0$, our treatment of the peg top was incorrect. It turns out to be OK, however, if the precession is slow, *i.e.* if $\Omega/\omega_3 \ll 1$.

On a level surface, $\cos\theta_0 > 0$, and therefore we must have

$$\omega_3 \geq \frac{2}{I_3}\sqrt{Mg\ell I_1\cos\theta_0} \,. \tag{13.119}$$

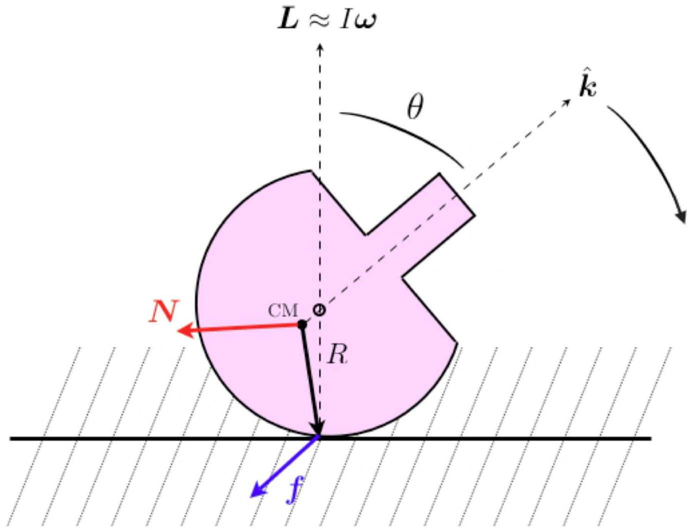


Figure 13.12: The tippie-top behaves in a counterintuitive way. Once started spinning with the peg end up, the peg axis rotates downward. Eventually the peg scrapes the surface and the top rises to the vertical in an inverted orientation.

Thus, if the top spins too slowly, it cannot maintain precession. Eqn. 13.118 says that there are two possible precession frequencies. When ω_3 is large, we have

$$\dot{\phi}_{\rm slow} = \frac{Mg\ell}{I_3\,\omega_3} + \mathcal{O}(\omega_3^{-1}) \qquad , \qquad \dot{\phi}_{\rm fast} = \frac{I_3\,\omega_3}{I_1\cos\theta_0} + \mathcal{O}(\omega_3^{-3}) \ . \tag{13.120}$$

Again, one usually observes slow precession.

A top with $\omega_3 > \frac{2}{I_3}\sqrt{Mg\ell I_1}$ may 'sleep' in the vertical position with $\theta_0 = 0$. Due to the constant action of frictional forces, ω_3 will eventually drop below this value, at which time the vertical position is no longer stable. The top continues to slow down and eventually falls.

13.7.3 Tippie-top

A particularly nice example from the Barger and Olsson book is that of the tippie-top, a truncated sphere with a peg end, sketched in Fig. 13.12 The CM is close to the center of curvature, which means that there is almost no gravitational torque acting on the top. The frictional force $\boldsymbol{f}$ opposes slipping, but as the top spins $\boldsymbol{f}$ rotates with it, and hence the time-averaged frictional force $\langle \boldsymbol{f} \rangle \approx 0$ has almost no effect on the motion of the CM. A similar argument shows that the frictional torque, which is nearly horizontal, also time averages to zero:

$$\left\langle \frac{d\boldsymbol{L}}{dt} \right\rangle_{\rm inertial} \approx \ 0 \ . \tag{13.121}$$

In the *body*-fixed frame, however, $\boldsymbol{N}$ is roughly constant, with magnitude $N \approx \mu MgR$, where R is the radius of curvature and μ the coefficient of sliding friction. Now we invoke

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt}\bigg|_{\rm body} + \ \boldsymbol{\omega} \times \boldsymbol{L} \ . \tag{13.122}$$

The second term on the RHS is very small, because the tippie-top is almost spherical, hence inertia tensor is very nearly diagonal, and this means

$$\boldsymbol{\omega} \times \boldsymbol{L} \approx \boldsymbol{\omega} \times I\boldsymbol{\omega} = 0 \ . \tag{13.123}$$

Thus, $\dot{\boldsymbol{L}}_{\rm body} \approx \boldsymbol{N}$, and taking the dot product of this equation with the unit vector $\hat{\boldsymbol{k}}$, we obtain

$$-N\sin\theta = \hat{\boldsymbol{k}} \cdot \boldsymbol{N} = \frac{d}{dt}\Big(\hat{\boldsymbol{k}} \cdot \boldsymbol{L}_{\rm body}\Big) = -L\sin\theta\,\dot{\theta} \ . \tag{13.124}$$

Thus,

$$\dot{\theta} = \frac{N}{L} \approx \frac{\mu M g R}{I\omega} \ . \tag{13.125}$$

Once the stem scrapes the table, the tippie-top rises to the vertical just like any other rising top.

Chapter 14

Continuum Mechanics

14.1 Strings

Consider a string of linear mass density $\mu(x)$ under tension $\tau(x)$.[1] Let the string move in a plane, such that its shape is described by a smooth function $y(x)$, the vertical displacement of the string at horizontal position x, as depicted in fig. 14.1. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. Consider a differential element of the string extending from x to $x+dx$. The change in length relative to the unstretched ($y=0$) configuration is

$$d\ell = \sqrt{dx^2 + dy^2} - dx = \frac{1}{2}\bigg(\frac{\partial y}{\partial x}\bigg)^2 dx + \mathcal{O}\big(dx^2\big) \ . \tag{14.1}$$

The differential potential energy is then

$$dU = \tau(x)\, d\ell = \tfrac{1}{2}\, \tau(x) \left(\frac{\partial y}{\partial x}\right)^2 dx \ . \tag{14.2}$$

The differential kinetic energy is simply

$$dT = \tfrac{1}{2}\, \mu(x) \left(\frac{\partial y}{\partial t}\right)^2 dx \ . \tag{14.3}$$

We can then write

$$L = \int\! dx\, \mathcal{L} \ , \tag{14.4}$$

where the *Lagrangian density* $\mathcal{L}$ is

$$\mathcal{L}(y, \dot{y}, y'; x, t) = \tfrac{1}{2}\, \mu(x) \bigg(\frac{\partial y}{\partial t}\bigg)^2 - \tfrac{1}{2}\, \tau(x) \bigg(\frac{\partial y}{\partial x}\bigg)^2 \ . \tag{14.5}$$

[1] As an example of a string with a position-dependent tension, consider a string of length ℓ freely suspended from one end at $z=0$ in a gravitational field. The tension is then $\tau(z) = \mu g\,(\ell - z)$.

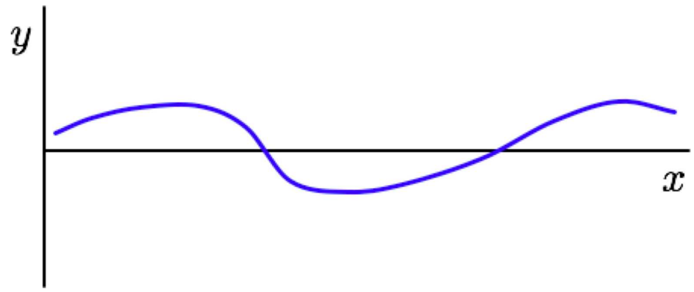


Figure 14.1: A string is described by the vertical displacement field $y(x,t)$.

The action for the string is now a double integral,

$$S = \int_{t_a}^{t_b} dt \int_{x_a}^{x_b} dx \, \mathcal{L}(y, \dot{y}, y'; x, t) \ , \tag{14.6}$$

where $y(x,t)$ is the vertical displacement field. Typically, we have $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2$. The first variation of S is

$$\delta S = \int_{x_a}^{x_b} dx \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right)\right] \delta y \tag{14.7}$$

$$+ \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial \dot{y}} \, \delta y \right]_{t=t_a}^{t=t_b} + \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y'} \, \delta y \right]_{x=x_b}^{x=x_a} \ , \tag{14.8}$$

which simply recapitulates the general result from eqn. 14.203. There are two boundary terms, one of which is an integral over time and the other an integral over space. The first boundary term vanishes provided $\delta y(x, t_a) = \delta y(x, t_b) = 0$. The second boundary term vanishes provided $\tau(x)\, y'(x)\, \delta y(x) = 0$ at $x = x_a$ and $x = x_b$, for all t. Assuming $\tau(x)$ does not vanish, this can happen in one of two ways: at each endpoint either $y(x)$ is fixed or $y'(x)$ vanishes.

Assuming that either $y(x)$ is fixed or $y'(x) = 0$ at the endpoints $x = x_a$ and $x = x_b$, the Euler-Lagrange equations for the string are obtained by setting $\delta S = 0$:

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= \frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) \\ &= \frac{\partial}{\partial x}\left[\tau(x)\, \frac{\partial y}{\partial x}\right] - \mu(x)\, \frac{\partial^2 y}{\partial t^2} \ , \end{aligned} \tag{14.9}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. When $\tau(x) = \tau$ and $\mu(x) = \mu$ are both constants, we obtain the Helmholtz equation,

$$\frac{1}{c^2}\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \ , \tag{14.10}$$

which is the wave equation for the string, where $c = \sqrt{\tau/\mu}$ has dimensions of velocity. We will now see that c is the speed of wave propagation on the string.

14.2 d'Alembert's Solution to the Wave Equation

Let us define two new variables,

$$u \equiv x - ct \qquad , \qquad v \equiv x + ct \ . \tag{14.11}$$

We then have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{14.12}$$

$$\frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\frac{\partial u}{\partial t}\frac{\partial}{\partial u} + \frac{1}{c}\frac{\partial v}{\partial t}\frac{\partial}{\partial v} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \ . \tag{14.13}$$

Thus,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -4\,\frac{\partial^2}{\partial u\,\partial v} \ . \tag{14.14}$$

Thus, the wave equation may be solved:

$$\frac{\partial^2 y}{\partial u\,\partial v} = 0 \qquad \Longrightarrow \qquad y(u,v) = f(u) + g(v) \ , \tag{14.15}$$

where $f(u)$ and $g(v)$ are arbitrary functions. For the moment, we work with an infinite string, so we have no spatial boundary conditions to satisfy. Note that $f(u)$ describes a right-moving disturbance, and $g(v)$ describes a left-moving disturbance:

$$y(x,t) = f(x - ct) + g(x + ct) \ . \tag{14.16}$$

We do, however, have boundary conditions in time. At $t = 0$, the configuration of the string is given by $y(x,0)$, and its instantaneous vertical velocity is $\dot{y}(x,0)$. We then have

$$y(x,0) = f(x) + g(x) \tag{14.17}$$

$$\dot{y}(x,0) = -c\,f'(x) + c\,g'(x) \ , \tag{14.18}$$

hence

$$f'(x) = \tfrac{1}{2}\,y'(x,0) - \tfrac{1}{2c}\,\dot{y}(x,0) \tag{14.19}$$

$$g'(x) = \tfrac{1}{2}\,y'(x,0) + \tfrac{1}{2c}\,\dot{y}(x,0) \ , \tag{14.20}$$

and integrating we obtain the right and left moving components

$$f(\xi) = \tfrac{1}{2}\, y(\xi,0) - \tfrac{1}{2c}\int\limits_0^\xi d\xi'\; \dot y(\xi',0) - \mathcal{C} \tag{14.21}$$

$$g(\xi) = \tfrac{1}{2}\, y(\xi,0) + \tfrac{1}{2c}\int\limits_0^\xi d\xi'\; \dot y(\xi',0) + \mathcal{C}\ , \tag{14.22}$$

where $\mathcal{C}$ is an arbitrary constant. Adding these together, we obtain the full solution

$$y(x,t) = \tfrac{1}{2}\Big[y(x-ct,0) + y(x+ct,0)\Big] + \tfrac{1}{2c}\int\limits_{x-ct}^{x+ct} d\xi\; \dot y(\xi,0)\ , \tag{14.23}$$

valid for all times.

14.2.1 Energy density and energy current

The Hamiltonian density for a string is

$$\mathcal{H} = \varPi\, \dot y - \mathcal{L}\ , \tag{14.24}$$

where

$$\varPi = \frac{\partial \mathcal{L}}{\partial \dot y} = \mu\, \dot y \tag{14.25}$$

is the momentum density. Thus,

$$\mathcal{H} = \frac{\varPi^2}{2\mu} + \tfrac{1}{2}\tau\, y'^2\ . \tag{14.26}$$

Expressed in terms of $\dot y$ rather than $\varPi$, this is the energy density $\mathcal{E}$,

$$\mathcal{E} = \tfrac{1}{2}\mu\, \dot y^2 + \tfrac{1}{2}\tau\, y'^2\ . \tag{14.27}$$

We now evaluate $\dot{\mathcal{E}}$ for a solution to the equations of motion:

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} &= \mu\, \frac{\partial y}{\partial t}\, \frac{\partial^2 y}{\partial t^2} + \tau\, \frac{\partial y}{\partial x}\, \frac{\partial^2 y}{\partial t\, \partial x} \\ &= \tau\, \frac{\partial y}{\partial t}\, \frac{\partial}{\partial x}\bigg(\tau\, \frac{\partial y}{\partial x}\bigg) + \tau\, \frac{\partial y}{\partial x}\, \frac{\partial^2 y}{\partial t\, \partial x} \\ &= \frac{\partial}{\partial x}\bigg[\tau\, \frac{\partial y}{\partial x}\, \frac{\partial y}{\partial t}\bigg] \equiv -\frac{\partial j_{\mathcal{E}}}{\partial x}\ ,\end{aligned} \tag{14.28}$$

where the *energy current density* (or energy flux) is

$$j_{\mathcal{E}} = -\tau\, \frac{\partial y}{\partial x}\, \frac{\partial y}{\partial t}\ . \tag{14.29}$$

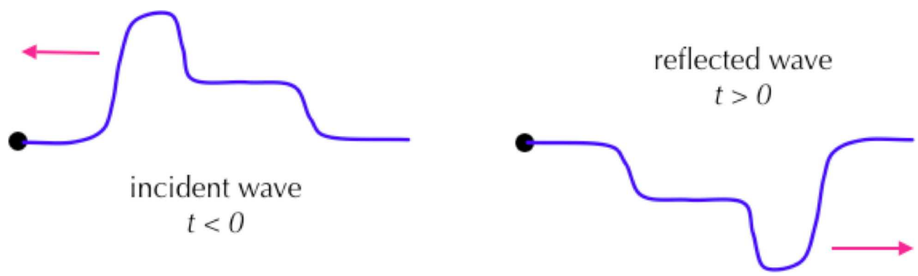


Figure 14.2: Reflection of a pulse at an interface at $x = 0$, with $y(0, t) = 0$.

We therefore have that solutions of the equation of motion also obey the energy *continuity equation*

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j_{\mathcal{E}}}{\partial x} = 0 \ . \tag{14.30}$$

Let us integrate the above equation between points x_1 and x_2. We obtain

$$\frac{\partial}{\partial t} \int_{x_1}^{x_2} dx \, \mathcal{E}(x,t) = -\int_{x_1}^{x_2} dx \, \frac{\partial j_{\mathcal{E}}(x,t)}{\partial x} = j_{\mathcal{E}}(x_1,t) - j_{\mathcal{E}}(x_2,t) \ , \tag{14.31}$$

which says that the time rate of change of the energy contained in the interval $\big[x_1, x_2\big]$ is equal to the difference between the entering and exiting energy flux.

When $\tau(x) = \tau$ and $\mu(x) = \mu$, we have

$$y(x,t) = f(x - ct) + g(x + ct) \tag{14.32}$$

and we find

$$\mathcal{E}(x,t) = \tau\big[f'(x - ct)\big]^2 + \tau\big[g'(x + ct)\big]^2 \tag{14.33}$$

$$j_{\mathcal{E}}(x,t) = c\tau\big[f'(x - ct)\big]^2 - c\tau\big[g'(x + ct)\big]^2 \ , \tag{14.34}$$

which are each sums over right-moving and left-moving contributions.

14.2.2 Reflection at an interface

Consider a semi-infinite string on the interval $\big[0, \infty\big]$, with $y(0, t) = 0$. We can still invoke d'Alembert's solution, $y(x, t) = f(x - ct) + g(x + ct)$, but we must demand

$$y(0,t) = f(-ct) + g(ct) = 0 \quad \Rightarrow \quad f(\xi) = -g(-\xi) \ . \tag{14.35}$$

Thus,

$$y(x,t) = g(ct + x) - g(ct - x) \ . \tag{14.36}$$

Now suppose $g(\xi)$ describes a pulse, and is nonzero only within a neighborhood of $\xi = 0$. For large negative values of t, the right-moving part, $-g(ct - x)$, is negligible everywhere,

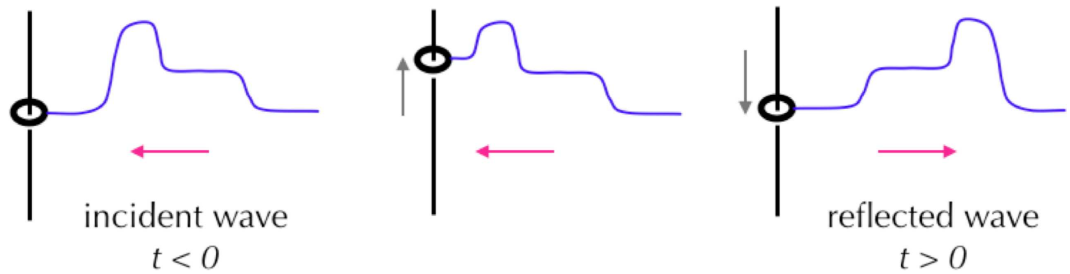


Figure 14.3: Reflection of a pulse at an interface at $x = 0$, with $y'(0, t) = 0$.

since $x > 0$ means that the argument $ct - x$ is always large and negative. On the other hand, the left moving part $g(ct + x)$ is nonzero for $x \approx -ct > 0$. Thus, for $t < 0$ we have a left-moving pulse incident from the right. For $t > 0$, the situation is reversed, and the left-moving component is negligible, and we have a right moving reflected wave. However, the minus sign in eqn. 14.35 means that the reflected wave is *inverted.*

If instead of fixing the endpoint at $x = 0$ we attach this end of the string to a massless ring which frictionlessly slides up and down a vertical post, then we must have $y'(0, t) = 0$, else there is a finite vertical force on the massless ring, resulting in infinite acceleration. We again write $y(x, t) = f(x - ct) + g(x + ct)$, and we invoke

$$y'(0, t) = f'(-ct) + g'(ct) \quad \Rightarrow \quad f'(\xi) = -g'(-\xi) \ , \tag{14.37}$$

which, upon integration, yields $f(\xi) = g(-\xi)$, and therefore

$$y(x, t) = g(ct + x) + g(ct - x) \ . \tag{14.38}$$

The reflected pulse is now 'right-side up', in contrast to the situation with a fixed endpoint.

14.2.3 Mass point on a string

Next, consider the case depicted in Fig. 14.4, where a point mass m is affixed to an infinite string at $x = 0$. Let us suppose that at large negative values of t, a right moving wave $f(ct - x)$ is incident from the left. The full solution may then be written as a sum of incident, reflected, and transmitted waves:

$$x < 0 \quad : \quad y(x, t) = f(ct - x) + g(ct + x) \tag{14.39}$$

$$x > 0 \quad : \quad y(x, t) = h(ct - x) \ . \tag{14.40}$$

At $x = 0$, we invoke Newton's second Law, $F = ma$:

$$m\, \ddot{y}(0, t) = \tau\, y'(0^+, t) - \tau\, y'(0^-, t) \ . \tag{14.41}$$

Any discontinuity in the derivative $y'(x, t)$ at $x = 0$ results in an acceleration of the point mass. Note that

$$y'(0^-, t) = -f'(ct) + g'(ct) \qquad , \qquad y'(0^+, t) = -h'(ct) \ . \tag{14.42}$$

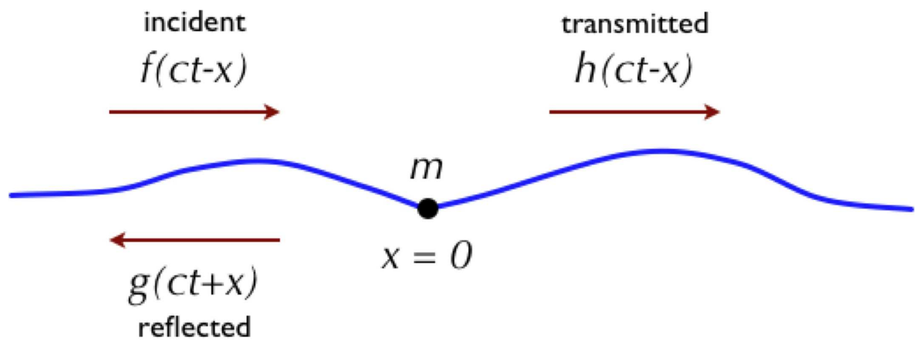


Figure 14.4: Reflection and transmission at an impurity. A point mass m is affixed to an infinite string at $x = 0$.

Further invoking continuity at $x = 0$, *i.e.* $y(0^-, t) = y(0^+, t)$, we have

$$h(\xi) = f(\xi) + g(\xi) \ , \tag{14.43}$$

and eqn. 14.41 becomes

$$g''(\xi) + \frac{2\tau}{mc^2}\, g'(\xi) = -f''(\xi) \ . \tag{14.44}$$

We solve this equation by Fourier analysis:

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad , \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \ . \tag{14.45}$$

Defining $\kappa \equiv 2\tau/mc^2 = 2\mu/m$, we have

$$\big[-k^2 + i\kappa k\big]\, \hat{g}(k) = k^2\, \hat{f}(k) \ . \tag{14.46}$$

We then have

$$\hat{g}(k) = -\frac{k}{k - i\kappa}\, \hat{f}(k) \equiv r(k)\, \hat{f}(k) \tag{14.47}$$

$$\hat{h}(k) = \frac{-i\kappa}{k - i\kappa}\, \hat{f}(k) \equiv t(k)\, \hat{f}(k) \ , \tag{14.48}$$

where $r(k)$ and $t(k)$ are the reflection and transmission amplitudes, respectively. Note that

$$t(k) = 1 + r(k) \ . \tag{14.49}$$

In real space, we have

$$h(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, \hat{f}(k)\, e^{ik\xi} \tag{14.50}$$

$$= \int\limits_{-\infty}^{\infty} d\xi' \left[\, \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')} \right] f(\xi') \tag{14.51}$$

$$\equiv \int\limits_{-\infty}^{\infty} d\xi'\, \mathcal{T}(\xi-\xi')\, f(\xi')\ , \tag{14.52}$$

where

$$\mathcal{T}(\xi-\xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')}\ , \tag{14.53}$$

is the transmission kernel in real space. For our example with $r(k) = -i\kappa/(k-i\kappa)$, the integral is done easily using the method of contour integration:

$$\mathcal{T}(\xi-\xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{-i\kappa}{k-i\kappa}\, e^{ik(\xi-\xi')} = \kappa\, e^{-\kappa(\xi-\xi')}\, \Theta(\xi-\xi')\ . \tag{14.54}$$

Therefore,

$$h(\xi) = \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, f(\xi')\ , \tag{14.55}$$

and of course $g(\xi) = h(\xi) - f(\xi)$. Note that $m = \infty$ means $\kappa = 0$, in which case $r(k) = -1$ and $t(k) = 0$. Thus we recover the inversion of the pulse shape under reflection found earlier.

For example, let the incident pulse shape be $f(\xi) = b\,\Theta\big(a - |\xi|\big)$. Then

$$\begin{aligned} h(\xi) &= \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, b\,\Theta(a-\xi')\,\Theta(a+\xi') \\ &= b\, e^{-\kappa\xi} \Big[e^{\kappa \min(a,\xi)} - e^{-\kappa a} \Big]\, \Theta(\xi+a)\ . \end{aligned} \tag{14.56}$$

Taking cases,

$$h(\xi) = \begin{cases} 0 & \text{if } \xi < -a \\ b\Big(1 - e^{-\kappa(a+\xi)}\Big) & \text{if } -a < \xi < a \\ 2b\, e^{-\kappa\xi}\, \sinh(\kappa a) & \text{if } \xi > a\ . \end{cases} \tag{14.57}$$

In Fig. 14.5 we show the reflection and transmission of this square pulse for two different values of κa.

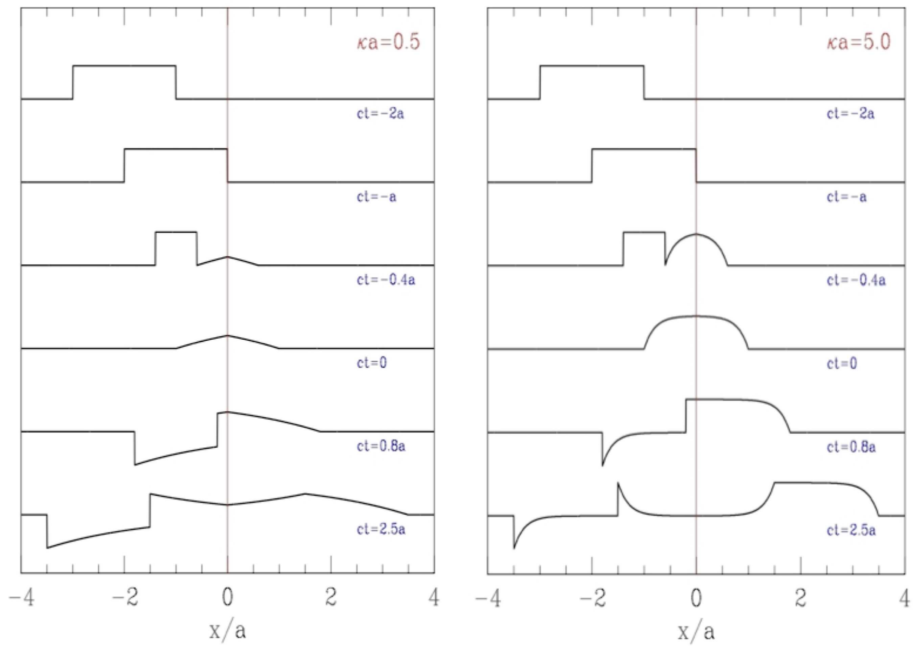


Figure 14.5: Reflection and transmission of a square wave pulse by a point mass at $x = 0$ The configuration of the string is shown for six different times, for $\kappa a = 0.5$ (left panel) and $\kappa a = 5.0$ (right panel). Note that the $\kappa a = 0.5$ case, which corresponds to a large mass $m = 2\mu/\kappa$, results in strong reflection with inversion, and weak transmission. For large κ, corresponding to small mass m, the reflection is weak and the transmission is strong.

14.2.4 Interface between strings of different mass density

Consider the situation in fig. 14.6, where the string for $x < 0$ is of density $\mu_{\rm L}$ and for $x > 0$ is of density $\mu_{\rm R}$. The d'Alembert solution in the two regions, with an incoming wave from the left, is

$$x < 0: \quad y(x,t) = f(c_{\rm L} t - x) + g(c_{\rm L} t + x) \tag{14.58}$$

$$x > 0: \quad y(x,t) = h(c_{\rm R} t - x) \ . \tag{14.59}$$

At $x = 0$ we have

$$f(c_{\rm L} t) + g(c_{\rm L} t) = h(c_{\rm R} t) \tag{14.60}$$

$$-f'(c_{\rm L} t) + g'(c_{\rm L} t) = -h'(c_{\rm R} t) \ , \tag{14.61}$$

where the second equation follows from $\tau\, y'(0^+,t) = \tau\, y'(0^-,t)$, so there is no finite vertical force on the infinitesimal interval bounding $x = 0$, which contains infinitesimal mass. Defining $\alpha \equiv c_{\rm R}/c_{\rm L}$, we integrate the second of these equations and have

$$f(\xi) + g(\xi) = h(\alpha\, \xi) \tag{14.62}$$

$$f(\xi) - g(\xi) = \alpha^{-1}\, h(\alpha\, \xi) \ . \tag{14.63}$$

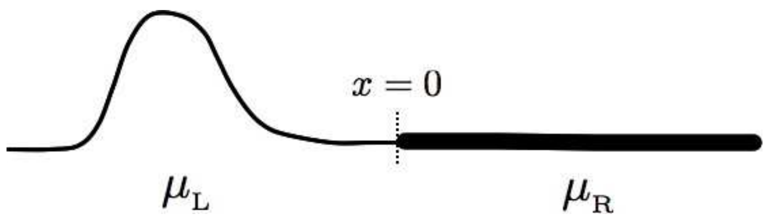


Figure 14.6: String formed from two semi-infinite regions of different densities..

Note that $y(\pm\infty, 0) = 0$ fixes the constant of integration. The solution is then

$$g(\xi) = \frac{\alpha - 1}{\alpha + 1}\, f(\xi) \tag{14.64}$$

$$h(\xi) = \frac{2\alpha}{\alpha + 1}\, f(\xi/\alpha) \ . \tag{14.65}$$

Thus,

$$x < 0 : \quad y(x,t) = f\big(c_{\rm L} t - x\big) + \left(\frac{\alpha - 1}{\alpha + 1}\right) f\big(c_{\rm L} t + x\big) \tag{14.66}$$

$$x > 0 : \quad y(x,t) = \frac{2\alpha}{\alpha + 1}\, f\big((c_{\rm R} t - x)/\alpha\big) \ . \tag{14.67}$$

It is instructive to compute the total energy in the string. For large negative values of the time t, the entire disturbance is confined to the region $x < 0$. The energy is

$$E(-\infty) = \tau \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \ . \tag{14.68}$$

For large positive times, the wave consists of the left-moving reflected $g(\xi)$ component in the region $x < 0$ and the right-moving transmitted component $h(\xi)$ in the region $x > 0$. The energy in the reflected wave is

$$E_{\rm L}(+\infty) = \tau \left(\frac{\alpha - 1}{\alpha + 1}\right)^2 \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \ . \tag{14.69}$$

For the transmitted portion, we use

$$y'(x > 0, t) = \frac{2}{\alpha + 1}\, f'\big((c_{\rm R} t - x)/\alpha\big) \tag{14.70}$$

to obtain

$$E_{\mathrm{R}}(\infty) = \frac{4\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi/\alpha)\big]^2$$

$$= \frac{4\alpha\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \ . \tag{14.71}$$

Thus, $E_{\mathrm{L}}(\infty) + E_{\mathrm{R}}(\infty) = E(-\infty)$, and energy is conserved.

14.3 Finite Strings : Bernoulli's Solution

Suppose $x_a = 0$ and $x_b = L$ are the boundaries of the string, where $y(0,t) = y(L,t) = 0$. Again we write

$$y(x,t) = f(x-ct) + g(x+ct) \ . \tag{14.72}$$

Applying the boundary condition at $x_a = 0$ gives, as earlier,

$$y(x,t) = g(ct+x) - g(ct-x) \ . \tag{14.73}$$

Next, we apply the boundary condition at $x_b = L$, which results in

$$g(ct+L) - g(ct-L) = 0 \quad \Longrightarrow \quad g(\xi) = g(\xi + 2L) \ . \tag{14.74}$$

Thus, $g(\xi)$ is periodic, with period $2L$. Any such function may be written as a Fourier sum,

$$g(\xi) = \sum_{n=1}^{\infty} \left\{ \mathcal{A}_n \, \cos\left(\frac{n\pi\xi}{L}\right) + \mathcal{B}_n \, \sin\left(\frac{n\pi\xi}{L}\right) \right\} . \tag{14.75}$$

The full solution for $y(x,t)$ is then

$$y(x,t) = g(ct+x) - g(ct-x)$$

$$= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ A_n \, \cos\left(\frac{n\pi ct}{L}\right) + B_n \, \sin\left(\frac{n\pi ct}{L}\right) \right\} , \tag{14.76}$$

where $A_n = \sqrt{2\mu L}\, \mathcal{B}_n$ and $B_n = -\sqrt{2\mu L}\, \mathcal{A}_n$. This is known as Bernoulli's solution.

We define the functions

$$\psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) . \tag{14.77}$$

We also write

$$k_n \equiv \frac{n\pi x}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad n = 1, 2, 3, \ldots, \infty \ . \tag{14.78}$$

Thus, $\psi_n(x) = \sqrt{2/\mu L}\,\sin(k_n x)$ has $(n+1)$ nodes at $x = jL/n$, for $j \in \{0, \ldots, n\}$. Note that

$$\langle\, \psi_m \,|\, \psi_n \,\rangle \equiv \int\limits_0^L \! dx\, \mu\, \psi_m(x)\, \psi_n(x) = \delta_{mn} \ . \tag{14.79}$$

Furthermore, this basis is complete:

$$\mu \sum_{n=1}^\infty \psi_n(x)\, \psi_n(x') = \delta(x - x') \ . \tag{14.80}$$

Our general solution is thus equivalent to

$$y(x,0) = \sum_{n=1}^\infty A_n\, \psi_n(x) \tag{14.81}$$

$$\dot y(x,0) = \sum_{n=1}^\infty \frac{n\pi c}{L}\, B_n\, \psi_n(x) \ . \tag{14.82}$$

The Fourier coefficients $\{A_n, B_n\}$ may be extracted from the initial data using the orthonormality of the basis functions and their associated resolution of unity:

$$A_n = \int\limits_0^L \! dx\, \mu\, \psi_n(x)\, y(x,0) \tag{14.83}$$

$$B_n = \frac{L}{n\pi c} \int\limits_0^L \! dx\, \mu\, \psi_n(x)\, \dot y(x,0) \ . \tag{14.84}$$

As an example, suppose our initial configuration is a triangle, with

$$y(x,0) = \begin{cases} \frac{2b}{L}\, x & \text{if } 0 \le x \le \frac{1}{2}L \\ \\ \frac{2b}{L}\,(L - x) & \text{if} \frac{1}{2}L \le x \le L \ , \end{cases} \tag{14.85}$$

and $\dot y(x,0) = 0$. Then $B_n = 0$ for all n, while

$$\begin{aligned} A_n &= \left(\frac{2\mu}{L}\right)^{1/2} \cdot \frac{2b}{L} \left\{ \int\limits_0^{L/2} \! dx\, x\, \sin\left(\frac{n\pi x}{L}\right) + \int\limits_{L/2}^{L} \! dx\, (L - x)\, \sin\left(\frac{n\pi x}{L}\right) \right\} \\ &= (2\mu L)^{1/2} \cdot \frac{4b}{n^2\pi^2}\, \sin\left(\tfrac{1}{2}n\pi\right) \delta_{n,\text{odd}} \ , \end{aligned} \tag{14.86}$$

after changing variables to $x = L\theta/n\pi$ and using $\theta \sin\theta\, d\theta = d\big(\sin\theta - \theta\, \cos\theta\big)$. Another way to write this is to separately give the results for even and odd coefficients:

$$A_{2k} = 0 \qquad , \qquad A_{2k+1} = \frac{4b}{\pi^2}\,(2\mu L)^{1/2} \cdot \frac{(-1)^k}{(2k+1)^2} \ . \tag{14.87}$$

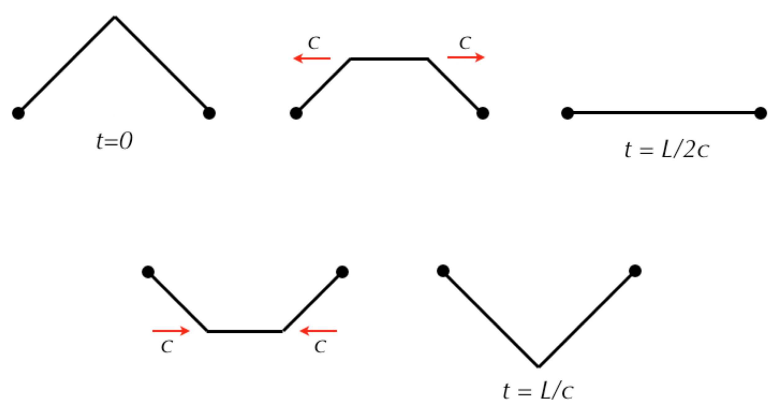


Figure 14.7: Evolution of a string with fixed ends starting from an isosceles triangle shape.

Note that each $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is antisymmetric about the midpoint $x = \frac{1}{2}L$, for all k. Since our initial conditions are that $y(x,0)$ is symmetric about $x = \frac{1}{2}L$, none of the even order eigenfunctions can enter into the expansion, precisely as we have found. The d'Alembert solution to this problem is particularly simple and is shown in Fig. 14.7. Note that $g(x) = \frac{1}{2}y(x,0)$ must be extended to the entire real line. We know that $g(x) = g(x+2L)$ is periodic with spatial period $2L$, but how to we extend $g(x)$ from the interval $\big[0,L\big]$ to the interval $\big[-L,0\big]$? To do this, we use $y(x,0) = g(x) - g(-x)$, which says that $g(x)$ must be *antisymmetric*, *i.e.* $g(x) = -g(-x)$. Equivalently, $\dot{y}(x,0) = cg'(x) - cg'(-x) = 0$, which integrates to $g(x) = -g(-x)$.

14.4 Sturm-Liouville Theory

Consider the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\,\mu(x)\,\dot{y}^2 - \tfrac{1}{2}\,\tau(x)\,y'^2 - \tfrac{1}{2}\,v(x)\,y^2 \ . \tag{14.88}$$

The last term is new and has the physical interpretation of a harmonic potential which attracts the string to the line $y=0$. The Euler-Lagrange equations are then

$$-\frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = -\mu(x)\,\frac{\partial^2 y}{\partial t^2} \ . \tag{14.89}$$

This equation is invariant under time translation. Thus, if $y(x,t)$ is a solution, then so is $y(x,t+t_0)$, for any t_0. This means that the solutions can be chosen to be eigenstates of the operator ∂_t, which is to say $y(x,t) = \psi(x)\,e^{-i\omega t}$. Because the coefficients are real, both y and y^* are solutions, and taking linear combinations we have

$$y(x,t) = \psi(x)\,\cos(\omega t + \phi) \ . \tag{14.90}$$

Plugging this into eqn. 14.89, we obtain

$$-\frac{d}{dx}\Big[\tau(x)\,\psi'(x)\Big] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \ . \tag{14.91}$$

This is the Sturm-Liouville equation. There are four types of boundary conditions that we shall consider:

1. Fixed endpoint: $\psi(x) = 0$, where $x = x_{a,b}$.
2. Natural: $\tau(x)\,\psi'(x) = 0$, where $x = x_{a,b}$.
3. Periodic: $\psi(x) = \psi(x + L)$, where $L = x_b - x_a$.
4. Mixed homogeneous: $\alpha\,\psi(x) + \beta\,\psi'(x) = 0$, where $x = x_{a,b}$.

The Sturm-Liouville equation is an eigenvalue equation. The eigenfunctions $\{\psi_n(x)\}$ satisfy

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_n'(x)\Big] + v(x)\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)\ . \tag{14.92}$$

Now suppose we a second solution $\psi_m(x)$, satisfying

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_m'(x)\Big] + v(x)\,\psi_m(x) = \omega_m^2\,\mu(x)\,\psi_m(x)\ . \tag{14.93}$$

Now multiply (14.92)* by $\psi_m(x)$ and (14.93) by $\psi_n^*(x)$ and subtract, yielding

$$\psi_n^*\,\frac{d}{dx}\Big[\tau\,\psi_m'\Big] - \psi_m\,\frac{d}{dx}\Big[\tau\,{\psi'}_n^*\Big] = \big(\omega_n^{*\,2} - \omega_m^2\big)\,\mu\,\psi_m\,\psi_n^* \tag{14.94}$$

$$= \frac{d}{dx}\Big[\tau\,\psi_n^*\,\psi_m' - \tau\,\psi_m\,{\psi'}_n^*\Big]\ . \tag{14.95}$$

We integrate this equation over the length of the string, to get

$$\big(\omega_n^{*\,2} - \omega_m^2\big)\int\limits_{x_a}^{x_b}\!dx\;\mu(x)\,\psi_n^*(x)\,\psi_m(x) = \Big[\tau(x)\,\psi_n^*(x)\,\psi_m'(x) - \tau(x)\,\psi_m(x)\,{\psi'}_n^*(x)\Big]_{x=x_a}^{x=x_b}$$

$$= 0\ . \tag{14.96}$$

The RHS vanishes for any of the four types of boundary conditions articulated above.

Thus, we have

$$\big(\omega_n^{*\,2} - \omega_m^2\big)\,\big\langle\, \psi_n \,\big|\, \psi_m \,\big\rangle = 0\ , \tag{14.97}$$

where the inner product is defined as

$$\big\langle\, \psi \,\big|\, \phi \,\big\rangle \equiv \int\limits_{x_a}^{x_b}\!dx\;\mu(x)\,\psi^*(x)\,\phi(x)\ . \tag{14.98}$$

Note that the distribution $\mu(x)$ is non-negative definite. Setting $m = n$, we have $\big\langle\, \psi_n \,\big|\, \psi_n \,\big\rangle \geq 0$, and hence $\omega_n^{*\,2} = \omega_n^2$, which says that $\omega_n^2 \in \mathbb{R}$. When $\omega_m^2 \neq \omega_n^2$, the eigenfunctions are orthogonal with respect to the above inner product. In the case of degeneracies, we may invoke the Gram-Schmidt procedure, which orthogonalizes the eigenfunctions within a given degenerate subspace. Since the Sturm-Liouville equation is linear, we may normalize the eigenfunctions, taking

$$\langle\, \psi_m \,|\, \psi_n \,\rangle = \delta_{mn}. \tag{14.99}$$

Finally, since the coefficients in the Sturm-Liouville equation are all real, we can and henceforth do choose the eigenfunctions themselves to be real.

Another important result, which we will not prove here, is the *completeness* of the eigenfunction basis. Completeness means

$$\mu(x) \sum_n \psi_n^*(x)\, \psi_n(x') = \delta(x - x') \ . \tag{14.100}$$

Thus, any function can be expanded in the eigenbasis, *viz.*

$$\phi(x) = \sum_n C_n\, \psi_n(x) \qquad , \qquad C_n = \langle\, \psi_n \,|\, \phi \,\rangle \ . \tag{14.101}$$

14.4.1 Variational method

Consider the functional

$$\omega^2\big[\psi(x)\big] = \frac{\frac{1}{2}\int\limits_{x_a}^{x_b}\!dx\, \Big\{ \tau(x)\, \psi'^2(x) + v(x)\, \psi^2(x) \Big\}}{\frac{1}{2}\int\limits_{x_a}^{x_b}\!dx\, \mu(x)\, \psi^2(x)} \equiv \frac{\mathcal{N}}{\mathcal{D}} \ . \tag{14.102}$$

The variation is

$$\begin{aligned} \delta\omega^2 &= \frac{\delta\mathcal{N}}{\mathcal{D}} - \frac{\mathcal{N}\,\delta\mathcal{D}}{\mathcal{D}^2} \\ &= \frac{\delta\mathcal{N} - \omega^2\,\delta\mathcal{D}}{\mathcal{D}} \ . \end{aligned} \tag{14.103}$$

Thus,

$$\delta\omega^2 = 0 \qquad \Longrightarrow \qquad \delta\mathcal{N} = \omega^2\,\delta\mathcal{D} \ , \tag{14.104}$$

which says

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \ , \tag{14.105}$$

which is the Sturm-Lioiuville equation. In obtaining this equation, we have dropped a boundary term, which is correct provided

$$\Big[\tau(x)\,\psi'(x)\,\psi(x)\Big]_{x=x_a}^{x=x_b} = 0 \ . \tag{14.106}$$

This condition is satisfied for any of the first three classes of boundary conditions: $\psi = 0$ (fixed endpoint), $\tau\,\psi' = 0$ (natural), or $\psi(x_a) = \psi(x_b)$, $\psi'(x_a) = \psi'(x_b)$ (periodic). For the fourth class of boundary conditions, $\alpha\psi + \beta\psi' = 0$ (mixed homogeneous), the Sturm-Liouville equation may still be derived, provided one uses a slightly different functional,

$$\omega^2\big[\psi(x)\big] = \frac{\widetilde{\mathcal{N}}}{\mathcal{D}} \qquad \text{with} \qquad \widetilde{\mathcal{N}} = \mathcal{N} + \frac{\alpha}{2\beta}\Big[\tau(x_b)\,\psi^2(x_b) - \tau(x_a)\,\psi^2(x_a)\Big] \ , \tag{14.107}$$

since then

$$\begin{aligned}\delta\widetilde{\mathcal{N}} - \widetilde{\mathcal{N}}\,\delta D = \int\limits_{x_a}^{x_b}\!\!dx\,\bigg\{ -\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) - \omega^2\mu(x)\,\psi(x)\bigg\}\,\delta\psi(x) \\ + \bigg[\tau(x)\bigg(\psi'(x) + \frac{\alpha}{\beta}\,\psi(x)\bigg)\,\delta\psi(x)\bigg]_{x=x_a}^{x=x_b} \ ,\end{aligned} \tag{14.108}$$

and the last term vanishes as a result of the boundary conditions.

For all four classes of boundary conditions we may write

$$\omega^2\big[\psi(x)\big] = \frac{\int\limits_{x_a}^{x_b}\!dx\,\psi(x)\,\overbrace{\Big[-\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x)\Big]}^{K}\,\psi(x)}{\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi^2(x)} \tag{14.109}$$

If we expand $\psi(x)$ in the basis of eigenfunctions of the Sturm-Liouville operator K,

$$\psi(x) = \sum_{n=1}^{\infty} \mathcal{C}_n\,\psi_n(x) \ , \tag{14.110}$$

we obtain

$$\omega^2\big[\psi(x)\big] = \omega^2(\mathcal{C}_1,\ldots,\mathcal{C}_\infty) = \frac{\sum_{j=1}^{\infty}|\mathcal{C}_j|^2\,\omega_j^2}{\sum_{k=1}^{\infty}|\mathcal{C}_k|^2} \ . \tag{14.111}$$

If $\omega_1^2 \le \omega_2^2 \le \ldots$, then we see that $\omega^2 \ge \omega_1^2$, so an arbitrary function $\psi(x)$ will always yield an upper bound to the lowest eigenvalue.

As an example, consider a violin string ($v = 0$) with a mass m affixed in the center. We write $\mu(x) = \mu + m\,\delta(x - \frac{1}{2}L)$, hence

$$\omega^2\big[\psi(x)\big] = \frac{\tau\int\limits_0^L\!dx\,\psi'^2(x)}{m\,\psi^2(\frac{1}{2}L) + \mu\int\limits_0^L\!dx\,\psi^2(x)} \tag{14.112}$$

Now consider a trial function

$$\psi(x) = \begin{cases} A\,x^\alpha & \text{if} \quad 0 \le x \le \frac{1}{2}L \\[2ex] A\,(L-x)^\alpha & \text{if} \quad \frac{1}{2}L \le x \le L \ . \end{cases} \tag{14.113}$$

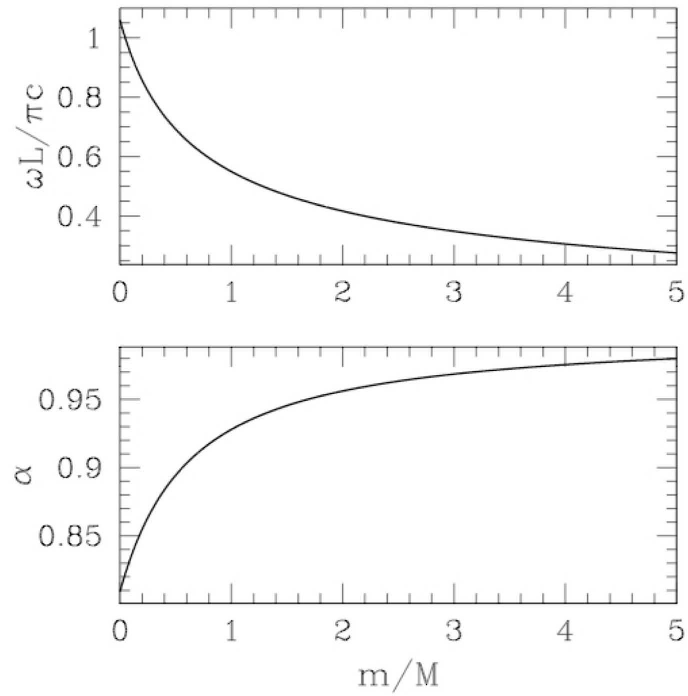


Figure 14.8: One-parameter variational solution for a string with a mass m affixed at $x = \frac{1}{2}L$.

The functional $\omega^2\big[\psi(x)\big]$ now becomes an ordinary function of the trial parameter α, with

$$\omega^2(\alpha) = \frac{2\tau\int_0^{L/2} dx\,\alpha^2\,x^{2\alpha-2}}{m\left(\frac{1}{2}L\right)^{2\alpha} + 2\mu\int\limits_0^{L/2} dx\,x^{2\alpha}} = \left(\frac{2c}{L}\right)^2 \cdot \frac{\alpha^2(2\alpha+1)}{(2\alpha-1)\big[1+(2\alpha+1)\frac{m}{M}\big]}\ , \qquad (14.114)$$

where $M = \mu L$ is the mass of the string alone. We minimize $\omega^2(\alpha)$ to obtain the optimal solution of this form:

$$\frac{d}{d\alpha}\,\omega^2(\alpha) = 0 \qquad \Longrightarrow \qquad 4\alpha^2 - 2\alpha - 1 + (2\alpha+1)^2\,(\alpha-1)\,\frac{m}{M} = 0\ . \qquad (14.115)$$

For $m/M \to 0$, we obtain $\alpha = \frac{1}{4}\big(1+\sqrt{5}\big) \approx 0.809$. The variational estimate for the eigenvalue is then 6.00% larger than the exact answer $\omega_1^0 = \pi c/L$. In the opposite limit, $m/M \to \infty$, the inertia of the string may be neglected. The normal mode is then piecewise linear, in the shape of an isosceles triangle with base L and height y. The equation of motion is then $m\ddot{y} = -2\tau\cdot(y/\frac{1}{2}L)$, assuming $|y/L| \ll 1$. Thus, $\omega_1 = (2c/L)\sqrt{M/m}$. This is reproduced exactly by the variational solution, for which $\alpha \to 1$ as $m/M \to \infty$.

14.5 Continua in Higher Dimensions

In higher dimensions, we generalize the operator K as follows:

$$K = -\frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x}) \ . \tag{14.116}$$

The eigenvalue equation is again

$$K\psi(\boldsymbol{x}) = \omega^2\,\mu(\boldsymbol{x})\,\psi(\boldsymbol{x}) \ , \tag{14.117}$$

and the Green's function again satisfies

$$\Big[K - \omega^2\,\mu(\boldsymbol{x})\Big]\,G_\omega(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \ , \tag{14.118}$$

and has the eigenfunction expansion,

$$G_\omega(\boldsymbol{x},\boldsymbol{x}') = \sum_{n=1}^\infty \frac{\psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}')}{\omega_n^2 - \omega^2} \ . \tag{14.119}$$

The eigenfunctions form a complete and orthonormal basis:

$$\mu(\boldsymbol{x})\sum_{n=1}^\infty \psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \tag{14.120}$$

$$\int\limits_\Omega d\boldsymbol{x}\,\mu(\boldsymbol{x})\,\psi_m(\boldsymbol{x})\,\psi_n(\boldsymbol{x}) = \delta_{mn} \ , \tag{14.121}$$

where Ω is the region of space in which the continuous medium exists. For purposes of simplicity, we consider here fixed boundary conditions $u(\boldsymbol{x},t)\big|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of Ω. The general solution to the wave equation

$$\left[\mu(\boldsymbol{x})\,\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x})\right] u(\boldsymbol{x},t) = 0 \tag{14.122}$$

is

$$u(\boldsymbol{x},t) = \sum_{n=1}^\infty \mathcal{C}_n\,\psi_n(\boldsymbol{x})\,\cos(\omega_n\,t + \delta_n) \ . \tag{14.123}$$

The variational approach generalizes as well. We define

$$\mathcal{N}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\left[\tau_{\alpha\beta}\,\frac{\partial\psi}{\partial x^\alpha}\,\frac{\partial\psi}{\partial x^\beta} + v\,\psi^2\right] \tag{14.124}$$

$$\mathcal{D}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\,\mu\,\psi^2 \ , \tag{14.125}$$

and

$$\omega^2\big[\psi(\boldsymbol{x})\big] = \frac{\mathcal{N}\big[\psi(\boldsymbol{x})\big]}{\mathcal{D}\big[\psi(\boldsymbol{x})\big]} \ . \tag{14.126}$$

Setting the variation $\delta\omega^2 = 0$ recovers the eigenvalue equation $K\psi = \omega^2\mu\,\psi$.

14.5.1 Membranes

Consider a surface where the height z is a function of the lateral coordinates x and y:

$$z = u(x, y) \ . \tag{14.127}$$

The equation of the surface is then

$$F(x, y, z) = z - u(x, y) = 0 \ . \tag{14.128}$$

Let the differential element of surface area be dS. The projection of this element onto the (x, y) plane is

$$\begin{aligned} dA &= dx\,dy \\ &= \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}\,dS \ . \end{aligned} \tag{14.129}$$

The unit normal $\hat{\boldsymbol{n}}$ is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} F}{\big|\boldsymbol{\nabla} F\big|} = \frac{\hat{\boldsymbol{z}} - \boldsymbol{\nabla} u}{\sqrt{1 + (\boldsymbol{\nabla} u)^2}} \ . \tag{14.130}$$

Thus,

$$dS = \frac{dx\,dy}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}} = \sqrt{1 + (\boldsymbol{\nabla} u)^2}\,dx\,dy \ . \tag{14.131}$$

The potential energy for a deformed surface can take many forms. In the case we shall consider here, we consider only the effect of surface tension σ, and we write the potential energy functional as

$$\begin{aligned} U\big[u(x, y, t)\big] &= \sigma \!\int\! dS \\ &= U_0 + \tfrac{1}{2}\!\int\! dA\,(\boldsymbol{\nabla} u)^2 + \ldots \ . \end{aligned} \tag{14.132}$$

The kinetic energy functional is

$$T\big[u(x, y, t)\big] = \tfrac{1}{2}\!\int\! dA\,\mu(\boldsymbol{x})\,(\partial_t u)^2 \ . \tag{14.133}$$

Thus, the action is

$$S\big[u(\boldsymbol{x}, t)\big] = \int\! d^2\!x\,\mathcal{L}(u, \boldsymbol{\nabla} u, \partial_t u, \boldsymbol{x}) \ , \tag{14.134}$$

where the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu(\boldsymbol{x})\,(\partial_t u)^2 - \tfrac{1}{2}\sigma(\boldsymbol{x})\,(\boldsymbol{\nabla} u)^2 \ , \tag{14.135}$$

where here we have allowed both $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ to depend on the spatial coordinates. The equations of motion are

$$0 = \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\,\partial_t u} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial\,\boldsymbol{\nabla} u} - \frac{\partial \mathcal{L}}{\partial u} \tag{14.136}$$

$$= \mu(\boldsymbol{x})\,\frac{\partial^2 u}{\partial t^2} - \boldsymbol{\nabla} \cdot \Big\{\sigma(\boldsymbol{x})\,\boldsymbol{\nabla} u\Big\} \ . \tag{14.137}$$

14.5.2 Helmholtz equation

When μ and σ are each constant, we obtain the Helmholtz equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{x},t) = 0 \ , \tag{14.138}$$

with $c = \sqrt{\sigma/\mu}$. The d'Alembert solution still works – waves of arbitrary shape can propagate *in a fixed direction* $\hat{\boldsymbol{k}}$:

$$u(\boldsymbol{x},t) = f(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct) \ . \tag{14.139}$$

This is called a *plane wave* because the three dimensional generalization of this wave has wavefronts which are planes. In our case, it might better be called a *line wave*, but people will look at you funny if you say that, so we'll stick with *plane wave*. Note that the locus of points of constant f satisfies

$$\phi(\boldsymbol{x},t) = \hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct = \text{constant} \ , \tag{14.140}$$

and setting $d\phi = 0$ gives

$$\hat{\boldsymbol{k}}\cdot\frac{d\boldsymbol{x}}{dt} = c \ , \tag{14.141}$$

which means that the velocity along $\hat{\boldsymbol{k}}$ is c. The component of $\boldsymbol{x}$ perpendicular to $\hat{\boldsymbol{k}}$ is arbitrary, hence the regions of constant ϕ correspond to lines which are orthogonal to $\hat{\boldsymbol{k}}$.

Owing to the linearity of the wave equation, we can construct arbitrary superpositions of plane waves. The most general solution is written

$$u(\boldsymbol{x},t) = \int\!\frac{d^2\!k}{(2\pi)^2}\,\Big[A(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-ckt)} + B(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+ckt)}\Big] \ . \tag{14.142}$$

The first term in the bracket on the RHS corresponds to a plane wave moving in the $+\hat{\boldsymbol{k}}$ direction, and the second term to a plane wave moving in the $-\hat{\boldsymbol{k}}$ direction.

14.5.3 Rectangles

Consider a rectangular membrane where $x \in [0,a]$ and $y \in [0,b]$, and subject to the boundary conditions $u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0$. We try a solution of the form

$$u(x,y,t) = X(x)\,Y(y)\,T(t) \ . \tag{14.143}$$

This technique is known as *separation of variables*. Dividing the Helmholtz equation by u then gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \ . \tag{14.144}$$

The first term on the LHS depends only on x. The second term on the LHS depends only on y. The RHS depends only on t. Therefore, each of these terms must individually be constant. We write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \ , \tag{14.145}$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2} \ . \tag{14.146}$$

Thus, $\omega = \pm c|\boldsymbol{k}|$. The most general solution is then

$$X(x) = A\,\cos(k_x x) + B\sin(k_x x) \tag{14.147}$$

$$Y(y) = C\,\cos(k_y y) + D\sin(k_y y) \tag{14.148}$$

$$T(t) = E\,\cos(\omega t) + B\sin(\omega t) \ . \tag{14.149}$$

The boundary conditions now demand

$$A = 0 \quad , \quad C = 0 \quad , \quad \sin(k_x a) = 0 \quad , \quad \sin(k_y b) = 0 \ . \tag{14.150}$$

Thus, the most general solution subject to the boundary conditions is

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathcal{A}_{mn} \sin\!\left(\frac{m\pi x}{a}\right) \sin\!\left(\frac{n\pi y}{b}\right) \cos\!\big(\omega_{mn} t + \delta_{mn}\big) \ , \tag{14.151}$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2} \ . \tag{14.152}$$

14.5.4 Circles

For a circular membrane, such as a drumhead, it is convenient to work in two-dimensional polar coordinates (r,φ). The Laplacian is then

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \ . \tag{14.153}$$

We seek a solution to the Helmholtz equation which satisfies the boundary conditions $u(r = a, \varphi, t) = 0$. Once again, we invoke the separation of variables method, writing

$$u(r,\varphi,t) = R(r)\,\Phi(\varphi)\,T(t) \ , \tag{14.154}$$

resulting in

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \ . \tag{14.155}$$

The azimuthal and temporal functions are

$$\Phi(\varphi) = e^{im\varphi} \quad , \quad T(t) = \cos(\omega t + \delta) \ , \tag{14.156}$$

where m is an integer in order that the function $u(r, \varphi, t)$ be single-valued. The radial equation is then

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right) R = 0 \ . \tag{14.157}$$

This is Bessel's equation, with solution

$$R(r) = A\, J_m\!\left(\frac{\omega r}{c}\right) + B\, N_m\!\left(\frac{\omega r}{c}\right) , \tag{14.158}$$

where $J_m(z)$ and $N_m(z)$ are the Bessel and Neumann functions of order m, respectively. Since the Neumann functions diverge at $r = 0$, we must exclude them, setting $B = 0$ for each m.

We now invoke the boundary condition $u(r = a, \varphi, t) = 0$. This requires

$$J_m\!\left(\frac{\omega a}{c}\right) = 0 \qquad \Longrightarrow \qquad \omega = \omega_{m\ell} = x_{m\ell}\,\frac{c}{a} , \tag{14.159}$$

where $J_m(x_{m\ell}) = 0$, *i.e.* $x_{m\ell}$ is the ℓ^{th} zero of $J_m(x)$. The mose general solution is therefore

$$u(r, \varphi, t) = \sum_{m=0}^{\infty}\sum_{\ell=1}^{\infty} \mathcal{A}_{m\ell}\, J_m\big(x_{m\ell}\, r/a\big)\, \cos\big(m\varphi + \beta_{m\ell}\big)\, \cos(\omega_{m\ell}\, t + \delta_{m\ell}\big) \ . \tag{14.160}$$

14.5.5 Sound in fluids

Let $\varrho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ be the density and velocity fields in a fluid. Mass conservation requires

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho\, \boldsymbol{v}) = 0 \ . \tag{14.161}$$

This is the continuity equation for mass.

Focus now on a small packet of fluid of infinitesimal volume dV. The total force on this fluid element is $d\boldsymbol{F} = \big(-\boldsymbol{\nabla} p + \varrho\, \boldsymbol{g}\big)\, dV$. By Newton's Second Law,

$$d\boldsymbol{F} = \big(\varrho\, dV\big)\, \frac{d\boldsymbol{v}}{dt} \tag{14.162}$$

Note that the chain rule gives

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v} \ . \tag{14.163}$$

Thus, dividing eqn, 14.162 by dV, we obtain

$$\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}\right) = -\boldsymbol{\nabla} p + \varrho\, \boldsymbol{g} \ . \tag{14.164}$$

This is the inviscid (*i.e.* zero viscosity) form of the Navier-Stokes equation.

Locally the fluid can also be described in terms of thermodynamic variables $p(\boldsymbol{x},t)$ (pressure) and $T(\boldsymbol{x},t)$ (temperature). For a one-component fluid there is necessarily an equation of state of the form $p = p(\varrho, T)$. Thus, we may write

$$dp = \frac{\partial p}{\partial \varrho}\bigg|_T d\varrho + \frac{\partial p}{\partial T}\bigg|_\varrho dT \ . \tag{14.165}$$

We now make the following approximations. First, we assume that the fluid is close to equilibrium at $\boldsymbol{v} = 0$, meaning we write $p = \bar{p} + \delta p$ and $\varrho = \bar{\varrho} + \delta\varrho$, and assume that δp, $\delta\varrho$, and $\boldsymbol{v}$ are small. The smallness of $\boldsymbol{v}$ means we can neglect the nonlinear term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in eqn. 14.164. Second, we neglect gravity (more on this later). The continuity equation then takes the form

$$\frac{\partial\, \delta\varrho}{\partial t} + \bar{\varrho}\, \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0 \ , \tag{14.166}$$

and the Navier-Stokes equation becomes

$$\bar{\varrho}\,\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\delta p \ . \tag{14.167}$$

Taking the time derivative of the former, and then invoking the latter of these equations yields

$$\frac{\partial^2\, \delta\varrho}{\partial t^2} = \nabla^2 p = \bigg(\frac{\partial p}{\partial \varrho}\bigg)\nabla^2\, \delta\varrho \equiv c^2\, \nabla^2 \delta\varrho \ . \tag{14.168}$$

The speed of wave propagation, *i.e.* the speed of sound, is given by

$$c = \sqrt{\frac{\partial p}{\partial \varrho}} \ . \tag{14.169}$$

Finally, we must make an assumption regarding the conditions under which the derivative $\partial p/\partial\varrho$ is computed. If the fluid is an excellent conductor of heat, then the temperature will equilibrate quickly and it is a good approximation to take the derivative at fixed temperature. The resulting value of c is called the *isothermal* sound speed c_T. If, on the other hand, the fluid is a poor conductor of heat, as is the case for air, then it is more appropriate to take the derivative at constant entropy, yielding the *adiabatic* sound speed. Thus,

$$c_T = \sqrt{\bigg(\frac{\partial p}{\partial \varrho}\bigg)_T} \quad , \quad c_S = \sqrt{\bigg(\frac{\partial p}{\partial \varrho}\bigg)_S} \ . \tag{14.170}$$

In an ideal gas, $c_S/c_T = \sqrt{\gamma}$, where $\gamma = c_p/c_V$ is the ratio of the specific heat at constant pressure to that at constant volume. For a (mostly) diatomic gas like air (comprised of N_2 and O_2 and just a little Ar), $\gamma = \frac{7}{5}$. Note that one can write $c^2 = 1/\varrho\kappa$, where

$$\kappa = \frac{1}{\varrho}\bigg(\frac{\partial \varrho}{\partial p}\bigg) \tag{14.171}$$

is the *compressibility*, which is the inverse of the *bulk modulus*. Again, one must specify whether one is talking about κ_T or κ_S. For reference in air at $T = 293\,\mathrm{K}$, using $M =$

28.8 g/mol, one obtains $c_T = 290.8\,\text{m/s}$ and $c_S = 344.0\,\text{m/s}$. In H_2O at 293 K, $c = 1482\,\text{m/s}$. In Al at 273 K, $c = 6420\,\text{m/s}$.

If we retain gravity, the wave equation becomes

$$\frac{\partial^2 \delta\varrho}{\partial t^2} = c^2\,\nabla^2 \delta\varrho - \boldsymbol{g} \cdot \boldsymbol{\nabla} \delta\varrho \ . \tag{14.172}$$

The dispersion relation is then

$$\omega(\boldsymbol{k}) = \sqrt{c^2 k^2 + i\boldsymbol{g} \cdot \boldsymbol{k}} \ . \tag{14.173}$$

We are permitted to ignore the effects of gravity so long as $c^2 k^2 \gg gk$. In terms of the wavelength $\lambda = 2\pi/k$, this requires

$$\lambda \ll \frac{2\pi c^2}{g} = 75.9\,\text{km (at } T = 293\,\text{K}) \ . \tag{14.174}$$

14.6 Dispersion

The one-dimensional Helmholtz equation $\partial_x^2\, y = c^{-2}\,\partial_t^2\, y$ is solved by a plane wave

$$y(x,t) = A\,e^{ikx}\,e^{-i\omega t} \ , \tag{14.175}$$

provided $\omega = \pm ck$. We say that there are *two branches* to the *dispersion relation* $\omega(k)$ for this equation. In general, we may add solutions, due to the linearity of the Helmholtz equation. The most general solution is then

$$\begin{aligned} y(x,t) &= \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \Big[\hat f(k)\, e^{ik(x-ct)} + \hat g(k)\, e^{ik(x+ct)} \Big] \\ &= f(x-ct) + g(x+ct) \ , \end{aligned} \tag{14.176}$$

which is consistent with d'Alembert's solution.

Consider now the free particle Schrödinger equation in one space dimension,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\, \frac{\partial^2 \psi}{\partial x^2} \ . \tag{14.177}$$

The function $\psi(x,t)$ is the quantum mechanical wavefunction for a particle of mass m moving freely along a one-dimensional line. The *probability density* for finding the particle at position x at time t is

$$\rho(x,t) = \big|\psi(x,t)\big|^2 \ . \tag{14.178}$$

Conservation of probability therefore requires

$$\int\limits_{-\infty}^{\infty} dx\, \big|\psi(x,t)\big|^2 = 1 \ . \tag{14.179}$$

This condition must hold at all times t.

As is the case with the Helmholtz equation, the Schrödinger equation is solved by a plane wave of the form

$$\psi(x,t) = A\, e^{ikx}\, e^{-i\omega t} \ , \tag{14.180}$$

where the dispersion relation now only has one branch, and is given by

$$\omega(k) = \frac{\hbar k^2}{2m} \ . \tag{14.181}$$

The most general solution is then

$$\psi(x,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{\psi}(k)\, e^{ikx}\, e^{-i\hbar k^2 t/2m} \ . \tag{14.182}$$

Let's suppose we start at time $t=0$ with a Gaussian wavepacket,

$$\psi(x,0) = \big(\pi \ell_0^2\big)^{-1/4}\, e^{-x^2/2\ell_0^2}\, e^{ik_0 x} \ . \tag{14.183}$$

To find the amplitude $\hat{\psi}(k)$, we perform the Fourier transform:

$$\begin{aligned} \hat{\psi}(k) &= \int\limits_{-\infty}^{\infty} dx\, \psi(x,0)\, e^{-ikx} \\ &= \sqrt{2}\, \big(\pi \ell_0^2\big)^{-1/4}\, e^{-(k-k_0)^2 \ell_0^2/2} \ . \end{aligned} \tag{14.184}$$

We now compute $\psi(x,t)$ valid for all times t:

$$\psi(x,t) = \sqrt{2}\, \big(\pi \ell_0^2\big)^{-1/4} \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx}\, e^{-(k-k_0)^2 \ell_0^2/2}\, e^{ikx}\, e^{-i\hbar k^2 t/2m} \tag{14.185}$$

$$\begin{aligned} &= \big(\pi \ell_0^2\big)^{-1/4} \big(1 + it/\tau\big)^{-1/2} \exp\left[-\frac{\big(x - \hbar k_0 t/m\big)^2}{2\,\ell_0^2\,\big(1 + t^2/\tau^2\big)} \right] \\ &\quad \times \exp\left[\frac{i\big(2k_0\, \ell_0^2\, x + x^2\, t/\tau - k_0^2\, \ell_0^4\, t/\tau\big)}{2\,\ell_0^2\,\big(1 + t^2/\tau^2\big)} \right] , \end{aligned} \tag{14.186}$$

where $\tau \equiv m\ell_0^2/\hbar$. The probability density is then the normalized Gaussian

$$\rho(x,t) = \frac{1}{\sqrt{\pi\, \ell^2(t)}}\, e^{-(x - v_0 t)^2/\ell^2(t)} \ , \tag{14.187}$$

where $v_0 = \hbar k_0/m$ and

$$\ell(t) = \ell_0\, \sqrt{1 + t^2/\tau^2} \ . \tag{14.188}$$

Note that $\ell(t)$ gives the width of the wavepacket, and that this width increases as a function of time, with $\ell(t \gg \tau) \simeq \ell_0\, t/\tau$.

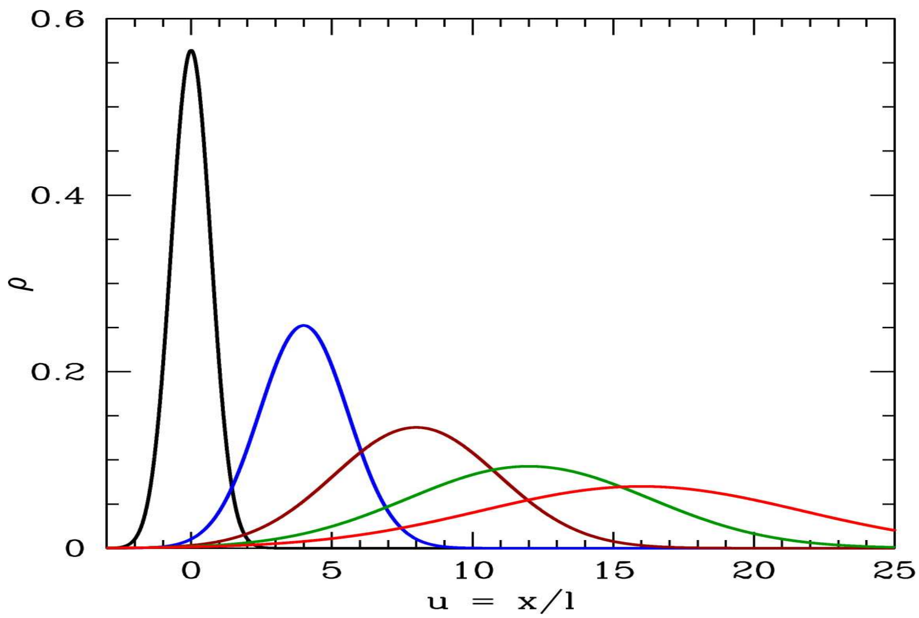


Figure 14.9: Wavepacket spreading for $k_0\,\ell_0 = 2$ with $t/\tau = 0$, 2, 4, 6, and 8.

Unlike the case of the Helmholtz equation, the solution to the Schrödinger equation does not retain its shape as it moves. This phenomenon is known as the *spreading of the wavepacket*. In fig. 14.9, we show the motion and spreading of the wavepacket.

For a given plane wave $e^{ikx}\,e^{-i\omega(k)t}$, the wavefronts move at the *phase velocity*

$$v_{\rm p}(k) = \frac{\omega(k)}{k} \ . \tag{14.189}$$

The center of the wavepacket, however, travels at the *group velocity*

$$v_{\rm g}(k) = \frac{d\omega}{dk}\bigg|_{k_0} \ , \tag{14.190}$$

where $k = k_0$ is the maximum of $\big|\hat{\psi}(k)\big|^2$.

14.7 Appendix I : Three Strings

Problem: Three identical strings are connected to a ring of mass m as shown in fig. 14.10. The linear mass density of each string is σ and each string is under identical tension τ. In equilibrium, all strings are coplanar. All motion on the string is in the $\hat{\boldsymbol{z}}$-direction, which is perpendicular to the equilibrium plane. The ring slides frictionlessly along a vertical pole.

It is convenient to describe each string as a half line $[-\infty, 0]$. We can choose coordinates x_1, x_2, and x_3 for the three strings, respectively. For each string, the ring lies at $x_i = 0$.

A pulse is sent down the first string. After a time, the pulse arrives at the ring. Transmitted waves are sent down the other two strings, and a reflected wave down the first string. The solution to the wave equation in the strings can be written as follows. In string #1, we have

$$z = f(ct - x_1) + g(ct + x_1) \ . \tag{14.191}$$

In the other two strings, we may write $z = h_{\rm A}(ct + x_2)$ and $z = h_{\rm B}(ct + x_3)$, as indicated in the figure.

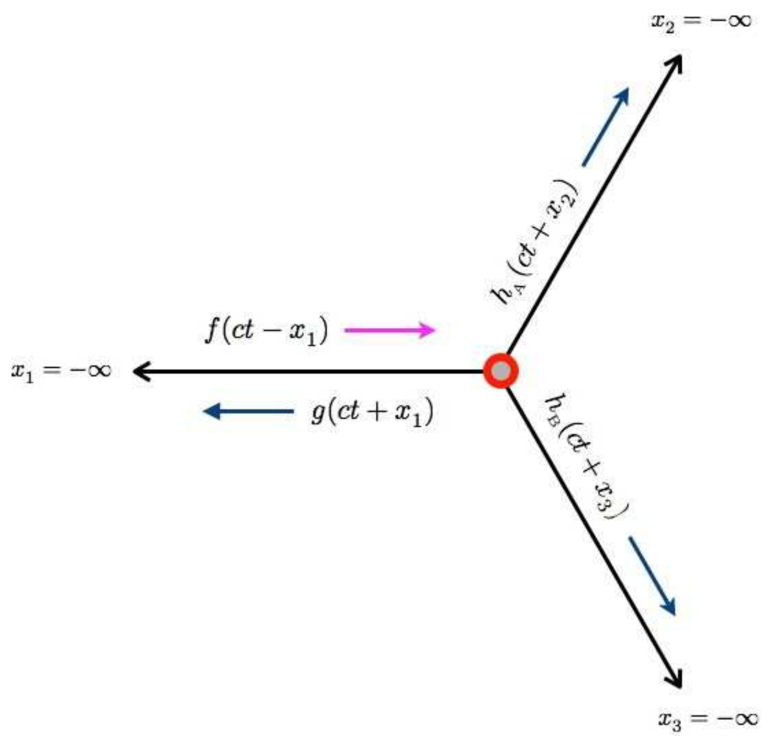


Figure 14.10: Three identical strings arranged symmetrically in a plane, attached to a common end. All motion is in the direction perpendicular to this plane. The red ring, whose mass is m, slides frictionlessly in this direction along a pole.

(a) Write the wave equation in string #1. Define all constants.

(b) Write the equation of motion for the ring.

(c) Solve for the reflected wave $g(\xi)$ in terms of the incident wave $f(\xi)$. You may write this relation in terms of the Fourier transforms $\hat{f}(k)$ and $\hat{g}(k)$.

(d) Suppose a very long wavelength pulse of maximum amplitude A is incident on the ring. What is the maximum amplitude of the reflected pulse? What do we mean by "very long wavelength"?

Solution:

(a) The wave equation is

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 z}{\partial t^2} \ , \tag{14.192}$$

where x is the coordinate along the string, and $c = \sqrt{\tau/\sigma}$ is the speed of wave propagation.

(b) Let Z be the vertical coordinate of the ring. Newton's second law says $m\ddot{Z} = F$, where the force on the ring is the sum of the vertical components of the tension in the three strings at $x = 0$:

$$F = -\tau\Big[-f'(ct) + g'(ct) + h'_{\rm A}(ct) + h'_{\rm B}(ct)\Big] \ , \tag{14.193}$$

where prime denotes differentiation with respect to argument.

(c) To solve for the reflected wave, we must eliminate the unknown functions $h_{\rm A,B}$ and then obtain g in terms of f. This is much easier than it might at first seem. We start by demanding continuity at the ring. This means

$$Z(t) = f(ct) + g(ct) = h_{\rm A}(ct) = h_{\rm B}(ct) \tag{14.194}$$

for all t. We can immediately eliminate $h_{\rm A,B}$:

$$h_{\rm A}(\xi) = h_{\rm B}(\xi) = f(\xi) + g(\xi) \ , \tag{14.195}$$

for all ξ. Newton's second law from part (b) may now be written as

$$mc^2\big[f''(\xi) + g''(\xi)\big] = -\tau\big[f'(\xi) + 3g'(\xi)\big] \ . \tag{14.196}$$

This linear ODE becomes a simple linear algebraic equation for the Fourier transforms,

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\,\hat{f}(k)\,e^{ik\xi} \ , \tag{14.197}$$

etc. We readily obtain

$$\hat{g}(k) = -\bigg(\frac{k - iQ}{k - 3iQ}\bigg)\,\hat{f}(k) \ , \tag{14.198}$$

where $Q \equiv \tau/mc^2$ has dimensions of inverse length. Since $h_{\rm A,B} = f + g$, we have

$$\hat{h}_{\rm A}(k) = \hat{h}_{\rm B}(k) = -\bigg(\frac{2iQ}{k - 3iQ}\bigg)\,\hat{f}(k) \ . \tag{14.199}$$

(d) For a very long wavelength pulse, composed of plane waves for which $|k| \ll Q$, we have $\hat{g}(k) \approx -\frac{1}{3}\,\hat{f}(k)$. Thus, the reflected pulse is inverted, and is reduced by a factor $\frac{1}{3}$ in amplitude. Note that for a very *short* wavelength pulse, for which $k \gg Q$, we have perfect reflection with inversion, and no transmission. This is due to the inertia of the ring.

It is straightforward to generalize this problem to one with n strings. The transmission into each of the $(n-1)$ channels is of course identical (by symmetry). One then finds the reflection and transmission amplitudes

$$r(k) = -\bigg(\frac{k - i(n-2)Q}{k - inQ}\bigg) \quad , \quad t(k) = -\bigg(\frac{2iQ}{k - inQ}\bigg) \ . \tag{14.200}$$

Conservation of energy means that the sum of the squares of the reflection amplitude and all the $(n-1)$ transmission amplitudes must be unity:

$$\big|r(k)\big|^2 + (n-1)\,\big|t(k)\big|^2 = 1 \ . \tag{14.201}$$

14.8 Appendix II : General Field Theoretic Formulation

Continuous systems possess an infinite number of degrees of freedom. They are described by a set of fields $\phi_a(\boldsymbol{x},t)$ which depend on space and time. These fields may represent local displacement, pressure, velocity, *etc.* The equations of motion of the fields are again determined by extremizing the action, which, in turn, is an integral of the *Lagrangian density* over all space and time. Extremization yields a set of (generally coupled) *partial* differential equations.

14.8.1 Euler-Lagrange equations for classical field theories

Suppose $\phi_a(\boldsymbol{x})$ depends on n independent variables, $\{x^1, x^2, \ldots, x^n\}$. Consider the functional

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_{\Omega} \! d\boldsymbol{x}\, \mathcal{L}(\phi_a\, \partial_\mu\phi_a, \boldsymbol{x}) \ , \tag{14.202}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^n. Then the first variation of S is

$$\begin{aligned} \delta S &= \int\limits_{\Omega} \! d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} \! d\Sigma\, n^\mu\, \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a + \int\limits_{\Omega} \! d\boldsymbol{x} \left\{ \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg) \right\} \delta\phi_a \ , \end{aligned} \tag{14.203}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg) \right]_{\boldsymbol{x}} \ , \tag{14.204}$$

where the subscript means we are to evaluate the term in brackets at $\boldsymbol{x}$. In a mechanical system, one of the n independent variables (usually x^0), is the time t. However, we may be interested in a time-independent context in which we wish to extremize the energy functional, for example. In any case, setting the first variation of S to zero yields the Euler-Lagrange equations,

$$\delta S = 0 \quad\Rightarrow\quad \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg) = 0 \tag{14.205}$$

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} - J_\mu\, A^\mu \ . \tag{14.206}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu\, F^{\mu\nu} = 4\pi J^\nu \ , \tag{14.207}$$

which are Maxwell's equations.

14.8.2 Conserved currents in field theory

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\sigma}\,\frac{\partial \tilde q_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{14.208}$$

where $\tilde q_\sigma = \tilde q_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t) \ , \tag{14.209}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $dQ/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde\phi_a}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{14.210}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde\phi_a}{\partial \zeta}\bigg|_{\zeta=0} \ . \tag{14.211}$$

We call $Q = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu\, J^\mu$ over the spatial region Ω gives

$$\frac{dQ}{dt} = \int\limits_\Omega d^3\!x\, \partial_0\, J^0 = -\int\limits_\Omega d^3\!x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \ , \tag{14.212}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[2]

$$\mathcal{L}(\psi,,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \ . \tag{14.213}$$

[2]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \text{diag}\,(+,-,-,-)$.

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \ , \tag{14.214}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)} \cdot (i\psi) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi^*\right)} \cdot (-i\psi^*) \\ &= \frac{K}{2i}\left(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\right) \ . \end{aligned} \tag{14.215}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

14.8.3 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi - g\left(|\psi|^2 - n_0\right)^2 \ . \tag{14.216}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x},t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned} \delta S[\psi^*,\psi] &= S[\psi^*+\delta\psi^*,\psi+\delta\psi] \\ &= \int\!dt\!\int\!d^d\!x\,\bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^*\cdot\boldsymbol{\nabla}\psi \\ &\qquad\qquad - 2g\left(|\psi|^2 - n_0\right)\left(\psi^*\delta\psi + \psi\delta\psi^*\right)\bigg\} \\ &= \int\!dt\!\int\!d^d\!x\,\Bigg\{\bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\left(|\psi|^2 - n_0\right)\psi^*\bigg]\,\delta\psi \\ &\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\left(|\psi|^2 - n_0\right)\psi\bigg]\,\delta\psi^*\Bigg\} \ , \end{aligned} \tag{14.217}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\left(|\psi|^2 - n_0\right)\psi \tag{14.218}$$

as well as its complex conjugate,

$$-i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\left(|\psi|^2 - n_0\right)\psi^* \ . \tag{14.219}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\frac{\delta S}{\delta \psi} = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi}\right) \tag{14.220}$$

$$\frac{\delta S}{\delta \psi^*} = \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*}\right) , \tag{14.221}$$

with $x^\mu = (t, \boldsymbol{x})^3$ Plugging in

$$\frac{\partial \mathcal{L}}{\partial \psi} = -2g\left(|\psi|^2 - n_0\right)\psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi} = i\hbar\, \psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{14.222}$$

and

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\hbar\, \psi - 2g\left(|\psi|^2 - n_0\right)\psi \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi^*} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi , \tag{14.223}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) . \tag{14.224}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi}\,\frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*}\,\frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0}$$

$$J^0 = -\hbar\, |\psi|^2 \tag{14.225}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{14.226}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 , \tag{14.227}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) . \tag{14.228}$$

are the particle density and the particle current, respectively.

[3] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

14.9 Appendix III : Green's Functions

Suppose we add a forcing term,

$$\mu(x)\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = \text{Re}\Big[\mu(x)\,f(x)\,e^{-i\omega t}\Big]\ . \tag{14.229}$$

We write the solution as

$$y(x,t) = \text{Re}\Big[y(x)\,e^{-i\omega t}\Big]\ , \tag{14.230}$$

where

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dy(x)}{dx}\bigg] + v(x)\,y(x) - \omega^2\,\mu(x)\,y(x) = \mu(x)\,f(x)\ , \tag{14.231}$$

or

$$\Big[K - \omega^2\mu(x)\Big]\,y(x) = \mu(x)\,f(x)\ , \tag{14.232}$$

where K is a differential operator,

$$K \equiv -\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x)\ . \tag{14.233}$$

Note that the eigenfunctions of K are the $\{\psi_n(x)\}$:

$$K\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)\ . \tag{14.234}$$

The formal solution to equation 14.232 is then

$$y(x) = \Big[\,K - \omega^2\mu\,\Big]^{-1}_{x,x'}\,\mu(x')\,f(x') \tag{14.235}$$

$$= \int\limits_{x_a}^{x_b} dx'\,\mu(x')\,G_\omega(x,x')\,f(x'). \tag{14.236}$$

What do we mean by the term in brackets? If we define the *Green's function*

$$G_\omega(x,x') \equiv \Big[\,K - \omega^2\mu\,\Big]^{-1}_{x,x'}\ , \tag{14.237}$$

what this means is

$$\Big[\,K - \omega^2\mu(x)\Big]\,G_\omega(x,x') = \delta(x-x')\ . \tag{14.238}$$

Note that the Green's function may be expanded in terms of the (real) eigenfunctions, as

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2}\ , \tag{14.239}$$

which follows from completeness of the eigenfunctions:

$$\mu(x)\sum_{n=1}^{\infty}\psi_n(x)\,\psi_n(x') = \delta(x-x')\ . \tag{14.240}$$

The expansion in eqn. 14.239 is formally exact, but difficult to implement, since it requires summing over an infinite set of eigenfunctions. It is more practical to construct the Green's function from solutions to the homogeneous Sturm Liouville equation, as follows. When $x \neq x'$, we have that $(K - \omega^2\mu)\, G_\omega(x,x') = 0$, which is a homogeneous ODE of degree two. Consider first the interval $x \in [x_a, x']$. A second order homogeneous ODE has two solutions, and further invoking the boundary condition at $x = x_a$, there is a unique solution, up to a multiplicative constant. Call this solution $y_1(x)$. Next, consider the interval $x \in [x', x_b]$. Once again, there is a unique solution to the homogeneous Sturm-Liouville equation, up to a multiplicative constant, which satisfies the boundary condition at $x = x_b$. Call this solution $y_2(x)$. We then can write

$$G_\omega(x,x') = \begin{cases} A(x')\, y_1(x) & \text{if } x_a \le x < x' \\ \\ B(x')\, y_2(x) & \text{if } x' < x \le x_b\ . \end{cases} \tag{14.241}$$

Here, $A(x')$ and $B(x')$ are undetermined functions. We now invoke the inhomogeneous Sturm-Liouville equation,

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dG_\omega(x,x')}{dx}\bigg] + v(x)\, G_\omega(x,x') - \omega^2\mu(x)\, G_\omega(x,x') = \delta(x-x')\ . \tag{14.242}$$

We integrate this from $x = x' - \epsilon$ to $x = x' + \epsilon$, where ϵ is a positive infinitesimal. This yields

$$\tau(x')\Big[A(x')\, y_1'(x') - B(x')\, y_2'(x')\Big] = 1\ . \tag{14.243}$$

Continuity of $G_\omega(x,x')$ itself demands

$$A(x')\, y_1(x') = B(x')\, y_2(x')\ . \tag{14.244}$$

Solving these two equations for $A(x')$ and $B(x')$, we obtain

$$A(x') = -\frac{y_2(x')}{\tau(x')\,\mathcal{W}_{y_1,y_2}(x')} \qquad , \qquad B(x') = -\frac{y_1(x')}{\tau(x')\,\mathcal{W}_{y_1,y_2}(x')}\ , \tag{14.245}$$

where $\mathcal{W}_{y_1,y_2}(x)$ is the *Wronskian*

$$\begin{aligned} \mathcal{W}_{y_1,y_2}(x) &= \det\begin{pmatrix} y_1(x) & y_2(x) \\ \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)\ . \end{aligned} \tag{14.246}$$

Now it is easy to show that $\mathcal{W}_{y_1,y_2}(x)\,\tau(x) = \mathcal{W}\,\tau$ is a constant. This follows from the fact that

$$\begin{aligned} 0 &= y_2\, K\, y_1 - y_2\, K\, y_1 \\ &= \frac{d}{dx}\bigg\{\tau(x)\Big[y_1\, y_2' - y_2\, y_1'\Big]\bigg\}\ . \end{aligned} \tag{14.247}$$

Thus, we have

$$G_\omega(x,x') = \begin{cases} -y_1(x)\,y_2(x')/\mathcal{W} & \text{if } x_a \le x < x' \\ \\ -y_1(x')\,y_2(x)/\mathcal{W} & \text{if } x' < x \le x_b \,, \end{cases} \tag{14.248}$$

or, in compact form,

$$G_\omega(x,x') = -\frac{y_1(x_<)\,y_2(x_>)}{\mathcal{W}\,\tau} \,, \tag{14.249}$$

where $x_< = \min(x,x')$ and $x_> = \max(x,x')$.

As an example, consider a uniform string (*i.e.* μ and τ constant, $v = 0$) with fixed endpoints at $x_a = 0$ and $x_b = L$. The normalized eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{\mu L}}\,\sin\left(\frac{n\pi x}{L}\right) \,, \tag{14.250}$$

and the eigenvalues are $\omega_n = n\pi c/L$. The Green's function is

$$G_\omega(x,x') = \frac{2}{\mu L}\sum_{n=1}^{\infty} \frac{\sin(n\pi x/L)\,\sin(n\pi x'/L)}{(n\pi c/L)^2 - \omega^2} \,. \tag{14.251}$$

Now construct the homogeneous solutions:

$$(K - \omega^2\mu)\,y_1 = 0 \quad , \quad y_1(0) = 0 \qquad \Longrightarrow \qquad y_1(x) = \sin\left(\frac{\omega x}{c}\right) \tag{14.252}$$

$$(K - \omega^2\mu)\,y_2 = 0 \quad , \quad y_2(L) = 0 \qquad \Longrightarrow \qquad y_2(x) = \sin\left(\frac{\omega(L-x)}{c}\right) \,. \tag{14.253}$$

The Wronskian is

$$\mathcal{W} = y_1\,y_2' - y_2\,y_1' = -\frac{\omega}{c}\,\sin\left(\frac{\omega L}{c}\right) \,. \tag{14.254}$$

Therefore, the Green's function is

$$G_\omega(x,x') = \frac{\sin\big(\omega x_</c\big)\,\sin\big(\omega(L-x_>)/c\big)}{(\omega\tau/c)\,\sin(\omega L/c)} \,. \tag{14.255}$$

14.9.1 Perturbation theory

Suppose we have solved for the Green's function for the linear operator K_0 and mass density $\mu_0(x)$. *I.e.* we have

$$\big(K_0 - \omega^2\mu_0(x)\big)\,G^0_\omega(x,x') = \delta(x-x') \,. \tag{14.256}$$

We now imagine perturbing $\tau_0 \to \tau_0 + \lambda\tau_1$, $v_0 \to v_0 + \lambda v_2$, $\mu_0 \to \mu_0 + \lambda\mu_1$. What is the new Green's function $G_\omega(x,x')$? We must solve

$$\big(L_0 + \lambda L_1\big)\,G_\omega(x,x') = \delta(x-x') \,, \tag{14.257}$$

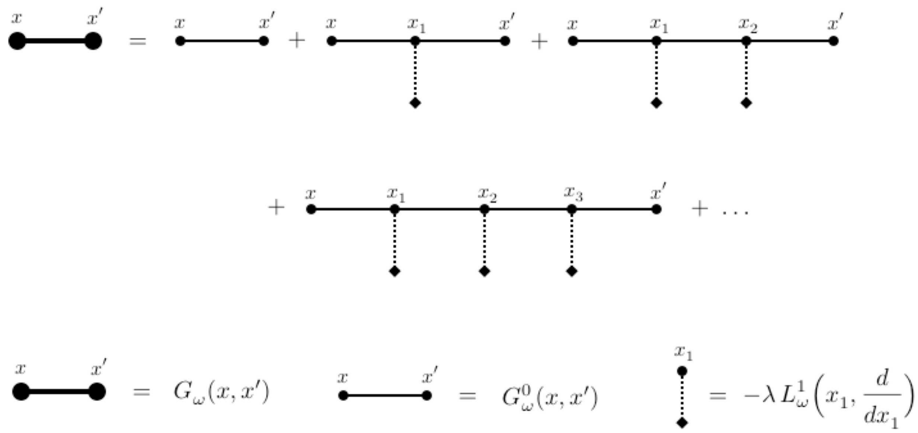


Figure 14.11: Diagrammatic representation of the perturbation expansion in eqn. 14.260..

where

$$L^0_\omega \equiv K_0 - \omega^2\,\mu_0 \tag{14.258}$$
$$L^1_\omega \equiv K_1 - \omega^2\,\mu_1\ . \tag{14.259}$$

Dropping the ω subscript for simplicity, the full Green's function is then given by

$$\begin{aligned} G_\omega &= \Big[L^0_\omega + \lambda L^1_\omega\Big]^{-1} \\ &= \Big[\,\big(G^0_\omega\big)^{-1} + \lambda L^1_\omega\,\Big]^{-1} \\ &= \Big[\,1 + \lambda\, G^0_\omega\, L^1_\omega\Big]^{-1}\, G^0_\omega \\ &= G^0_\omega - \lambda\, G^0_\omega\, L^1_\omega\, G^0_\omega + \lambda^2\, G^0_\omega\, L^1_\omega\, G^0_\omega\, L^1_\omega\, G^0_\omega + \ldots\ . \end{aligned} \tag{14.260}$$

The 'matrix multiplication' is of course a convolution, *i.e.*

$$G_\omega(x,x') = G^0_\omega(x,x') - \lambda \int\limits_{x_a}^{x_b} \! dx_1\, G^0_\omega(x,x_1)\, L^1_\omega\big(x_1, \tfrac{d}{dx_1}\big)\, G^0_\omega(x_1,x') + \ldots\ . \tag{14.261}$$

Each term in the perturbation expansion of eqn. 14.260 may be represented by a diagram, as depicted in Fig. 14.11.

As an example, consider a string with $x_a = 0$ and $x_b = L$ with a mass point m affixed at the point $x = d$. Thus, $\mu_1(x) = m\,\delta(x-d)$, and $L^1_\omega = -m\omega^2\,\delta(x-d)$, with $\lambda = 1$. The perturbation expansion gives

$$\begin{aligned} G_\omega(x,x') &= G^0_\omega(x,x') + m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x') + m^2\omega^4\, G^0_\omega(x,d)\, G^0_\omega(d,d)\, G^0_\omega(d,x') + \ldots \\ &= G^0_\omega(x,x') + \frac{m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x')}{1 - m\omega^2\, G^0_\omega(d,d)}\ . \end{aligned} \tag{14.262}$$

Note that the eigenfunction expansion,

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \ , \tag{14.263}$$

says that the exact eigenfrequencies are poles of $G_\omega(x,x')$, and furthermore the residue at each pole is

$$\operatorname*{Res}_{\omega=\omega_n} G_\omega(x,x') = -\frac{1}{2\omega_n}\,\psi_n(x)\,\psi_n(x') \ . \tag{14.264}$$

According to eqn. 14.262, the poles of $G_\omega(x,x')$ are located at solutions to[4]

$$m\omega^2\, G^0_\omega(d,d) = 1 \ . \tag{14.265}$$

For simplicity let us set $d = \frac{1}{2}L$, so the mass point is in the middle of the string. Then according to eqn. 14.255,

$$\begin{aligned} G^0_\omega\big(\tfrac{1}{2}L,\tfrac{1}{2}L\big) &= \frac{\sin^2(\omega L/2c)}{(\omega\tau/c)\,\sin(\omega L/c)} \\ &= \frac{c}{2\omega\tau}\,\tan\!\left(\frac{\omega L}{2c}\right) \ . \end{aligned} \tag{14.266}$$

The eigenvalue equation is therefore

$$\tan\!\left(\frac{\omega L}{2c}\right) = \frac{2\tau}{m\omega c} \ , \tag{14.267}$$

which can be manipulated to yield

$$\frac{m}{M}\,\lambda = \operatorname{ctn}\lambda \ , \tag{14.268}$$

where $\lambda = \omega L/2c$ and $M = \mu L$ is the total mass of the string. When $m = 0$, the LHS vanishes, and the roots lie at $\lambda = (n+\frac{1}{2})\pi$, which gives $\omega = \omega_{2n+1}$. Why don't we see the poles at the even mode eigenfrequencies ω_{2n}? The answer is that these poles are present in the Green's function. They do not cancel for $d = \frac{1}{2}L$ because the perturbation does not couple to the even modes, which all have $\psi_{2n}(\frac{1}{2}L) = 0$. The case of general d may be instructive in this regard. One finds the eigenvalue equation

$$\frac{\sin(2\lambda)}{2\lambda\,\sin\big(2\epsilon\lambda\big)\,\sin\big(2(1-\epsilon)\lambda\big)} = \frac{m}{M} \ , \tag{14.269}$$

where $\epsilon = d/L$. Now setting $m = 0$ we recover $2\lambda = n\pi$, which says $\omega = \omega_n$, and all the modes are recovered.

[4]Note in particular that there is no longer any divergence at the location of the original poles of $G^0_\omega(x,x')$. These poles are cancelled.

14.9.2 Perturbation theory for eigenvalues and eigenfunctions

We wish to solve

$$\big(K_0 + \lambda K_1\big)\,\psi = \omega^2\big(\mu_0 + \lambda\mu_1\big)\,\psi \ , \tag{14.270}$$

which is equivalent to

$$L^0_\omega\,\psi = -\lambda L^1_\omega\,\psi \ . \tag{14.271}$$

Multiplying by $\big(L^0_\omega\big)^{-1} = G^0_\omega$ on the left, we have

$$\psi(x) = -\lambda \int\limits_{x_a}^{x_b} \! dx'\, G_\omega(x,x')\, L^1_\omega\, \psi(x') \tag{14.272}$$

$$= \lambda \sum_{m=1}^\infty \frac{\psi_m(x)}{\omega^2 - \omega_m^2} \int\limits_{x_a}^{x_b} \! dx'\, \psi_m(x')\, L^1_\omega\, \psi(x') \ . \tag{14.273}$$

We are free to choose any normalization we like for $\psi(x)$. We choose

$$\langle\, \psi \,|\, \psi_n \,\rangle = \int\limits_{x_a}^{x_b} \! dx\, \mu_0(x)\, \psi_n(x)\, \psi(x) = 1 \ , \tag{14.274}$$

which entails

$$\omega^2 - \omega_n^2 = \lambda \int\limits_{x_a}^{x_b} \! dx\, \psi_n(x)\, L^1_\omega\, \psi(x) \tag{14.275}$$

as well as

$$\psi(x) = \psi_n(x) + \lambda \sum_{\substack{k \\ (k\neq n)}} \frac{\psi_k(x)}{\omega^2 - \omega_k^2} \int\limits_{x_a}^{x_b} \! dx'\, \psi_k(x')\, L^1_\omega\, \psi(x') \ . \tag{14.276}$$

By expanding ψ and ω^2 in powers of λ, we can develop an order by order perturbation series.

To lowest order, we have

$$\omega^2 = \omega_n^2 + \lambda \int\limits_{x_a}^{x_b} \! dx\, \psi_n(x)\, L^1_{\omega_n} \psi_n(x) \ . \tag{14.277}$$

For the case $L^1_\omega = -m\,\omega^2\,\delta(x-d)$, we have

$$\begin{aligned} \frac{\delta\omega_n}{\omega_n} &= -\tfrac{1}{2} m \big[\psi_n(d)\big]^2 \\ &= -\frac{m}{M}\,\sin^2\!\Big(\frac{n\pi d}{L}\Big) \ . \end{aligned} \tag{14.278}$$

For $d = \frac{1}{2}L$, only the odd n modes are affected, as the even n modes have a node at $x = \frac{1}{2}L$.

Carried out to second order, one obtains for the eigenvalues,

$$\omega^2 = \omega_n^2 + \lambda \int_{x_a}^{x_b} dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) + \lambda^2 \sum_{\substack{k \\ (k \neq n)}} \frac{\left| \int_{x_a}^{x_b} dx\, \psi_k(x)\, L^1_{\omega_n}\, \psi_n(x) \right|^2}{\omega_n^2 - \omega_k^2} + \mathcal{O}(\lambda^3) - \lambda^2 \int_{x_a}^{x_b} dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) \cdot \int_{x_a}^{x_b} dx'\, \mu_1(x') \left[\psi_n(x')\right]^2 + \mathcal{O}(\lambda^3)\ . \tag{14.279}$$

Chapter 15

Special Relativity

For an extraordinarily lucid, if characteristically brief, discussion, see chs. 1 and 2 of L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (*Course of Theoretical Physics, vol. 2*).

15.1 Introduction

All distances are relative in physics. They are measured with respect to a fixed *frame of reference*. Frames of reference in which free particles move with constant velocity are called *inertial frames*. The *principle of relativity* states that the laws of Nature are identical in all inertial frames.

15.1.1 Michelson-Morley experiment

We learned how sound waves in a fluid, such as air, obey the Helmholtz equation. Let us restrict our attention for the moment to solutions of the form $\phi(x,t)$ which do not depend on y or z. We then have a one-dimensional wave equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2}\ . \tag{15.1}$$

The fluid in which the sound propagates is assumed to be at rest. But suppose the fluid is not at rest. We can investigate this by shifting to a moving frame, defining $x' = x - ut$, with $y' = y$, $z' = z$ and of course $t' = t$. This is a Galilean transformation. In terms of the new variables, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad , \qquad \frac{\partial}{\partial t} = -u\,\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}\ . \tag{15.2}$$

The wave equation is then

$$\left(1 - \frac{u^2}{c^2}\right)\frac{\partial^2\phi}{\partial x'^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t'^2} - \frac{2u}{c^2}\,\frac{\partial^2\phi}{\partial x'\,\partial t'}\ . \tag{15.3}$$

Clearly the wave equation acquires a different form when expressed in the new variables (x', t'), *i.e.* in a frame in which the fluid is not at rest. The general solution is then of the modified d'Alembert form,

$$\phi(x', t') = f(x' - c_{\text{R}} t') + g(x' + c_{\text{L}} t') \ , \tag{15.4}$$

where $c_{\text{R}} = c - u$ and $c_{\text{L}} = c + u$ are the speeds of rightward and leftward propagating disturbances, respectively. Thus, there is a *preferred frame of reference* – the frame in which the fluid is at rest. In the rest frame of the fluid, sound waves travel with velocity c in either direction.

Light, as we know, is a wave phenomenon in classical physics. The propagation of light is described by Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\,\frac{\partial \boldsymbol{B}}{\partial t} \tag{15.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\,\boldsymbol{j} + \frac{1}{c}\,\frac{\partial \boldsymbol{E}}{\partial t} \ , \tag{15.6}$$

where ρ and $\boldsymbol{j}$ are the local charge and current density, respectively. Taking the curl of Faraday's law, and restricting to free space where $\rho = \boldsymbol{j} = 0$, we once again have (using a Cartesian system for the fields) the wave equation,

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\,\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \ . \tag{15.7}$$

(We shall discuss below, in section 15.8, the beautiful properties of Maxwell's equations under general coordinate transformations.)

In analogy with the theory of sound, it was assumed prior to Einstein that there was in fact a preferred reference frame for electromagnetic radiation – one in which the medium which was excited during the EM wave propagation was at rest. This notional medium was called the *lumineferous ether*. Indeed, it was generally assumed during the 19^{th} century that light, electricity, magnetism, and heat (which was not understood until Boltzmann's work in the late 19^{th} century) all had separate ethers. It was Maxwell who realized that light, electricity, and magnetism were all unified phenomena, and accordingly he proposed a single ether for electromagnetism. It was believed at the time that the earth's motion through the ether would result in a drag on the earth.

In 1887, Michelson and Morley set out to measure the changes in the speed of light on earth due to the earth's movement through the ether (which was generally assumed to be at rest in the frame of the Sun). The Michelson interferometer is shown in fig. 15.1, and works as follows. Suppose the apparatus is moving with velocity $u\,\hat{\boldsymbol{x}}$ through the ether. Then the time it takes a light ray to travel from the half-silvered mirror to the mirror on the right and back again is

$$t_x = \frac{\ell}{c+u} + \frac{\ell}{c-u} = \frac{2\ell c}{c^2 - u^2} \ . \tag{15.8}$$

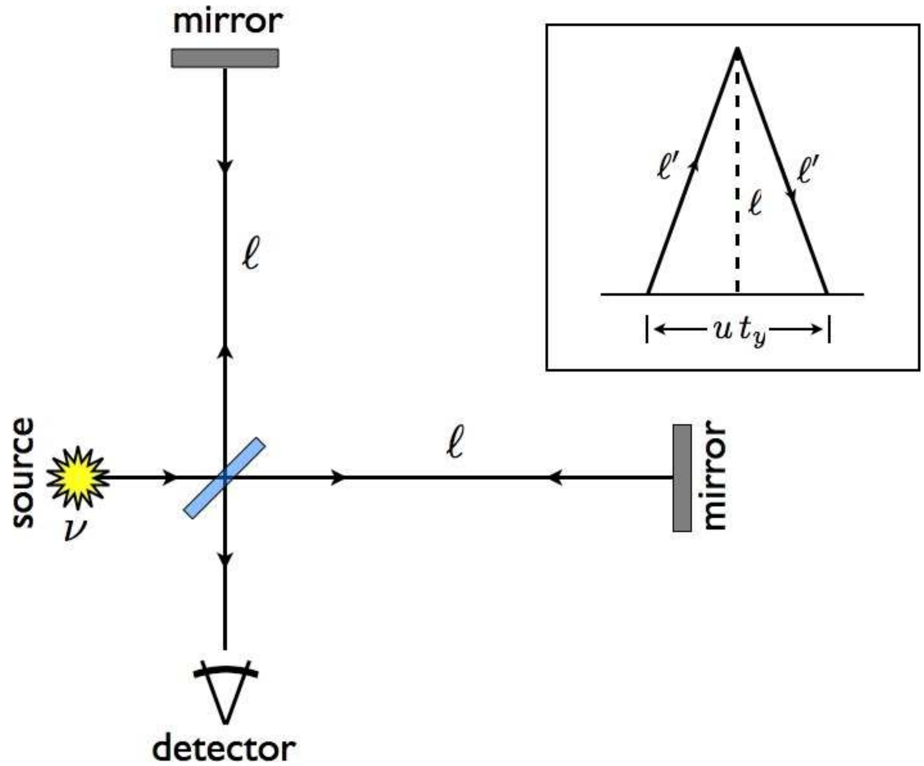


Figure 15.1: The Michelson-Morley experiment (1887) used an interferometer to effectively measure the time difference for light to travel along two different paths. Inset: analysis for the y-directed path.

For motion along the other arm of the interferometer, the geometry in the inset of fig. 15.1 shows $\ell' = \sqrt{\ell^2 + \frac{1}{4}u^2 t_y^2}$, hence

$$t_y = \frac{2\ell'}{c} = \frac{2}{c}\sqrt{\ell^2 + \tfrac{1}{4}u^2 t_y^2} \qquad \Rightarrow \qquad t_y = \frac{2\ell}{\sqrt{c^2 - u^2}} \ . \tag{15.9}$$

Thus, the difference in times along these two paths is

$$\Delta t = t_x - t_y = \frac{2\ell c}{c^2} - \frac{2\ell}{\sqrt{c^2 - u^2}} \approx \frac{\ell}{c} \cdot \frac{u^2}{c^2} \ . \tag{15.10}$$

Thus, the difference in phase between the two paths is

$$\frac{\Delta\phi}{2\pi} = \nu\,\Delta t \approx \frac{\ell}{\lambda} \cdot \frac{u^2}{c^2} \ , \tag{15.11}$$

where λ is the wavelength of the light. We take $u \approx 30\,\mathrm{km/s}$, which is the earth's orbital velocity, and $\lambda \approx 5000\,\mathrm{Å}$. From this we find that $\Delta\phi \approx 0.02 \times 2\pi$ if $\ell = 1\,\mathrm{m}$. Michelson and Morley found that the observed fringe shift $\Delta\phi/2\pi$ was approximately 0.02 times the expected value. The inescapable conclusion was that the speed of light did not depend on the motion of the source. This was very counterintuitive!

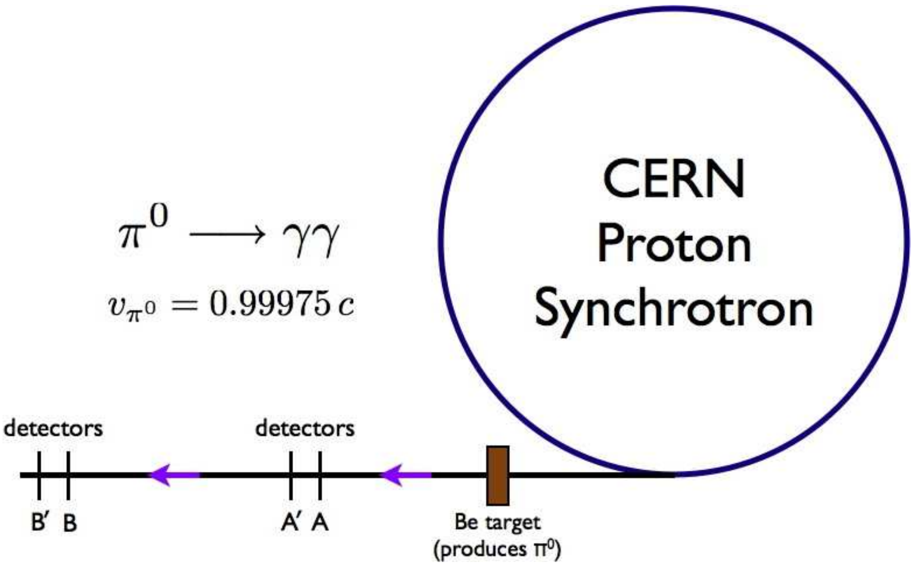


Figure 15.2: Experimental setup of Alvager *et al.* (1964), who used the decay of high energy neutral pions to test the source velocity dependence of the speed of light.

The history of the development of special relativity is quite interesting, but we shall not have time to dwell here on the many streams of scientific thought during those exciting times. Suffice it to say that the Michelson-Morley experiment, while a landmark result, was not the last word. It had been proposed that the ether could be dragged, either entirely or partially, by moving bodies. If the earth dragged the ether along with it, then there would be no ground-level 'ether wind' for the MM experiment to detect. Other experiments, however, such as stellar aberration, in which the apparent position of a distant star varies due to the earth's orbital velocity, rendered the "ether drag" theory untenable – the notional 'ether bubble' dragged by the earth could not reasonably be expected to extend to the distant stars.

A more recent test of the effect of a moving source on the speed of light was performed by T. Alvåger *et al.*, *Phys. Lett.* **12**, 260 (1964), who measured the velocity of γ-rays (photons) emitted from the decay of highly energetic neutral pions (π^0). The pion energies were in excess of 6 GeV, which translates to a velocity of $v = 0.99975\,c$, according to special relativity. Thus, photons emitted in the direction of the pions should be traveling at close to $2c$, if the source and photon velocities were to add. Instead, the velocity of the photons was found to be $c = 2.9977 \pm 0.0004 \times 10^{10}$ cm/s, which is within experimental error of the best accepted value.

15.1.2 Einsteinian and Galilean relativity

The *Principle of Relativity* states that the laws of nature are the same when expressed in any inertial frame. This principle can further be refined into two classes, depending on whether one takes the velocity of the propagation of interactions to be finite or infinite.

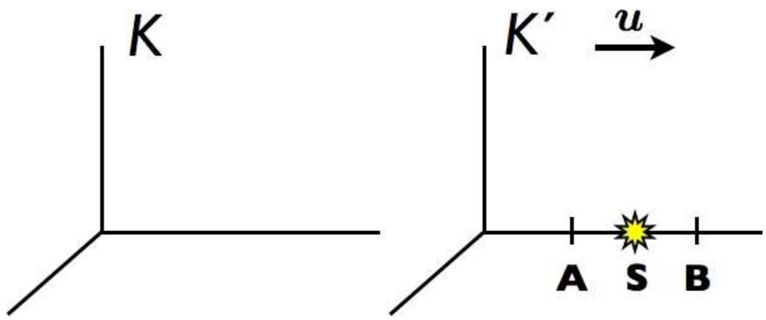


Figure 15.3: Two reference frames.

The interaction of matter in classical mechanics is described by a potential function $U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$. Typically, one has two-body interactions in which case one writes $U = \sum_{i<j} U(\boldsymbol{r}_i, \boldsymbol{r}_j)$. These interactions are thus assumed to be instantaneous, which is unphysical. The interaction of particles is mediated by the exchange of gauge bosons, such as the photon (for electromagnetic interactions), gluons (for the strong interaction, at least on scales much smaller than the 'confinement length'), or the graviton (for gravity). Their velocity of propagation, according to the principle of relativity, is the same in all reference frames, and is given by the speed of light, $c = 2.998 \times 10^8$ m/s.

Since c is so large in comparison with terrestrial velocities, and since d/c is much shorter than all other relevant time scales for typical interparticle separations d, the assumption of an instantaneous interaction is usually quite accurate. The combination of the principle of relativity with finiteness of c is known as Einsteinian relativity. When $c = \infty$, the combination comprises Galilean relativity:

$$\begin{aligned} c < \infty \quad &: \quad \text{Einsteinian relativity} \\ c = \infty \quad &: \quad \text{Galilean relativity} \ . \end{aligned}$$

Consider a train moving at speed u. In the rest frame of the train track, the speed of the light beam emanating from the train's headlight is $c + u$. This would contradict the principle of relativity. This leads to some very peculiar consequences, foremost among them being the fact that events which are simultaneous in one inertial frame will not in general be simultaneous in another. In Newtonian mechanics, on the other hand, time is absolute, and is independent of the frame of reference. If two events are simultaneous in one frame then they are simultaneous in all frames. This is not the case in Einsteinian relativity!

We can begin to apprehend this curious feature of simultaneity by the following *Gedankenexperiment* (a long German word meaning "thought experiment")[1]. Consider the case in fig. 15.3 in which frame K' moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to frame K. Let a source at S emit a signal (a light pulse) at $t = 0$. In the frame K' the signal's arrival at equidistant locations A and B is simultaneous. In frame K, however, A moves toward left-propagating

[1] Unfortunately, many important physicists were German and we have to put up with a legacy of long German words like *Gedankenexperiment*, *Zitterbewegung*, *Brehmsstrahlung*, *Stosszahlansatz*, *Kartoffelsalat*, *etc.*

emitted wavefront, and B moves away from the right-propagating wavefront. For classical sound, the speed of the left-moving and right-moving wavefronts is $c \mp u$, taking into account the motion of the source, and thus the relative velocities of the signal and the detectors remain at c. But according to the principle of relativity, the speed of light is c in all frames, and is so in frame K for both the left-propagating and right-propagating signals. Therefore, the relative velocity of A and the left-moving signal is $c+u$ and the relative velocity of B and the right-moving signal is $c-u$. Therefore, A 'closes in' on the signal and receives it before B, which is moving away from the signal. We might expect the arrival times to be $t^*_{\mathrm{A}} = d/(c+u)$ and $t^*_{\mathrm{B}} = d/(c-u)$, where d is the distance between the source S and either detector A or B in the K' frame. Later on we shall analyze this problem and show that

$$t^*_{\mathrm{A}} = \sqrt{\frac{c-u}{c+u}} \cdot \frac{d}{c} \qquad , \qquad t^*_{\mathrm{B}} = \sqrt{\frac{c+u}{c-u}} \cdot \frac{d}{c} \ . \tag{15.12}$$

Our naïve analysis has omitted an important detail – the *Lorentz contraction* of the distance d as seen by an observer in the K frame.

15.2 Intervals

Now let us express mathematically the constancy of c in all frames. An *event* is specified by the time and place where it occurs. Thus, an event is specified by *four* coordinates, (t, x, y, z). The four-dimensional space spanned by these coordinates is called *spacetime*. The *interval* between two events in spacetime at (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is defined to be

$$s_{12} = \sqrt{c^2(t_1-t_2)^2 - (x_1-x_2)^2 - (y_1-y_2)^2 - (z_1-z_2)^2} \ . \tag{15.13}$$

For two events separated by an infinitesimal amount, the interval ds is infinitesimal, with

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \ . \tag{15.14}$$

Now when the two events denote the emission and reception of an electromagnetic signal, we have $ds^2 = 0$. This must be true in any frame, owing to the invariance of c, hence since ds and ds' are differentials of the same order, we must have $ds'^2 = ds^2$. This last result requires homogeneity and isotropy of space as well. Finally, if infinitesimal intervals are invariant, then integrating we obtain $s = s'$, and we conclude that *the interval between two space-time events is the same in all inertial frames*.

When $s_{12}^2 > 0$, the interval is said to be *time-like*. For timelike intervals, we can always find a reference frame in which the two events occur at the same *locations*. As an example, consider a passenger sitting on a train. Event #1 is the passenger yawning at time t_1. Event #2 is the passenger yawning again at some later time t_2. To an observer sitting in the train station, the two events take place at different locations, but in the frame of the passenger, they occur at the same location.

When $s_{12}^2 < 0$, the interval is said to be *space-like*. Note that $s_{12} = \sqrt{s_{12}^2} \in i\mathbb{R}$ is pure imaginary, so one says that imaginary intervals are spacelike. As an example, at this

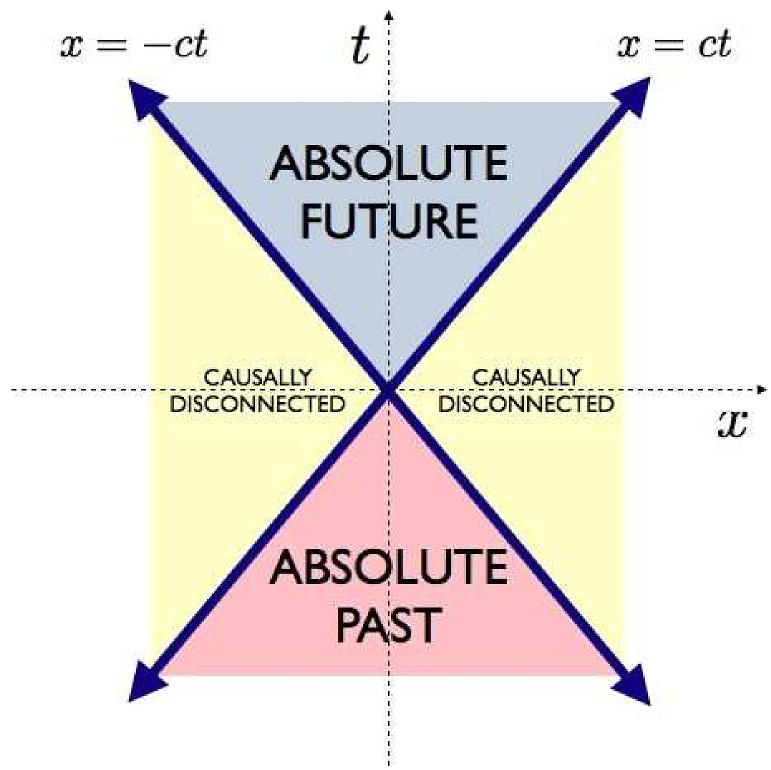


Figure 15.4: A $(1+1)$–dimensional light cone. The forward light cone consists of timelike events with $\Delta t > 0$. The backward light cone consists of timelike events with $\Delta t < 0$. The causally disconnected regions are time-like, and intervals connecting the origin to any point on the light cone itself are light-like.

moment, in the frame of the reader, the North and South poles of the earth are separated by a space-like interval. If the interval between two events is space-like, a reference frame can always be found in which the events are simultaneous.

An interval with $s_{12} = 0$ is said to be *light-like*.

This leads to the concept of the *light cone*, depicted in fig. 15.4. Consider an event E. In the frame of an inertial observer, all events with $s^2 > 0$ and $\Delta t > 0$ are in E's *forward light cone* and are part of his *absolute future*. Events with $s^2 > 0$ and $\Delta t < 0$ lie in E's *backward light cone* are are part of his *absolute past*. Events with spacelike separations $s^2 < 0$ are *causally disconnected* from E. Two events which are causally disconnected can not possible influence each other. Uniform rectilinear motion is represented by a line $t = x/v$ with constant slope. If $v < c$, this line is contained within E's light cone. E is potentially influenced by all events in its backward light cone, *i.e.* its absolute past. It is impossible to find a frame of reference which will transform past into future, or spacelike into timelike intervals.

15.2.1 Proper time

Proper time is the time read on a clock traveling with a moving observer. Consider two observers, one at rest and one in motion. If dt is the differential time elapsed in the rest frame, then

$$\begin{aligned} ds^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 && (15.15)\\ &= c^2\,dt'^2\ , && (15.16)\end{aligned}$$

where dt' is the differential time elapsed on the moving clock. Thus,

$$dt' = dt\,\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}\ , \tag{15.17}$$

and the time elapsed on the moving observer's clock is

$$t'_2 - t'_1 = \int\limits_{t_1}^{t_2}\!dt\,\sqrt{1-\frac{\boldsymbol{v}^2(t)}{c^2}}\ . \tag{15.18}$$

Thus, *moving clocks run slower*. This is an essential feature which is key to understanding many important aspects of particle physics. A particle with a brief lifetime can, by moving at speeds close to c, appear to an observer in our frame to be long-lived. It is customary to define two dimensionless measures of a particle's velocity:

$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{v}}{c} \qquad , \qquad \gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}}\ . \tag{15.19}$$

As $v \to c$, we have $\beta \to 1$ and $\gamma \to \infty$.

Suppose we wish to compare the elapsed time on two clocks. We keep one clock at rest in an inertial frame, while the other executes a closed path in space, returning to its initial location after some interval of time. When the clocks are compared, the moving clock will show a smaller elapsed time. This is often stated as the "twin paradox." The total elapsed time on a moving clock is given by

$$\tau = \frac{1}{c}\int\limits_a^b\!ds\ , \tag{15.20}$$

where the integral is taken over the *world line* of the moving clock. The elapsed time τ takes on a minimum value when the path from a to b is a straight line. To see this, one can express $\tau\big[\boldsymbol{x}(t)\big]$ as a functional of the path $\boldsymbol{x}(t)$ and extremize. This results in $\ddot{\boldsymbol{x}} = 0$.

15.2.2 Irreverent problem from Spring 2002 final exam

Flowers for Algernon – Bob's beloved hamster, Algernon, is very ill. He has only three hours to live. The veterinarian tells Bob that Algernon can be saved only through a gallbadder transplant. A suitable donor gallbladder is available from a hamster recently pronounced brain dead after a blender accident in New York (miraculously, the gallbladder was unscathed), but it will take Life Flight five hours to bring the precious rodent organ to San Diego.

Bob embarks on a bold plan to save Algernon's life. He places him in a cage, ties the cage to the end of a strong meter-long rope, and whirls the cage above his head while the Life Flight team is *en route*. Bob reasons that *if he can make time pass more slowly for Algernon*, the gallbladder will arrive in time to save his life.

(a) At how many revolutions per second must Bob rotate the cage in order that the gallbladder arrive in time for the life-saving surgery? What is Algernon's speed v_0?

Solution : We have $\beta(t) = \omega_0 R/c$ is constant, therefore, from eqn. 15.18,

$$\Delta t = \gamma\,\Delta t' \ . \tag{15.21}$$

Setting $\Delta t' = 3\,\text{hr}$ and $\Delta t = 5\,\text{hr}$, we have $\gamma = \frac{5}{3}$, which entails $\beta = \sqrt{1-\gamma^{-2}} = \frac{4}{5}$. Thus, $v_0 = \frac{4}{5}\,c$, which requires a rotation frequency of $\omega_0/2\pi = 38.2\,\text{MHz}$.

(b) Bob finds that he cannot keep up the pace! Assume Algernon's speed is given by

$$v(t) = v_0\sqrt{1-\frac{t}{T}} \tag{15.22}$$

where v_0 is the speed from part (a), and $T = 5\,\text{h}$. As the plane lands at the pet hospital's emergency runway, Bob peers into the cage to discover that Algernon is dead! In order to fill out his death report, the veterinarian needs to know: *when did Algernon die?* Assuming he died after his own hamster watch registered three hours, derive an expression for the elapsed time on the veterinarian's clock at the moment of Algernon's death.

Solution : ⟨Sniffle⟩. We have $\beta(t) = \frac{4}{5}\left(1-\frac{t}{T}\right)^{1/2}$. We set

$$T' = \int\limits_0^{T^*}\!dt\,\sqrt{1-\beta^2(t)} \tag{15.23}$$

where $T' = 3\,\text{hr}$ and T^* is the time of death in Bob's frame. We write $\beta_0 = \frac{4}{5}$ and $\gamma_0 = (1-\beta_0^2)^{-1/2} = \frac{5}{3}$. Note that $T'/T = \sqrt{1-\beta_0^2} = \gamma_0^{-1}$.

Rescaling by writing $\zeta = t/T$, we have

$$\begin{aligned}\frac{T'}{T} = \gamma_0^{-1} &= \int\limits_0^{T^*/T}\!d\zeta\,\sqrt{1-\beta_0^2+\beta_0^2\,\zeta} \\ &= \frac{2}{3\beta_0^2}\left[\left(1-\beta_0^2+\beta_0^2\,\frac{T^*}{T}\right)^{3/2} - (1-\beta_0^2)^{3/2}\right] \\ &= \frac{2}{3\gamma_0}\cdot\frac{1}{\gamma_0^2-1}\left[\left(1+(\gamma_0^2-1)\,\frac{T^*}{T}\right)^{3/2} - 1\right] \ .\end{aligned} \tag{15.24}$$

Solving for T^*/T we have

$$\frac{T^*}{T} = \frac{\left(\frac{3}{2}\,\gamma_0^2 - \frac{1}{2}\right)^{2/3} - 1}{\gamma_0^2 - 1} \ . \tag{15.25}$$

With $\gamma_0 = \frac{5}{3}$ we obtain

$$\frac{T^*}{T} = \frac{9}{16}\left[\left(\tfrac{11}{3}\right)^{2/3} - 1\right] = 0.77502\ldots \tag{15.26}$$

Thus, $T^* = 3.875\,\text{hr} = 3$ hr 52 min 50.5 sec after Bob starts swinging.

(c) Identify at least three practical problems with Bob's scheme.

Solution : As you can imagine, student responses to this part were varied and generally sarcastic. *E.g.* "the atmosphere would ignite," or "Bob's arm would fall off," or "Algernon's remains would be found on the inside of the far wall of the cage, squashed flatter than a coat of semi-gloss paint," *etc.*

15.3 Four-Vectors and Lorentz Transformations

We have spoken thus far about different reference frames. So how precisely do the coordinates (t, x, y, z) transform between frames K and K'? In classical mechanics, we have $t = t'$ and $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{u}\,t$, according to fig. 15.3. This yields the *Galilean transformation*,

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_x & 1 & 0 & 0 \\ u_y & 0 & 1 & 0 \\ u_z & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \ . \tag{15.27}$$

Such a transformation does not leave intervals invariant.

Let us define the *four-vector* x^μ as

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} \ . \tag{15.28}$$

Thus, $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. In order for intervals to be invariant, the transformation between x^μ in frame K and $x'^{\,\mu}$ in frame K' must be linear:

$$x^\mu = L^\mu_{\ \nu}\, x'^{\,\nu} \ , \tag{15.29}$$

where we are using the Einstein convention of summing over repeated indices. We define the *Minkowski metric tensor* $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{15.30}$$

Clearly $g = g^{\rm t}$ is a symmetric matrix.

Note that the matrix $L^\alpha_{\ \beta}$ has one raised index and one lowered index. For the notation we are about to develop, it is very important to distinguish raised from lowered indices. To raise or lower an index, we use the metric tensor. For example,

$$x_\mu = g_{\mu\nu}\, x^\nu = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} \ . \tag{15.31}$$

The act of summing over an identical raised and lowered index is called *index contraction*. Note that

$$g^\mu_{\ \nu} = g^{\mu\rho}\, g_{\rho\nu} = \delta^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{15.32}$$

Now let's investigate the invariance of the interval. We must have $x'^{\,\mu}\, x'_\mu = x^\mu\, x_\mu$. Note that

$$\begin{aligned} x^\mu\, x_\mu &= L^\mu_{\ \alpha}\, x'^{\,\alpha}\, L_\mu^{\ \beta}\, x'_\beta \\ &= \big(L^\mu_{\ \alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta}\big)\, x'^{\,\alpha}\, x'^{\,\beta} \ , \end{aligned} \tag{15.33}$$

from which we conclude

$$L^\mu_{\ \alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta} = g_{\alpha\beta} \ . \tag{15.34}$$

This result also may be written in other ways:

$$L^{\mu\alpha}\, g_{\mu\nu}\, L^{\nu\beta} = g^{\alpha\beta} \quad , \quad L^{{\rm t}\,\mu}_{\alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta} = g_{\alpha\beta} \tag{15.35}$$

Another way to write this equation is $L^{\rm t}\, g\, L = g$. A rank-4 matrix which satisfies this constraint, with $g = {\rm diag}(+,-,-,-)$ is an element of the group ${\rm O}(3,1)$, known as the *Lorentz group*.

Let us now count the freedoms in L. As a 4×4 real matrix, it contains 16 elements. The matrix $L^{\rm t}\, g\, L$ is a symmetric 4×4 matrix, which contains 10 independent elements: 4 along the diagonal and 6 above the diagonal. Thus, there are 10 constraints on 16 elements of L, and we conclude that the group ${\rm O}(3,1)$ is 6-dimensional. This is also the dimension of the four-dimensional orthogonal group ${\rm O}(4)$, by the way. Three of these six parameters may be taken to be the Euler angles. That is, the group ${\rm O}(3)$ constitutes a three-dimensional *subgroup* of the Lorentz group ${\rm O}(3,1)$, with elements

$$L^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} \ , \tag{15.36}$$

where $R^{\mathrm{t}}R = MI$, *i.e.* $R \in \mathrm{O}(3)$ is a rank-3 orthogonal matrix, parameterized by the three Euler angles (ϕ, θ, ψ). The remaining three parameters form a vector $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ and define a second class of Lorentz transformations, called boosts:[2]

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1 + (\gamma-1)\,\hat\beta_x\,\hat\beta_x & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & 1 + (\gamma-1)\,\hat\beta_y\,\hat\beta_y & (\gamma-1)\,\hat\beta_y\,\hat\beta_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat\beta_x\,\hat\beta_z & (\gamma-1)\,\hat\beta_y\,\hat\beta_z & 1 + (\gamma-1)\,\hat\beta_z\,\hat\beta_z \end{pmatrix} \quad , \tag{15.37}$$

where

$$\widehat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} \qquad , \qquad \gamma = \big(1 - \boldsymbol{\beta}^2\big)^{-1/2} \ . \tag{15.38}$$

IMPORTANT : Since the components of $\boldsymbol{\beta}$ are not the spatial components of a four vector, we will only write these components with a lowered index, as β_i, with $i = 1, 2, 3$. We will not write β^i with a raised index, but if we did, we'd mean the same thing, *i.e.* $\beta^i = \beta_i$. Note that for the spatial components of a 4-vector like x^μ, we have $x_i = -x^i$.

Let's look at a simple example, where $\beta_x = \beta$ and $\beta_y = \beta_z = 0$. Then

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{15.39}$$

The effect of this Lorentz transformation $x^\mu = L^{\mu}{}_{\nu}\, x'^{\,\nu}$ is thus

$$ct = \gamma ct' + \gamma\beta x' \tag{15.40}$$

$$x = \gamma\beta ct' + \gamma x' \ . \tag{15.41}$$

How fast is the origin of K' moving in the K frame? We have $dx' = 0$ and thus

$$\frac{1}{c}\,\frac{dx}{dt} = \frac{\gamma\beta\, c\, dt'}{\gamma\, c\, dt'} = \beta \ . \tag{15.42}$$

Thus, $u = \beta c$, *i.e.* $\beta = u/c$.

It is convenient to take advantage of the fact that $\mathrm{P}^{\boldsymbol{\beta}}_{ij} \equiv \hat\beta_i\, \hat\beta_j$ is a *projection operator*, which satisfies $\big(\mathrm{P}^{\boldsymbol{\beta}}\big)^2 = \mathrm{P}^{\boldsymbol{\beta}}$. The action of $\mathrm{P}^{\boldsymbol{\beta}}_{ij}$ on any vector $\boldsymbol{\xi}$ is to project that vector onto the $\hat{\boldsymbol{\beta}}$ direction:

$$\mathrm{P}^{\boldsymbol{\beta}}\,\boldsymbol{\xi} = (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi})\,\hat{\boldsymbol{\beta}} \ . \tag{15.43}$$

We may now write the general Lorentz boost, with $\boldsymbol{\beta} = \boldsymbol{u}/c$, as

$$L = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\,\mathrm{P}^{\boldsymbol{\beta}} \end{pmatrix} \ , \tag{15.44}$$

[2]Unlike rotations, the boosts do not themselves define a subgroup of $\mathrm{O}(3,1)$.

where I is the 3×3 unit matrix, and where we write column and row vectors

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} \qquad , \qquad \boldsymbol{\beta}^{\rm t} = \begin{pmatrix} \beta_x & \beta_y & \beta_z \end{pmatrix} \tag{15.45}$$

as a mnemonic to help with matrix multiplications. We now have

$$\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\rm t} \\ \gamma\boldsymbol{\beta} & {\rm I} + (\gamma-1)\,{\rm P}^{\boldsymbol{\beta}} \end{pmatrix} \begin{pmatrix} ct' \\ \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} \gamma ct' + \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}' \\ \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,{\rm P}^{\boldsymbol{\beta}}\,\boldsymbol{x}' \end{pmatrix} . \tag{15.46}$$

Thus,

$$ct = \gamma ct' + \gamma\boldsymbol{\beta}\!\cdot\!\boldsymbol{x}' \tag{15.47}$$

$$\boldsymbol{x} = \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,(\hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{x}')\,\hat{\boldsymbol{\beta}} \ . \tag{15.48}$$

If we resolve $\boldsymbol{x}$ and $\boldsymbol{x}'$ into components parallel and perpendicular to $\boldsymbol{\beta}$, writing

$$x_{\parallel} = \hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{x} \qquad , \qquad \boldsymbol{x}_{\perp} = \boldsymbol{x} - (\hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{x})\,\hat{\boldsymbol{\beta}} \ , \tag{15.49}$$

with corresponding definitions for $x'_{\parallel}$ and $\boldsymbol{x}'_{\perp}$, the general Lorentz boost may be written as

$$ct = \gamma ct' + \gamma\beta x'_{\parallel} \tag{15.50}$$

$$x_{\parallel} = \gamma\beta ct' + \gamma x'_{\parallel} \tag{15.51}$$

$$\boldsymbol{x}_{\perp} = \boldsymbol{x}'_{\perp} \ . \tag{15.52}$$

Thus, the components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are parallel to $\boldsymbol{\beta}$ enter into a one-dimensional Lorentz boost along with t and t', as described by eqn. 15.41. The components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are perpendicular to $\boldsymbol{\beta}$ are unaffected by the boost.

Finally, the Lorentz group $O(3,1)$ is a group under multiplication, which means that if $L_{\rm a}$ and $L_{\rm b}$ are elements, then so is the product $L_{\rm a}\,L_{\rm b}$. Explicitly, we have

$$(L_{\rm a}\,L_{\rm b})^{\rm t}\,g\,L_{\rm a}\,L_{\rm b} = L_{\rm b}^{\rm t}\,(L_{\rm a}^{\rm t}\,g\,L_{\rm a})\,L_{\rm b} = L_{\rm b}^{\rm t}\,g\,L_{\rm b} = g \ . \tag{15.53}$$

15.3.1 Covariance and contravariance

Note that

$$\begin{aligned} L^{{\rm t}\,\mu}_{\alpha}\; g_{\mu\nu}\, L^{\nu}{}_{\beta} &= \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\alpha\beta} \ , \end{aligned} \tag{15.54}$$

since $\gamma^2\,(1-\beta^2)=1$. This is in fact the general way that tensors transform under a Lorentz transformation:

$$\text{covariant vectors}: \ x^\mu = L^\mu_{\ \nu}\, x'^{\,\nu} \tag{15.55}$$

$$\text{covariant tensors}: \ F^{\mu\nu} = L^\mu_{\ \alpha}\, L^\nu_{\ \beta}\, F'^{\,\alpha\beta} = L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} \tag{15.56}$$

Note how index contractions always involve one raised index and one lowered index. Raised indices are called *contravariant indices* and lowered indiced are called *covariant indices*. The transformation rules for contravariant vectors and tensors are

$$\text{contravariant vectors}: \ x_\mu = L_\mu^{\ \nu}\, x'_\nu \tag{15.57}$$

$$\text{contravariant tensors}: \ F_{\mu\nu} = L_\mu^{\ \alpha}\, L_\nu^{\ \beta}\, F'_{\ \alpha\beta} = L_\mu^{\ \alpha}\, F'_{\ \alpha\beta}\, L^{\mathrm{t}\beta}_{\ \ \nu} \tag{15.58}$$

A *Lorentz scalar* has no indices at all. For example,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \ , \tag{15.59}$$

is a Lorentz scalar. In this case, we have contracted a tensor with two four-vectors. The dot product of two four-vectors is also a Lorentz scalar:

$$\begin{aligned} a\cdot b \equiv a^\mu\, b_\mu &= g_{\mu\nu}\, a^\mu\, b^\nu \\ &= a^0\, b^0 - a^1\, b^1 - a^2\, b^2 - a^3\, b^3 \\ &= a^0\, b^0 - \boldsymbol{a}\cdot\boldsymbol{b} \ . \end{aligned} \tag{15.60}$$

Note that the dot product $a\cdot b$ of four-vectors is invariant under a simultaneous Lorentz transformation of both a^μ and b^μ, *i.e.* $a\cdot b = a'\cdot b'$. Indeed, this invariance is the very definition of what it means for something to be a Lorentz scalar. Derivatives with respect to covariant vectors yield contravariant vectors:

$$\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f \quad , \quad \frac{\partial A^\mu}{\partial x^\nu} = \partial_\nu A^\mu \equiv B^\mu_{\ \nu} \quad , \quad \frac{\partial B^\mu_{\ \nu}}{\partial x^\lambda} = \partial_\lambda B^\mu_{\ \nu} \equiv C^\mu_{\ \nu\lambda}$$

et cetera. Note that differentiation with respect to the *co*variant vector x^μ is expressed by the *contra*variant differential operator ∂_μ:

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\, , \, \frac{\partial}{\partial x}\, , \, \frac{\partial}{\partial y}\, , \, \frac{\partial}{\partial z}\right) \tag{15.61}$$

$$\frac{\partial}{\partial x_\mu} \equiv \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\, , \, -\frac{\partial}{\partial x}\, , \, -\frac{\partial}{\partial y}\, , \, -\frac{\partial}{\partial z}\right) \ . \tag{15.62}$$

The contraction $\Box \equiv \partial^\mu \partial_\mu$ is a Lorentz scalar differential operator, called the *D'Alembertian*:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \ . \tag{15.63}$$

The Helmholtz equation for scalar waves propagating with speed c can thus be written in compact form as $\Box\phi = 0$.

15.3.2 What to do if you hate raised and lowered indices

Admittedly, this covariant and contravariant business takes some getting used to. Ultimately, it helps to keep straight which indices transform according to L (covariantly) and which transform according to L^{t} (contravariantly). If you find all this irksome, the raising and lowering can be safely ignored. We define the position four-vector as before, but with no difference between raised and lowered indices. In fact, we can just represent all vectors and tensors with lowered indices exclusively, writing *e.g.* $\mathrm{x}_\mu = (ct, x, y, z)$. The metric tensor is $\mathrm{g} = \mathrm{diag}(+,-,-,-)$ as before. The dot product of two four-vectors is

$$\mathrm{x}\cdot\mathrm{y} = \mathrm{g}_{\mu\nu}\,\mathrm{x}_\mu\,\mathrm{y}_\nu \ . \tag{15.64}$$

The Lorentz transformation is

$$\mathrm{x}_\mu = \mathrm{L}_{\mu\nu}\,\mathrm{x}'_\nu \ . \tag{15.65}$$

Since this preserves intervals, we must have

$$\begin{aligned}\mathrm{g}_{\mu\nu}\,\mathrm{x}_\mu\,\mathrm{y}_\nu &= \mathrm{g}_{\mu\nu}\,\mathrm{L}_{\mu\alpha}\,\mathrm{x}'_\alpha\,\mathrm{L}_{\nu\beta}\,\mathrm{y}'_\beta \\ &= \big(\mathrm{L}^{\mathrm{t}}_{\alpha\mu}\,\mathrm{g}_{\mu\nu}\,\mathrm{L}_{\nu\beta}\big)\,\mathrm{x}'_\alpha\,\mathrm{y}'_\beta \ ,\end{aligned} \tag{15.66}$$

which entails

$$\mathrm{L}^{\mathrm{t}}_{\alpha\mu}\,\mathrm{g}_{\mu\nu}\,\mathrm{L}_{\nu\beta} = \mathrm{g}_{\alpha\beta} \ . \tag{15.67}$$

In terms of the quantity $L^\mu{}_\nu$ defined above, we have $\mathrm{L}_{\mu\nu} = L^\mu{}_\nu$. In this convention, we could completely avoid raised indices, or we could simply make no distinction, taking $\mathrm{x}^\mu = \mathrm{x}_\mu$ and $\mathrm{L}_{\mu\nu} = \mathrm{L}^\mu{}_\nu = \mathrm{L}^{\mu\nu}$, *etc.*

15.3.3 Comparing frames

Suppose in the K frame we have a measuring rod which is at rest. What is its length as measured in the K' frame? Recall K' moves with velocity $\boldsymbol{u} = u\,\hat{\boldsymbol{x}}$ with respect to K. From the Lorentz transformation in eqn. 15.41, we have

$$x_1 = \gamma(x'_1 + \beta c\,t'_1) \tag{15.68}$$

$$x_2 = \gamma(x'_2 + \beta c\,t'_2) \ , \tag{15.69}$$

where $x_{1,2}$ are the positions of the ends of the rod in frame K. The rod's length in any frame is the instantaneous spatial separation of its ends. Thus, we set $t'_1 = t'_2$ and compute the separation $\Delta x' = x'_2 - x'_1$:

$$\Delta x = \gamma\,\Delta x' \qquad \Longrightarrow \qquad \Delta x' = \gamma^{-1}\Delta x = \big(1-\beta^2\big)^{1/2}\,\Delta x \ . \tag{15.70}$$

The *proper length* ℓ_0 of a rod is its instantaneous end-to-end separation in its rest frame. We see that

$$\ell(\beta) = \big(1-\beta^2\big)^{1/2}\,\ell_0 \ , \tag{15.71}$$

so the length is always greatest in the rest frame. This is an example of a *Lorentz-Fitzgerald contraction*. Note that the *transverse* dimensions do not contract:

$$\Delta y' = \Delta y \qquad , \qquad \Delta z' = \Delta z \ . \tag{15.72}$$

Thus, the *volume contraction* of a bulk object is given by its length contraction: $\mathcal{V}' = \gamma^{-1}\,\mathcal{V}$.

A striking example of relativistic issues of length, time, and simultaneity is the famous 'pole and the barn' paradox, described in the Appendix (section). Here we illustrate some essential features via two examples.

15.3.4 Example I

Next, let's analyze the situation depicted in fig. 15.3. In the K' frame, we'll denote the following spacetime points:

$$A' = \begin{pmatrix} ct' \\ -d \end{pmatrix} \quad , \quad B' = \begin{pmatrix} ct' \\ +d \end{pmatrix} \quad , \quad S'_- = \begin{pmatrix} ct' \\ -ct' \end{pmatrix} \quad , \quad S'_- = \begin{pmatrix} ct' \\ +ct' \end{pmatrix} \ . \tag{15.73}$$

Note that the origin in K' is given by $O' = (ct', 0)$. Here we are setting $y = y' = z = z' = 0$ and dealing only with one spatial dimension. The points $S'_\pm$ denote the left-moving (S'_-) and right-moving (S'_+) wavefronts. We see that the arrival of the signal S'_1 at A' requires $S'_1 = A'$, hence $ct' = d$. The same result holds when we set $S'_2 = B'$ for the arrival of the right-moving wavefront at B'.

We now use the Lorentz transformation

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta \\ \gamma\,\beta & \gamma \end{pmatrix} \tag{15.74}$$

to transform to the K frame. Thus,

$$A = \begin{pmatrix} ct^*_A \\ x^*_A \end{pmatrix} = LA' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} d \\ -d \end{pmatrix} = \gamma(1-\beta)d \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{15.75}$$

$$B = \begin{pmatrix} ct^*_B \\ x^*_B \end{pmatrix} = LB' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} d \\ +d \end{pmatrix} = \gamma(1+\beta)d \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ . \tag{15.76}$$

Thus, $t^*_A = \gamma(1-\beta)d/c$ and $t^*_B = \gamma(1+\beta)d/c$. Thus, the two events are *not* simultaneous in K. The arrival at A is first.

15.3.5 Example II

Consider a rod of length ℓ_0 extending from the origin to the point $\ell_0\,\hat{\boldsymbol{x}}$ at rest in frame K. In the frame K, the two ends of the rod are located at spacetime coordinates

$$A = \begin{pmatrix} ct \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ct \\ \ell_0 \end{pmatrix} \ , \tag{15.77}$$

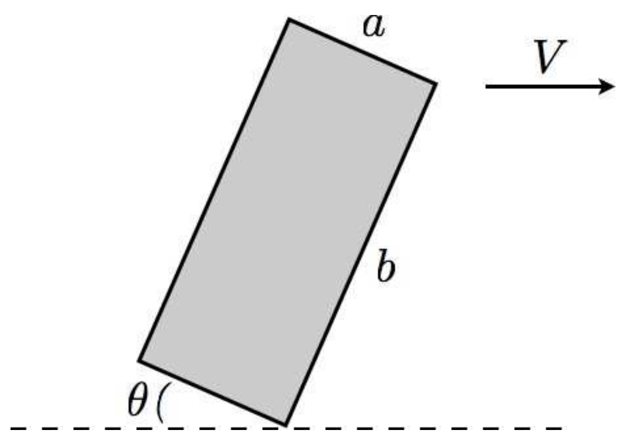


Figure 15.5: A rectangular plate moving at velocity $\boldsymbol{V} = V\,\hat{\boldsymbol{x}}$.

respectively. Now consider the origin in frame K'. Its spacetime coordinates are

$$C' = \begin{pmatrix} ct' \\ 0 \end{pmatrix} \ . \tag{15.78}$$

To an observer in the K frame, we have

$$C = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct' \\ \gamma\beta ct' \end{pmatrix} \ . \tag{15.79}$$

Now consider two events. The first event is the coincidence of A with C, *i.e.* the origin of K' instantaneously coincides with the origin of K. Setting $A = C$ we obtain $t = t' = 0$. The second event is the coincidence of B with C. Setting $B = C$ we obtain $t = l_0/\beta c$ and $t' = \ell_0/\gamma\beta c$. Note that $t = \ell(\beta)/\beta c$, *i.e.* due to the Lorentz-Fitzgerald contraction of the rod as seen in the K' frame, where $\ell(\beta) = \ell_0/\gamma$.

15.3.6 Deformation of a rectangular plate

Problem: A rectangular plate of dimensions $a \times b$ moves at relativistic velocity $\boldsymbol{V} = V\hat{\boldsymbol{x}}$ as shown in fig. 15.5. In the rest frame of the rectangle, the a side makes an angle θ with respect to the $\hat{\boldsymbol{x}}$ axis. Describe in detail and sketch the shape of the plate as measured by an observer in the laboratory frame. Indicate the lengths of all sides and the values of all interior angles. Evaluate your expressions for the case $\theta = \frac{1}{4}\pi$ and $V = \sqrt{\frac{2}{3}}\,c$.

Solution: An observer in the laboratory frame will measure lengths parallel to $\hat{\boldsymbol{x}}$ to be Lorentz contracted by a factor γ^{-1}, where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = V/c$. Lengths perpendicular to $\hat{\boldsymbol{x}}$ remain unaffected. Thus, we have the situation depicted in fig. 15.6. Simple trigonometry then says

$$\tan\phi = \gamma\tan\theta \quad , \quad \tan\tilde{\phi} = \gamma^{-1}\tan\theta \ ,$$

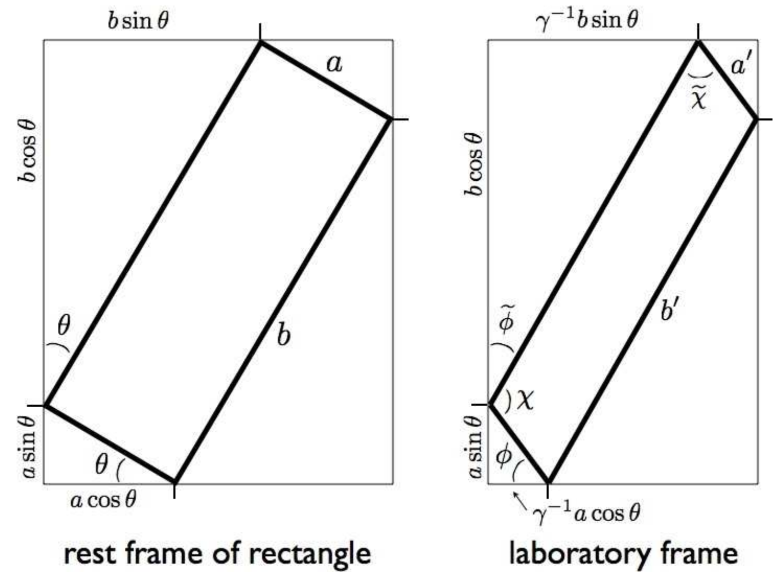


Figure 15.6: Relativistic deformation of the rectangular plate.

as well as

$$
\begin{aligned}
a' &= a\sqrt{\gamma^{-2}\cos^2\!\theta + \sin^2\!\theta} = a\sqrt{1-\beta^2\cos^2\!\theta} \\
b' &= b\sqrt{\gamma^{-2}\sin^2\!\theta + \cos^2\!\theta} = b\sqrt{1-\beta^2\sin^2\!\theta}\ .
\end{aligned}
$$

The plate deforms to a parallelogram, with internal angles

$$
\begin{aligned}
\chi &= \tfrac{1}{2}\pi + \tan^{-1}(\gamma\tan\theta) - \tan^{-1}(\gamma^{-1}\tan\theta) \\
\tilde{\chi} &= \tfrac{1}{2}\pi - \tan^{-1}(\gamma\tan\theta) + \tan^{-1}(\gamma^{-1}\tan\theta)\ .
\end{aligned}
$$

Note that the area of the plate as measured in the laboratory frame is

$$
\begin{aligned}
\varOmega' &= a'\,b'\,\sin\chi = a'\,b'\,\cos(\phi-\tilde{\phi}) \\
&= \gamma^{-1}\,\varOmega\ ,
\end{aligned}
$$

where $\varOmega = ab$ is the proper area. The area contraction factor is γ^{-1} and not γ^{-2} (or γ^{-3} in a three-dimensional system) because only the parallel dimension gets contracted.

Setting $V=\sqrt{\frac{2}{3}}\,c$ gives $\gamma=\sqrt{3}$, and with $\theta=\frac{1}{4}\pi$ we have $\phi=\frac{1}{3}\pi$ and $\tilde{\phi}=\frac{1}{6}\pi$. The interior angles are then $\chi=\frac{2}{3}\pi$ and $\tilde{\chi}=\frac{1}{3}\pi$. The side lengths are $a'=\sqrt{\frac{2}{3}}\,a$ and $b'=\sqrt{\frac{2}{3}}\,b$.

15.3.7 Transformation of velocities

Let K' move at velocity $\boldsymbol{u} = c\boldsymbol{\beta}$ relative to K. The transformation from K' to K is given by the Lorentz boost,

$$L^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_x & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & 1+(\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z & 1+(\gamma-1)\,\hat{\beta}_z\,\hat{\beta}_z \end{pmatrix} . \tag{15.80}$$

Applying this, we have

$$dx^{\mu} = L^{\mu}_{\ \nu}\, dx'^{\,\nu} . \tag{15.81}$$

This yields

$$dx^0 = \gamma\, dx'^{\,0} + \gamma\, \boldsymbol{\beta} \cdot d\boldsymbol{x}' \tag{15.82}$$

$$d\boldsymbol{x} = \gamma\, \boldsymbol{\beta}\, dx'^{\,0} + d\boldsymbol{x}' + (\gamma-1)\,\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\cdot d\boldsymbol{x}' . \tag{15.83}$$

We then have

$$\begin{aligned} \boldsymbol{V} = c\,\frac{d\boldsymbol{x}}{dx^0} &= \frac{c\,\gamma\,\boldsymbol{\beta}\,dx'^{\,0} + c\,d\boldsymbol{x}' + c\,(\gamma-1)\,\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\cdot d\boldsymbol{x}'}{\gamma\,dx'^{\,0} + \gamma\,\boldsymbol{\beta}\cdot d\boldsymbol{x}'} \\ &= \frac{\boldsymbol{u} + \gamma^{-1}\,\boldsymbol{V}' + (1-\gamma^{-1})\,\hat{\boldsymbol{u}}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{V}'}{1 + \boldsymbol{u}\cdot\boldsymbol{V}'/c^2} . \end{aligned} \tag{15.84}$$

The second line is obtained by dividing both numerator and denominator by $dx'^{\,0}$, and then writing $\boldsymbol{V}' = d\boldsymbol{x}'/dx'^{\,0}$. There are two special limiting cases:

$$\text{velocities parallel } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 1\big) \quad \Longrightarrow \quad \boldsymbol{V} = \frac{(u+V')\,\hat{\boldsymbol{u}}}{1 + u\,V'/c^2} \tag{15.85}$$

$$\text{velocities perpendicular } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 0\big) \quad \Longrightarrow \quad \boldsymbol{V} = \boldsymbol{u} + \gamma^{-1}\boldsymbol{V}' . \tag{15.86}$$

Note that if either u or V' is equal to c, the resultant expression has $|\boldsymbol{V}| = c$ as well. One can't boost the speed of light!

Let's revisit briefly the example in section 15.3.4. For an observer, in the K frame, the relative velocity of S and A is $c+u$, because even though we must boost the velocity $-c\,\hat{\boldsymbol{x}}$ of the left-moving light wave by $u\,\hat{\boldsymbol{x}}$, the result is still $-c\,\hat{\boldsymbol{x}}$, according to our velocity addition formula. The distance between the emission and detection points is $d(\beta) = d/\gamma$. Thus,

$$t^*_A = \frac{d(\beta)}{c+u} = \frac{d}{\gamma} \cdot \frac{1}{c+u} = \frac{d}{\gamma c} \cdot \frac{1-\beta}{1-\beta^2} = \gamma\,(1-\beta)\,\frac{d}{c} . \tag{15.87}$$

This result is exactly as found in section 15.3.4 by other means. A corresponding analysis yields $t^*_B = \gamma\,(1+\beta)\,d/c$. again in agreement with the earlier result. Here, it is crucial to account for the Lorentz contraction of the distance between the source S and the observers A and B as measured in the K frame.

15.3.8 Four-velocity and four-acceleration

In nonrelativistic mechanics, the velocity $\boldsymbol{V} = \frac{d\boldsymbol{x}}{dt}$ is locally tangent to a particle's trajectory. In relativistic mechanics, one defines the *four-velocity*,

$$u^\alpha \equiv \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{\sqrt{1-\boldsymbol{\beta}^2}\, c\, dt} = \begin{pmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{pmatrix} \ , \tag{15.88}$$

which is locally tangent to the world line of a particle. Note that

$$g_{\alpha\beta}\, u^\alpha\, u^\beta = 1 \ . \tag{15.89}$$

The four-acceleration is defined as

$$w^\nu \equiv \frac{du^\nu}{ds} = \frac{d^2x^\nu}{ds^2} \ . \tag{15.90}$$

Note that $u \cdot w = 0$, so the 4-velocity and 4-acceleration are orthogonal with respect to the Minkowski metric.

15.4 Three Kinds of Relativistic Rockets

15.4.1 Constant acceleration model

Consider a rocket which undergoes constant acceleration along $\hat{\boldsymbol{x}}$. Clearly the rocket has no rest frame *per se*, because its velocity is changing. However, this poses no serious obstacle to discussing its relativistic motion. We consider a frame K' in which the rocket is *instantaneously* at rest. In such a frame, the rocket's 4-acceleration is $w'^{\,\alpha} = (0, a/c^2)$, where we suppress the transverse coordinates y and z. In an inertial frame K, we have

$$w^\alpha = \frac{d}{ds}\begin{pmatrix} \gamma \\ \gamma\beta \end{pmatrix} = \frac{\gamma}{c}\begin{pmatrix} \dot\gamma \\ \gamma\dot\beta + \dot\gamma\beta \end{pmatrix} \ . \tag{15.91}$$

Transforming $w'^{\,\alpha}$ into the K frame, we have

$$w^\alpha = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ a/c^2 \end{pmatrix} = \begin{pmatrix} \gamma\beta a/c^2 \\ \gamma a/c^2 \end{pmatrix} \ . \tag{15.92}$$

Taking the upper component, we obtain the equation

$$\dot\gamma = \frac{\beta a}{c} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{a}{c} \ , \tag{15.93}$$

the solution of which, with $\beta(0) = 0$, is

$$\beta(t) = \frac{at}{\sqrt{c^2 + a^2t^2}} \qquad , \qquad \gamma(t) = \sqrt{1 + \left(\frac{at}{c}\right)^2} \ . \tag{15.94}$$

The proper time for an observer moving with the rocket is thus

$$\tau = \int_0^t \frac{c\,dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a}\,\sinh^{-1}\left(\frac{at}{c}\right) .$$

For large times $t \gg c/a$, the proper time grows logarithmically in t, which is parametrically slower. To find the position of the rocket, we integrate $\dot{x} = c\beta$, and obtain, with $x(0) = 0$,

$$x(t) = \int_0^t \frac{a\,ct_1\,dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a}\left(\sqrt{c^2 + a^2\,t^2} \; - \; c\right) . \tag{15.95}$$

It is interesting to consider the situation in the frame K'. We then have

$$\beta(\tau) = \tanh(a\tau/c) \quad , \quad \gamma(\tau) = \cosh(a\tau/c) . \tag{15.96}$$

For an observer in the frame K', the distance he has traveled is $\Delta x'(\tau) = \Delta x(\tau)/\gamma(\tau)$, as we found in eqn. 15.70. Now $x(\tau) = (c^2/a)\big(\cosh(a\tau/c) - 1\big)$, hence

$$\Delta x'(\tau) = \frac{c^2}{a}\Big(1 - \text{sech}(a\tau/c)\Big) . \tag{15.97}$$

For $\tau \ll c/a$, we expand $\text{sech}(a\tau/c) \approx 1 - \frac{1}{2}(a\tau/c)^2$ and find $x'(\tau) = \frac{1}{2}a\tau^2$, which clearly is the nonrelativistic limit. For $\tau \to \infty$, however, we have $\Delta x'(\tau) \to c^2/a$ is *finite*! Thus, while the entire Universe is falling behind the accelerating observer, it all piles up at a *horizon* a distance c^2/a behind it, in the frame of the observer. The light from these receding objects is increasingly red-shifted (see section 15.6 below), until it is no longer visible. Thus, as John Baez describes it, the horizon is “a dark plane that appears to be swallowing the entire Universe!” In the frame of the inertial observer, however, nothing strange appears to be happening at all!

15.4.2 Constant force with decreasing mass

Suppose instead the rocket is subjected to a constant force F_0 in its instantaneous rest frame, and furthermore that the rocket's mass satisfies $m(\tau) = m_0(1 - \alpha\tau)$, where τ is the proper time for an observer moving with the rocket. Then from eqn. 15.93, we have

$$\begin{aligned}\frac{F_0}{m_0(1-\alpha\tau)} &= \frac{d(\gamma\beta)}{dt} = \gamma^{-1}\,\frac{d(\gamma\beta)}{d\tau} \\ &= \frac{1}{1-\beta^2}\,\frac{d\beta}{d\tau} = \frac{d}{d\tau}\,\tfrac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right) \quad ,\end{aligned} \tag{15.98}$$

after using the chain rule, and with $d\tau/dt = \gamma^{-1}$. Integrating, we find

$$\ln\left(\frac{1+\beta}{1-\beta}\right) = \frac{2F_0}{\alpha m_0 c}\,\ln\big(1 - \alpha\tau\big) \quad \Longrightarrow \quad \beta(\tau) = \frac{1-(1-\alpha\tau)^r}{1+(1-\alpha\tau)^r} \quad , \tag{15.99}$$

with $r = 2F_0/\alpha m_0 c$. As $\tau \to \alpha^{-1}$, the rocket loses all its mass, and it asymptotically approaches the speed of light.

It is convenient to write

$$\beta(\tau) = \tanh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] , \tag{15.100}$$

in which case

$$\gamma = \frac{dt}{d\tau} = \cosh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] \tag{15.101}$$

$$\frac{1}{c}\,\frac{dx}{d\tau} = \sinh\left[\frac{r}{2}\,\ln\left(\frac{1}{1-\alpha\tau}\right)\right] . \tag{15.102}$$

Integrating the first of these from $\tau = 0$ to $\tau = \alpha^{-1}$, we find $t^* \equiv t\big(\tau = \alpha^{-1}\big)$ is

$$t^* = \frac{1}{2\alpha}\int\limits_0^1 d\sigma\,\Big(\sigma^{-r/2} + \sigma^{r/2}\Big) = \begin{cases} \Big[\alpha^2 - \big(\frac{F_0}{mc}\big)^2\Big]^{-1}\alpha & \text{if } \alpha > \frac{F_0}{mc} \\ \\ \infty & \text{if } \alpha \le \frac{F_0}{mc} \ . \end{cases} \tag{15.103}$$

Since $\beta(\tau = \alpha^{-1}) = 1$, this is the time in the K frame when the rocket reaches the speed of light.

15.4.3 Constant *ejecta* velocity

Our third relativistic rocket model is a generalization of what is commonly known as the *rocket equation* in classical physics. The model is one of a rocket which is continually ejecting burnt fuel at a velocity $-u$ in the instantaneous rest frame of the rocket. The nonrelativistic rocket equation follows from overall momentum conservation:

$$dp_{\text{rocket}} + dp_{\text{fuel}} = d(mv) + (v-u)\,(-dm) = 0 \ , \tag{15.104}$$

since if $dm < 0$ is the differential change in rocket mass, the differential *ejecta* mass is $-dm$. This immediately gives

$$m\,dv + u\,dm = 0 \quad\Longrightarrow\quad v = u\,\ln\left(\frac{m_0}{m}\right) , \tag{15.105}$$

where the rocket is assumed to begin at rest, and where m_0 is the initial mass of the rocket. Note that as $m \to 0$ the rocket's speed increases without bound, which of course violates special relativity.

In relativistic mechanics, as we shall see in section 15.5, the rocket's momentum, as described by an inertial observer, is $p = \gamma m v$, and its energy is $\gamma m c^2$. We now write two equations for overall conservation of momentum and energy:

$$d(\gamma m v) + \gamma_{\rm e} v_{\rm e}\, dm_{\rm e} = 0 \tag{15.106}$$

$$d(\gamma m c^2) + \gamma_{\rm e}(dm_{\rm e}\, c^2) = 0 \ , \tag{15.107}$$

where $v_{\rm e}$ is the velocity of the *ejecta* in the inertial frame, $dm_{\rm e}$ is the differential mass of the *ejecta*, and $\gamma_{\rm e} = \big(1 - \frac{v_{\rm e}^2}{c^2}\big)^{-1/2}$. From the second of these equations, we have

$$\gamma_{\rm e}\, dm_{\rm e} = -d(\gamma m) \ , \tag{15.108}$$

which we can plug into the first equation to obtain

$$(v - v_{\rm e})\, d(\gamma m) + \gamma m\, dv = 0 \ . \tag{15.109}$$

Before solving this, we remark that eqn. 15.108 implies that $dm_{\rm e} < |dm|$ – the differential mass of the *ejecta* is less than the mass lost by the rocket! This is Einstein's famous equation $E = mc^2$ at work – more on this later.

To proceed, we need to use the parallel velocity addition formula of eqn. 15.85 to find $v_{\rm e}$:

$$v_{\rm e} = \frac{v - u}{1 - \frac{uv}{c^2}} \qquad \Longrightarrow \qquad v - v_{\rm e} = \frac{u\big(1 - \frac{v^2}{c^2}\big)}{\big(1 - \frac{uv}{c^2}\big)} \ . \tag{15.110}$$

We now define $\beta_u = u/c$, in which case eqn, 15.109 becomes

$$\beta_u\,(1 - \beta^2)\, d(\gamma m) + (1 - \beta\beta_u)\, \gamma m\, d\beta = 0 \ . \tag{15.111}$$

Using $d\gamma = \gamma^3 \beta\, d\beta$, we observe a felicitous cancellation of terms, leaving

$$\beta_u\, \frac{dm}{m} + \frac{d\beta}{1 - \beta^2} = 0 \ . \tag{15.112}$$

Integrating, we obtain

$$\beta = \tanh\left(\beta_u\, \ln \frac{m_0}{m}\right) \ . \tag{15.113}$$

Note that this agrees with the result of eqn. 15.100, if we take $\beta_u = F_0/\alpha m c$.

15.5 Relativistic Mechanics

Relativistic particle dynamics follows from an appropriately extended version of Hamilton's principle $\delta S = 0$. The action S must be a Lorentz scalar. The action for a free particle is

$$S\big[\boldsymbol{x}(t)\big] = -mc\int\limits_a^b ds = -mc^2 \int\limits_{t_a}^{t_b} dt\, \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \ . \tag{15.114}$$

Thus, the free particle Lagrangian is

$$L = -mc^2\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} = -mc^2 + \tfrac{1}{2}m\boldsymbol{v}^2 + \tfrac{1}{8}mc^2\left(\frac{\boldsymbol{v}^2}{c^2}\right)^2 + \ldots \ . \tag{15.115}$$

Thus, L can be written as an expansion in powers of $\boldsymbol{v}^2/c^2$. Note that $L(\boldsymbol{v}=0) = -mc^2$. We interpret this as $-U_0$, where $U_0 = mc^2$ is the *rest energy* of the particle. As a constant, it has no consequence for the equations of motion. The next term in L is the familiar nonrelativistic kinetic energy, $\frac{1}{2}m\boldsymbol{v}^2$. Higher order terms are smaller by increasing factors of $\beta^2 = v^2/c^2$.

We can add a potential $U(\boldsymbol{x},t)$ to obtain

$$L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = -mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}} - U(\boldsymbol{x},t) \ . \tag{15.116}$$

The momentum of the particle is

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m \dot{\boldsymbol{x}} \ . \tag{15.117}$$

The force is $\boldsymbol{F} = -\boldsymbol{\nabla} U$ as usual, and Newton's Second Law still reads $\dot{\boldsymbol{p}} = \boldsymbol{F}$. Note that

$$\dot{\boldsymbol{p}} = \gamma m\left(\dot{\boldsymbol{v}} + \frac{v\dot{v}}{c^2}\,\gamma^2\boldsymbol{v}\right) \ . \tag{15.118}$$

Thus, the force $\boldsymbol{F}$ is not necessarily in the direction of the acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$. The Hamiltonian, recall, is a function of coordinates and momenta, and is given by

$$H = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2} + U(\boldsymbol{x},t) \ . \tag{15.119}$$

Since $\partial L/\partial t = 0$ for our case, H is conserved by the motion of the particle. There are two limits of note:

$$|\boldsymbol{p}| \ll mc \quad \text{(non-relativistic)} \qquad : \ H = mc^2 + \frac{\boldsymbol{p}^2}{2m} + U + \mathcal{O}(p^4/m^4c^4) \tag{15.120}$$

$$|\boldsymbol{p}| \gg mc \quad \text{(ultra-relativistic)} \qquad : \ H = c|\boldsymbol{p}| + U + \mathcal{O}(mc/p) \ . \tag{15.121}$$

Expressed in terms of the coordinates and velocities, we have $H = E$, the total energy, with

$$E = \gamma mc^2 + U \ . \tag{15.122}$$

In particle physics applications, one often defines the kinetic energy T as

$$T = E - U - mc^2 = (\gamma - 1)mc^2 \ . \tag{15.123}$$

When electromagnetic fields are included,

$$\begin{aligned} L(\boldsymbol{x},\dot{\boldsymbol{x}},t) &= -mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}} - q\,\phi + \frac{q}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{x}} \\ &= -\gamma mc^2 - \frac{q}{c}\,A_\mu\,\frac{dx^\mu}{dt} \ , \end{aligned} \tag{15.124}$$

where the electromagnetic *4-potential* is $A^\mu = (\phi\,,\,\boldsymbol{A})$. Recall $A_\mu = g_{\mu\nu}\,A^\nu$ has the sign of its spatial components reversed. One the has

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m \dot{\boldsymbol{x}} + \frac{q}{c}\,\boldsymbol{A}\ , \tag{15.125}$$

and the Hamiltonian is

$$H = \sqrt{m^2c^4 + \Big(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\Big)^2} + q\,\phi\ . \tag{15.126}$$

15.5.1 Relativistic harmonic oscillator

From $E = \gamma mc^2 + U$, we have

$$\dot{x}^2 = c^2\left[1 - \left(\frac{mc^2}{E - U(x)}\right)^2\right]\ . \tag{15.127}$$

Consider the one-dimensional harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$. We define the turning points as $x = \pm b$, satisfying

$$E - mc^2 = U(\pm b) = \tfrac{1}{2}kb^2\ . \tag{15.128}$$

Now define the angle θ via $x \equiv b\cos\theta$, and further define the dimensionless parameter $\epsilon = kb^2/4mc^2$. Then, after some manipulations, one obtains

$$\dot{\theta} = \omega_0\ \frac{\sqrt{1+\epsilon\sin^2\!\theta}}{1+2\epsilon\sin^2\!\theta}\ , \tag{15.129}$$

with $\omega_0 = \sqrt{k/m}$ as in the nonrelativistic case. Hence, the problem is reduced to quadratures (a quaint way of saying 'doing an an integral'):

$$t(\theta) - t_0 = \omega_0^{-1}\!\int\limits_{\theta_0}^{\theta}\! d\vartheta\ \frac{1+2\epsilon\sin^2\!\vartheta}{\sqrt{1+\epsilon\sin^2\!\vartheta}}\ . \tag{15.130}$$

While the result can be expressed in terms of elliptic integrals, such an expression is not particularly illuminating. Here we will content ourselves with computing the period $T(\epsilon)$:

$$T(\epsilon) = \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}}\! d\vartheta\ \frac{1+2\epsilon\sin^2\!\vartheta}{\sqrt{1+\epsilon\sin^2\!\vartheta}} \tag{15.131}$$

$$= \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}}\! d\vartheta\,\Big(1+\tfrac{3}{2}\epsilon\sin^2\!\vartheta - \tfrac{5}{8}\epsilon^2\sin^4\!\vartheta + \ldots\Big)$$

$$= \frac{2\pi}{\omega_0}\cdot\Big\{1+\tfrac{3}{4}\epsilon - \tfrac{15}{64}\epsilon^2 + \ldots\Big\}\ . \tag{15.132}$$

Thus, for the relativistic harmonic oscillator, the period does depend on the amplitude, unlike the nonrelativistic case.

15.5.2 Energy-momentum 4-vector

Let's focus on the case where $U(\boldsymbol{x}) = 0$. This is in fact a realistic assumption for subatomic particles, which propagate freely between collision events.

The differential proper time for a particle is given by

$$d\tau = \frac{ds}{c} = \gamma^{-1}\, dt \ , \tag{15.133}$$

where $x^\mu = (ct, \boldsymbol{x})$ are coordinates for the particle in an inertial frame. Thus,

$$\boldsymbol{p} = \gamma m \dot{\boldsymbol{x}} = m\,\frac{d\boldsymbol{x}}{d\tau} \qquad , \qquad \frac{E}{c} = mc\gamma = m\,\frac{dx^0}{d\tau} \ , \tag{15.134}$$

with $x^0 = ct$. Thus, we can write the *energy-momentum 4-vector* as

$$p^\mu = m\,\frac{dx^\mu}{d\tau} = \begin{pmatrix} E/c \\ p^x \\ p^y \\ p^z \end{pmatrix} \ . \tag{15.135}$$

Note that $p^\nu = mcu^\nu$, where u^ν is the 4-velocity of eqn. 15.88. The four-momentum satisfies the relation

$$p^\mu\, p_\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2 c^2 \ . \tag{15.136}$$

The relativistic generalization of force is

$$f^\mu = \frac{dp^\mu}{d\tau} = \big(\gamma \boldsymbol{F}\!\cdot\!\boldsymbol{v}/c \,,\, \gamma\boldsymbol{F}\big) \ , \tag{15.137}$$

where $\boldsymbol{F} = d\boldsymbol{p}/dt$ as usual.

The energy-momentum four-vector transforms covariantly under a Lorentz transformation. This means

$$p^\mu = L^\mu_{\ \nu}\, p'^{\,\nu} \ . \tag{15.138}$$

If frame K' moves with velocity $\boldsymbol{u} = c\beta\,\hat{\boldsymbol{x}}$ relative to frame K, then

$$\frac{E}{c} = \frac{c^{-1}E' + \beta\, p'^x}{\sqrt{1-\beta^2}} \quad , \quad p^x = \frac{p'^x + \beta c^{-1} E'}{\sqrt{1-\beta^2}} \quad , \quad p^y = p'^y \quad , \quad p^z = p'^z \ . \tag{15.139}$$

In general, from eqns. 15.50, 15.51, and 15.52, we have

$$\frac{E}{c} = \gamma\frac{E'}{c} + \gamma\beta p'_{\|} \tag{15.140}$$

$$p_{\|} = \gamma\beta\frac{E}{c} + \gamma p'_{\|} \tag{15.141}$$

$$\boldsymbol{p}_\perp = \boldsymbol{p}'_\perp \tag{15.142}$$

where $p_{\|} = \hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{p}$ and $\boldsymbol{p}_\perp = \boldsymbol{p} - (\hat{\boldsymbol{\beta}}\!\cdot\!\boldsymbol{p})\,\hat{\boldsymbol{\beta}}$.

15.5.3 4-momentum for massless particles

For a massless particle, such as a photon, we have $p^\mu p_\mu = 0$, which means $E^2 = \boldsymbol{p}^2 c^2$. The 4-momentum may then be written $p^\mu = \big(|\boldsymbol{p}|\,,\,\boldsymbol{p}\big)$. We define the *4-wavevector* k^μ by the relation $p^\mu = \hbar k^\mu$, where $\hbar = h/2\pi$ and h is Planck's constant. We also write $\omega = ck$, with $E = \hbar\omega$.

15.6 Relativistic Doppler Effect

The 4-wavevector $k^\mu = \big(\omega/c\,,\,\boldsymbol{k}\big)$ for electromagnetic radiation satisfies $k^\mu k_\mu = 0$. The energy-momentum 4-vector is $p^\mu = \hbar k^\mu$. The phase $\phi(x^\mu) = -k_\mu x^\mu = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ of a plane wave is a Lorentz scalar. This means that the total number of wave crests (*i.e.* $\phi = 2\pi n$) emitted by a source will be the total number observed by a detector.

Suppose a moving source emits radiation of angular frequency ω' in its rest frame. Then

$$\begin{aligned} k'^{\,\mu} &= L^\mu_{\ \nu}(-\boldsymbol{\beta})\,k^\nu \\ &= \begin{pmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\,\hat\beta_x\,\hat\beta_x & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_z \\ -\gamma\,\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & 1+(\gamma-1)\,\hat\beta_y\,\hat\beta_y & (\gamma-1)\,\hat\beta_y\,\hat\beta_z \\ -\gamma\,\beta_z & (\gamma-1)\,\hat\beta_x\,\hat\beta_z & (\gamma-1)\,\hat\beta_y\,\hat\beta_z & 1+(\gamma-1)\,\hat\beta_z\,\hat\beta_z \end{pmatrix} \begin{pmatrix} \omega/c \\ k^x \\ k^y \\ k^z \end{pmatrix} . \end{aligned} \tag{15.143}$$

This gives

$$\frac{\omega'}{c} = \gamma\,\frac{\omega}{c} - \gamma\,\boldsymbol{\beta}\cdot\boldsymbol{k} = \gamma\,\frac{\omega}{c}\,(1-\beta\cos\theta)\ , \tag{15.144}$$

where $\theta = \cos^{-1}(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{k}})$ is the angle measured in K between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{k}}$. Solving for ω, we have

$$\omega = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\,\omega_0\ , \tag{15.145}$$

where $\omega_0 = \omega'$ is the angular frequency in the rest frame of the moving source. Thus,

$$\theta = 0 \quad\Rightarrow\quad \text{source approaching} \quad\Rightarrow\quad \omega = \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_0 \tag{15.146}$$

$$\theta = \tfrac{1}{2}\pi \quad\Rightarrow\quad \text{source perpendicular} \quad\Rightarrow\quad \omega = \sqrt{1-\beta^2}\,\omega_0 \tag{15.147}$$

$$\theta = \pi \quad\Rightarrow\quad \text{source receding} \quad\Rightarrow\quad \omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega_0\ . \tag{15.148}$$

Recall the non-relativistic Doppler effect:

$$\omega = \frac{\omega_0}{1-(V/c)\cos\theta}\ . \tag{15.149}$$

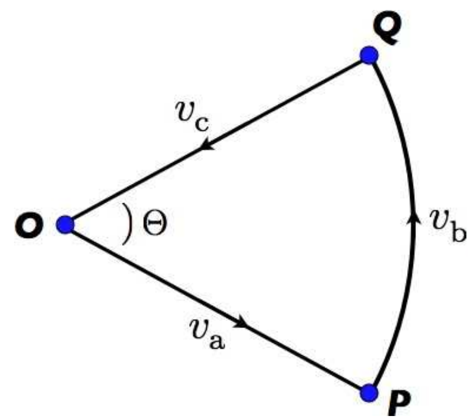


Figure 15.7: Alice's big adventure.

We see that approaching sources have their frequencies shifted higher; this is called the *blue shift*, since blue light is on the high frequency (short wavelength) end of the optical spectrum. By the same token, receding sources are *red-shifted* to lower frequencies.

15.6.1 Romantic example

Alice and Bob have a "May-December" thang going on. Bob is May and Alice December, if you get my drift. The social stigma is too much to bear! To rectify this, they decide that Alice should take a ride in a space ship. Alice's itinerary takes her along a sector of a circle of radius R and angular span of $\Theta = 1$ radian, as depicted in fig. 15.7. Define $\text{O} \equiv (r = 0)$, $\text{P} \equiv (r = R, \phi = -\frac{1}{2}\Theta)$, and $\text{Q} \equiv (r = R, \phi = \frac{1}{2}\Theta)$. Alice's speed along the first leg (straight from O to P) is $v_a = \frac{3}{5}\,c$. Her speed along the second leg (an arc from P to Q) is $v_b = \frac{12}{13}\,c$. The final leg (straight from Q to O) she travels at speed $v_c = \frac{4}{5}\,c$. Remember that the length of an circular arc of radius R and angular spread α (radians) is $\ell = \alpha R$.

`(a)` Alice and Bob synchronize watches at the moment of Alice's departure. What is the elapsed time on Bob's watch when Alice returns? What is the elapsed time on Alice's watch? What must R be in order for them to erase their initial 30 year age difference?

Solution : In Bob's frame, Alice's trip takes a time

$$\begin{aligned}\Delta t &= \frac{R}{c\beta_a} + \frac{R\Theta}{c\beta_b} + \frac{R}{c\beta_c} \\ &= \frac{R}{c}\left(\tfrac{5}{3} + \tfrac{13}{12} + \tfrac{5}{4}\right) = \frac{4R}{c}\ .\end{aligned} \tag{15.150}$$

The elapsed time on Alice's watch is

$$\begin{aligned}\Delta t' &= \frac{R}{c\gamma_a\beta_a} + \frac{R\Theta}{c\gamma_b\beta_b} + \frac{R}{c\gamma_c\beta_c} \\ &= \frac{R}{c}\left(\tfrac{5}{3}\cdot\tfrac{4}{5} + \tfrac{13}{12}\cdot\tfrac{5}{13} + \tfrac{5}{4}\cdot\tfrac{3}{5}\right) = \frac{5R}{2c}\ .\end{aligned} \tag{15.151}$$

Thus, $\Delta T = \Delta t - \Delta t' = 3R/2c$ and setting $\Delta T = 30\,\text{yr}$, we find $R = 20\,\text{ly}$. So Bob will have aged 80 years and Alice 50 years upon her return. (Maybe this isn't such a good plan after all.)

(b) As a signal of her undying love for Bob, Alice continually shines a beacon throughout her trip. The beacon produces monochromatic light at wavelength $\lambda_0 = 6000\,\text{Å}$ (frequency $f_0 = c/\lambda_0 = 5 \times 10^{14}$ Hz). Every night, Bob peers into the sky (with a radiotelescope), hopefully looking for Alice's signal. What frequencies f_a, f_b, and f_c does Bob see?

Solution : Using the relativistic Doppler formula, we have

$$\begin{aligned} f_\text{a} &= \sqrt{\frac{1-\beta_\text{a}}{1+\beta_\text{a}}} \times f_0 = \tfrac{1}{2} f_0 \\ f_\text{b} &= \sqrt{1-\beta_\text{b}^2} \times f_0 = \tfrac{5}{13} f_0 \\ f_\text{c} &= \sqrt{\frac{1+\beta_\text{c}}{1-\beta_\text{c}}} \times f_0 = 3 f_0 \ . \end{aligned} \tag{15.152}$$

(c) Show that the total number of wave crests counted by Bob is the same as the number emitted by Alice, over the entire trip.

Solution : Consider first the O–P leg of Alice's trip. The proper time elapsed on Alice's watch during this leg is $\Delta t'_a = R/c\gamma_a\beta_a$, hence she emits $N'_a = Rf_0/c\gamma_a\beta_a$ wavefronts during this leg. Similar considerations hold for the P–Q and Q–O legs, so $N'_b = R\Theta f_0/c\gamma_b\beta_b$ and $N'_c = Rf_0/c\gamma_c\beta_c$.

Although the duration of the O–P segment of Alice's trip takes a time $\Delta t_\text{a} = R/c\beta_\text{a}$ in Bob's frame, he keeps receiving the signal at the Doppler-shifted frequency f_a until the wavefront emitted when Alice arrives at P makes its way back to Bob. That takes an extra time R/c, hence the number of crests emitted for Alice's O–P leg is

$$N_\text{a} = \left(\frac{R}{c\beta_\text{a}} + \frac{R}{c}\right)\sqrt{\frac{1-\beta_\text{a}}{1+\beta_\text{a}}} \times f_0 = \frac{Rf_0}{c\gamma_a\beta_a} = N'_\text{a} \ , \tag{15.153}$$

since the source is receding from the observer.

During the P–Q leg, we have $\theta = \frac{1}{2}\pi$, and Alice's velocity is orthogonal to the wavevector $\boldsymbol{k}$, which is directed radially inward. Bob's first signal at frequency f_b arrives a time R/c after Alice passes P, and his last signal at this frequency arrives a time R/c after Alice passes Q. Thus, the total time during which Bob receives the signal at the Doppler-shifted frequency f_b is $\Delta t_\text{b} = R\Theta/c$, and

$$N_\text{b} = \frac{R\Theta}{c\beta_\text{b}} \cdot \sqrt{1-\beta_\text{b}^2} \times f_0 = \frac{R\Theta f_0}{c\gamma_b\beta_b} = N'_\text{b} \ . \tag{15.154}$$

Finally, during the Q–O home stretch, Bob first starts to receive the signal at the Doppler-shifted frequency $f_{\rm c}$ a time R/c after Alice passes Q, and he continues to receive the signal until the moment Alice rushes into his open and very flabby old arms when she makes it back to O. Thus, Bob receives the frequency $f_{\rm c}$ signal for a duration $\Delta t_{\rm c} - R/c$, where $\Delta t_{\rm c} = R/c\beta_{\rm c}$. Thus,

$$N_{\rm c} = \left(\frac{R}{c\beta_{\rm c}} - \frac{R}{c}\right)\sqrt{\frac{1+\beta_{\rm c}}{1-\beta_{\rm c}}} \times f_0 = \frac{Rf_0}{c\gamma_c\beta_c} = N'_{\rm c} \ , \qquad (15.155)$$

since the source is approaching.

Therefore, the number of wavelengths emitted by Alice will be precisely equal to the number received by Bob – none of the waves gets lost.

15.7 Relativistic Kinematics of Particle Collisions

As should be expected, special relativity is essential toward the understanding of subatomic particle collisions, where the particles themselves are moving at close to the speed of light. In our analysis of the kinematics of collisions, we shall find it convenient to adopt the standard convention on units, where we set $c \equiv 1$. Energies will typically be given in GeV, where $1\,{\rm GeV} = 10^9\,{\rm eV} = 1.602 \times 10^{-10}\,R_{\rm J}$. Momenta will then be in units of GeV$/c$, and masses in units of GeV$/c^2$. With $c \equiv 1$, it is then customary to quote masses in energy units. For example, the mass of the proton in these units is $m_{\rm p} = 938\,{\rm MeV}$, and $m_{\pi^-} = 140\,{\rm MeV}$.

For a particle of mass M, its 4-momentum satisfies $P_\mu\, P^\mu = M^2$ (remember $c = 1$). Consider now an observer with 4-velocity U^μ. The energy of the particle, in the rest frame of the observer is $E = P^\mu\, U_\mu$. For example, if $P^\mu = (M, 0, 0, 0)$ is its rest frame, and $U^\mu = (\gamma\, , \gamma\boldsymbol{\beta})$, then $E = \gamma M$, as we have already seen.

Consider next the emission of a photon of 4-momentum $P^\mu = (\hbar\omega/c, \hbar\boldsymbol{k})$ from an object with 4-velocity V^μ, and detected in a frame with 4-velocity U^μ. In the frame of the detector, the photon energy is $E = P^\mu\, U_\mu$, while in the frame of the emitter its energy is $E' = P^\mu\, V_\mu$. If $U^\mu = (1, 0, 0, 0)$ and $V^\mu = (\gamma\, , \gamma\boldsymbol{\beta})$, then $E = \hbar\omega$ and $E' = \hbar\omega' = \gamma\hbar(\omega - \boldsymbol{\beta}\cdot\boldsymbol{k}) = \gamma\hbar\omega(1 - \beta\cos\theta)$, where $\theta = \cos^{-1}\big(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{k}}\big)$. Thus, $\omega = \gamma^{-1}\omega'/(1 - \beta\cos\theta)$. This recapitulates our earlier derivation in eqn. 15.144.

Consider next the interaction of several particles. If in a given frame the 4-momenta of the reactants are P^μ_i, where n labels the reactant 'species', and the 4-momenta of the products are Q^μ_j, then if the collision is elastic, we have that total 4-momentum is conserved, *i.e.*

$$\sum_{i=1}^{N} P^\mu_i = \sum_{j=1}^{\bar N} Q^\mu_j \ , \qquad (15.156)$$

where there are N reactants and $\bar N$ products. For massive particles, we can write

$$P^\mu_i = \gamma_i\, m_i\, (1\, , \boldsymbol{v}_i) \quad , \quad Q^\mu_j = \bar\gamma_j\, \bar m_j\, (1\, , \bar v_j) \ , \qquad (15.157)$$

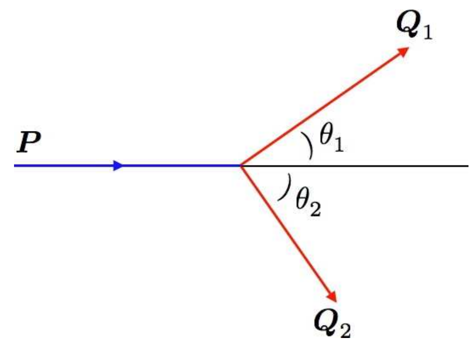


Figure 15.8: Spontaneous decay of a single reactant into two products.

while for massless particles,

$$P_i^\mu = \hbar k_i \left(1\,,\,\hat{\boldsymbol{k}}\right) \quad , \quad Q_j^\mu = \hbar \bar{k}_j \left(1\,,\,\hat{\bar{\boldsymbol{k}}}\right) \,. \tag{15.158}$$

15.7.1 Spontaneous particle decay into two products

Consider first the decay of a particle of mass M into two particles. We have $P^\mu = Q_1^\mu + Q_2^\mu$, hence in the rest frame of the (sole) reactant, which is also called the 'center of mass' (CM) frame since the total 3-momentum vanishes therein, we have $M = E_1 + E_2$. Since $E_i^{\rm CM} = \gamma^{\rm CM} m_i$, and $\gamma_i \geq 1$, clearly we must have $M > m_1 + m_2$, or else the decay cannot possibly conserve energy. To analyze further, write $P^\mu - Q_1^\mu = Q_2^\mu$. Squaring, we obtain

$$M^2 + m_1^2 - 2P_\mu\, Q_1^\mu = m_2^2 \,. \tag{15.159}$$

The dot-product $P \cdot Q_1$ is a Lorentz scalar, and hence may be evaluated in any frame.

Let us first consider the CM frame, where $P^\mu = M(1,0,0,0)$, and $P_\mu\, Q_1^\mu = M E_1^{\rm CM}$, where $E_1^{\rm CM}$ is the energy of $n = 1$ product in the rest frame of the reactant. Thus,

$$E_1^{\rm CM} = \frac{M^2 + m_1^2 - m_2^2}{2M} \qquad , \qquad E_2^{\rm CM} = \frac{M^2 + m_2^2 - m_1^2}{2M} \,, \tag{15.160}$$

where the second result follows merely from switching the product labels. We may now write $Q_1^\mu = (E_1^{\rm CM}, \boldsymbol{p}^{\rm CM})$ and $Q_2^\mu = (E_2^{\rm CM}, -\boldsymbol{p}^{\rm CM})$, with

$$\begin{aligned}(\boldsymbol{p}^{\rm CM})^2 &= (E_1^{\rm CM})^2 - m_1^2 = (E_2^{\rm CM})^2 - m_2^2 \\ &= \left(\frac{M^2 - m_1^2 - m_2^2}{2M}\right)^2 - \left(\frac{m_1\, m_2}{M}\right)^2 \,. \end{aligned} \tag{15.161}$$

In the laboratory frame, we have $P^\mu = \gamma M \,(1\,,\,\boldsymbol{V})$ and $Q_i^\mu = \gamma_i\, m_i\,(1\,,\,\boldsymbol{V}_i)$. Energy and momentum conservation then provide four equations for the six unknowns $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Thus, there is a two-parameter family of solutions, assuming we regard the reactant velocity $\boldsymbol{V}^{\rm K}$ as fixed, corresponding to the freedom to choose $\hat{\boldsymbol{p}}^{\text{CM}}$ in the CM frame solution above. Clearly the three vectors $\boldsymbol{V}$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$ must lie in the same plane, and with $\boldsymbol{V}$ fixed, only one additional parameter is required to fix this plane. The other free parameter may be taken to be the relative angle $\theta_1 = \cos^{-1}\big(\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{V}}_1\big)$ (see fig. 15.8). The angle θ_2 as well as the speed V_2 are then completely determined. We can use eqn. 15.159 to relate θ_1 and V_1:

$$M^2 + m_1^2 - m_2^2 = 2Mm_1\gamma\gamma_1\big(1 - VV_1\cos\theta_1\big) \ . \tag{15.162}$$

It is convenient to express both γ_1 and V_1 in terms of the energy E_1:

$$\gamma_1 = \frac{E_1}{m_1} \qquad , \qquad V_1 = \sqrt{1-\gamma_1^{-2}} = \sqrt{1-\frac{m_1^2}{E_1^2}} \ . \tag{15.163}$$

This results in a quadratic equation for E_1, which may be expressed as

$$(1-V^2\cos^2\!\theta_1)E_1^2 - 2\sqrt{1-V^2}\,E_1^{\text{CM}}E_1 + (1-V^2)(E_1^{\text{CM}})^2 + m_1^2\,V^2\cos^2\!\theta_1 = 0 \ , \tag{15.164}$$

the solutions of which are

$$E_1 = \frac{\sqrt{1-V^2}\,E_1^{\text{CM}} \pm V\cos\theta_1\sqrt{(1-V^2)(E_1^{\text{CM}})^2 - (1-V^2\cos^2\!\theta_1)m_1^2}}{1-V^2\cos^2\!\theta_1} \ . \tag{15.165}$$

The discriminant is positive provided

$$\left(\frac{E_1^{\text{CM}}}{m_1}\right)^2 > \frac{1-V^2\cos^2\!\theta_1}{1-V^2} \ , \tag{15.166}$$

which means

$$\sin^2\!\theta_1 < \frac{V^{-2}-1}{(V_1^{\text{CM}})^{-2}-1} \equiv \sin^2\!\theta_1^* \ , \tag{15.167}$$

where

$$V_1^{\text{CM}} = \sqrt{1-\left(\frac{m_1}{E_1^{\text{CM}}}\right)^2} \tag{15.168}$$

is the speed of product 1 in the CM frame. Thus, for $V < V_1^{\text{CM}} < 1$, the scattering angle θ_1 may take on any value, while for larger reactant speeds $V_1^{\text{CM}} < V < 1$ the quantity $\sin^2\!\theta_1$ cannot exceed a critical value.

15.7.2 Miscellaneous examples of particle decays

Let us now consider some applications of the formulae in eqn. 15.160:

- Consider the decay $\pi^0 \to \gamma\gamma$, for which $m_1 = m_2 = 0$. We then have $E_1^{\text{CM}} = E_2^{\text{CM}} = \frac{1}{2}M$. Thus, with $M = m_{\pi^0} = 135\,\text{MeV}$, we have $E_1^{\text{CM}} = E_2^{\text{CM}} = 67.5\,\text{MeV}$ for the photon energies in the CM frame.

- For the reaction $K^+ \longrightarrow \mu^+ + \nu_\mu$ we have $M = m_{K^+} = 494\,\text{MeV}$ and $m_1 = m_{\mu^-} = 106\,\text{MeV}$. The neutrino mass is $m_2 \approx 0$, hence $E_2^{\text{CM}} = 236\,\text{MeV}$ is the emitted neutrino's energy in the CM frame.

- A Λ^0 hyperon with a mass $M = m_{\Lambda^0} = 1116\,\text{MeV}$ decays into a proton ($m_1 = m_\text{p} = 938\,\text{MeV}$) and a pion $m_2 = m_{\pi^-} = 140\,\text{MeV}$). The CM energy of the emitted proton is $E_1^{\text{CM}} = 943\,\text{MeV}$ and that of the emitted pion is $E_2^{\text{CM}} = 173\,\text{MeV}$.

15.7.3 Threshold particle production with a stationary target

Consider now a particle of mass M_1 moving with velocity $\boldsymbol{V}_1 = V_1\,\hat{\boldsymbol{x}}$, incident upon a stationary target particle of mass M_2, as indicated in fig. 15.9. Let the product masses be $m_1, m_2, \ldots, m_{N'}$. The 4-momenta of the reactants and products are

$$P_1^\mu = \big(E_1\,, \boldsymbol{P}_1\big) \quad , \quad P_2^\mu = M_2\,\big(1\,, \boldsymbol{0}\big) \quad , \quad Q_j^\mu = \big(\varepsilon_j\,, \boldsymbol{p}_j\big)\ . \tag{15.169}$$

Note that $E_1^2 - \boldsymbol{P}_1^2 = M_1^2$ and $\varepsilon_j^2 - \boldsymbol{p}_j^2 = m_j^2$, with $j \in \{1, 2, \ldots, N'\}$.

Conservation of momentum means that

$$P_1^\mu + P_2^\mu = \sum_{j=1}^{N'} Q_j^\mu\ . \tag{15.170}$$

In particular, taking the $\mu = 0$ component, we have

$$E_1 + M_2 = \sum_{j=1}^{N'} \varepsilon_j\ , \tag{15.171}$$

which certainly entails

$$E_1 \geq \sum_{j=1}^{N'} m_j - M_2 \tag{15.172}$$

since $\varepsilon_j = \gamma_j\,m_j \geq m_j$. But can the equality ever be achieved? This would only be the case if $\gamma_j = 1$ for all j, *i.e.* the final velocities are all zero. But this itself is quite impossible, since the initial state momentum is $\boldsymbol{P}$.

To determine the threshold energy E_1^{thr}, we compare the length of the total momentum vector in the LAB and CM frames:

$$(P_1 + P_2)^2 = M_1^2 + M_2^2 + 2E_1\,M_2 \qquad \text{(LAB)} \tag{15.173}$$

$$= \Bigg(\sum_{j=1}^{N'} \varepsilon_j^{\text{CM}}\Bigg)^2 \qquad \text{(CM)}\ . \tag{15.174}$$

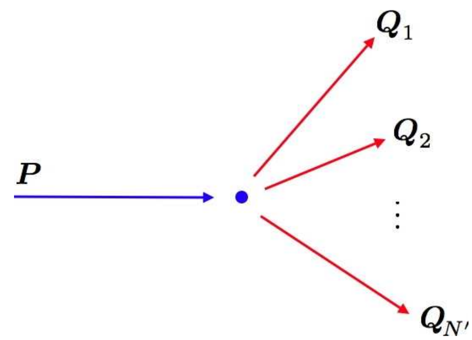


Figure 15.9: A two-particle initial state, with a stationary target in the LAB frame, and an N'-particle final state.

Thus,

$$E_1 = \frac{\left(\sum_{j=1}^{N'} \varepsilon_j^{\text{CM}}\right)^2 - M_1^2 - M_2^2}{2M_2} \tag{15.175}$$

and we conclude

$$E_1 \geq E_1^{\text{THR}} = \frac{\left(\sum_{j=1}^{N'} m_j\right)^2 - M_1^2 - M_2^2}{2M_2} \ . \tag{15.176}$$

Note that in the CM frame it *is* possible for each $\varepsilon_j^{\text{CM}} = m_j$.

Finally, we must have $E_1^{\text{THR}} \geq \sum_{j=1}^{N'} m_j - M_2$. This then requires

$$M_1 + M_2 \leq \sum_{j=1}^{N'} m_j \ . \tag{15.177}$$

15.7.4 Transformation between frames

Consider a particle with 4-velocity u^μ in frame K and consider a Lorentz transformation between this frame and a frame K' moving relative to K with velocity $\boldsymbol{V}$. We may write

$$u^\mu = \big(\gamma\,,\,\gamma v\cos\theta\,,\,\gamma v\sin\theta\,\hat{\boldsymbol{n}}_\perp\big) \quad , \quad u'^\mu = \big(\gamma'\,,\,\gamma' v'\cos\theta'\,,\,\gamma' v'\sin\theta'\,\hat{\boldsymbol{n}}'_\perp\big) \ . \tag{15.178}$$

According to the general transformation rules of eqns. 15.50, 15.51, and 15.52, we may write

$$\gamma = \Gamma\,\gamma' + \Gamma V\,\gamma' v' \cos\theta' \tag{15.179}$$

$$\gamma v\cos\theta = \Gamma V\,\gamma' + \Gamma\,\gamma' v'\cos\theta' \tag{15.180}$$

$$\gamma v\sin\theta = \gamma' v'\sin\theta' \tag{15.181}$$

$$\hat{\boldsymbol{n}}_\perp = \hat{\boldsymbol{n}}'_\perp \ , \tag{15.182}$$

where the $\hat{\boldsymbol{x}}$ axis is taken to be $\hat{\boldsymbol{V}}$, and where $\Gamma \equiv (1 - V^2)^{-1/2}$. Note that the last two of these equations may be written as a single vector equation for the transverse components.

Dividing the eqn. 15.181 by eqn. 15.180, we obtain the result

$$\tan\theta = \frac{\sin\theta'}{\varGamma\Big(\frac{V}{v'} + \cos\theta'\Big)} \ . \tag{15.183}$$

We can then use eqn. 15.179 to relate v' and $\cos\theta'$:

$$\gamma'^{-1} = \sqrt{1 - v'^2} = \frac{\varGamma}{\gamma}\left(1 + V v' \cos\theta'\right) \ . \tag{15.184}$$

Squaring both sides, we obtain a quadratic equation whose roots are

$$v' = \frac{-\varGamma^2\, V \cos\theta' \pm \sqrt{\gamma^4 - \varGamma^2\,\gamma^2\,(1 - V^2\cos^2\!\theta')}}{\gamma^2 + \varGamma^2\, V^2 \cos^2\!\theta'} \ . \tag{15.185}$$

CM frame mass and velocity

To find the velocity of the CM frame, simply write

$$P^\mu_{\rm tot} = \sum_{i=1}^N P^\mu_i = \left(\sum_{i=1}^N \gamma_i\, m_i \,,\, \sum_{i=1}^N \gamma_i\, m_i\, \boldsymbol{v}_i\right) \tag{15.186}$$

$$\equiv \varGamma M\,(1\,,\,\boldsymbol{V}) \ . \tag{15.187}$$

Then

$$M^2 = \left(\sum_{i=1}^N \gamma_i\, m_i\right)^2 - \left(\sum_{i=1}^N \gamma_i\, m_i\, \boldsymbol{v}_i\right)^2 \tag{15.188}$$

and

$$\boldsymbol{V} = \frac{\sum_{i=1}^N \gamma_i\, m_i\, \boldsymbol{v}_i}{\sum_{i=1}^N \gamma_i\, m_i} \ . \tag{15.189}$$

15.7.5 Compton scattering

An extremely important example of relativistic scattering occurs when a photon scatters off an electron: $e^- + \gamma \longrightarrow e^- + \gamma$ (see fig. 15.10). Let us work in the rest frame of the reactant electron. Then we have

$$P^\mu_e = m_{\rm e}\,(1,0) \qquad , \qquad \widetilde{P}^\mu_e = m_{\rm e}\,(\gamma\,,\,\gamma\boldsymbol{V}) \tag{15.190}$$

for the initial and final 4-momenta of the electron. For the photon, we have

$$P^\mu_\gamma = (\omega\,,\,\boldsymbol{k}) \qquad , \qquad \widetilde{P}^\mu_\gamma = (\widetilde{\omega}\,,\,\widetilde{\boldsymbol{k}}) \ , \tag{15.191}$$

where we've set $\hbar = 1$ as well. Conservation of 4-momentum entails

$$P^\mu_\gamma - \widetilde{P}^\mu_\gamma = \widetilde{P}^\mu_e - P^\mu_e \ . \tag{15.192}$$

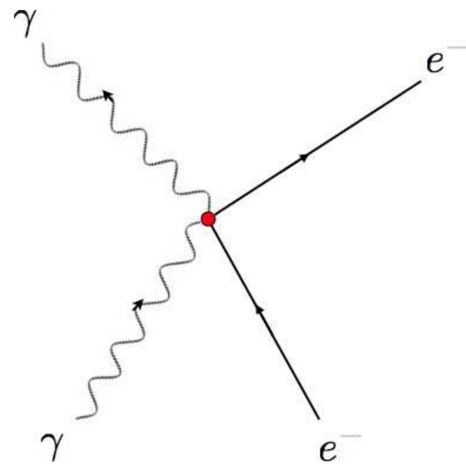


Figure 15.10: Compton scattering of a photon and an electron.

Thus,

$$\big(\omega - \tilde{\omega}\,,\,\boldsymbol{k} - \tilde{\boldsymbol{k}}\big) = m_{\rm e}\,\big(\gamma - 1\,,\gamma\boldsymbol{V}\big)\ . \tag{15.193}$$

Squaring each side, we obtain

$$\begin{aligned}\big(\omega - \tilde{\omega}\big)^2 - \big(\boldsymbol{k} - \tilde{\boldsymbol{k}}\big)^2 &= 2\omega\,\tilde{\omega}\,(\cos\theta - 1)\\ &= m_{\rm e}^2\Big((\gamma-1)^2 - \gamma^2\boldsymbol{V}^2\Big)\\ &= 2m_{\rm e}^2(1-\gamma)\\ &= 2m_{\rm e}\,\big(\tilde{\omega} - \omega\big)\ .\end{aligned} \tag{15.194}$$

Here we have used $|\boldsymbol{k}| = \omega$ for photons, and also $(\gamma - 1)\,m_{\rm e} = \omega - \tilde{\omega}$, from eqn. 15.193.

Restoring the units $\hbar$ and c, we find the Compton formula

$$\frac{1}{\tilde{\omega}} - \frac{1}{\omega} = \frac{\hbar}{m_{\rm e}c^2}\,\big(1 - \cos\theta\big)\ . \tag{15.195}$$

This is often expressed in terms of the photon wavelengths, as

$$\tilde{\lambda} - \lambda = \frac{4\pi\hbar}{m_{\rm e}c}\,\sin^2\!\big(\tfrac{1}{2}\theta\big)\ , \tag{15.196}$$

showing that the wavelength of the scattered light increases with the scattering angle in the rest frame of the target electron.

15.8 Covariant Electrodynamics

We begin with the following expression for the Lagrangian density of charged particles coupled to an electromagnetic field, and then show that the Euler-Lagrange equations recapitulate Maxwell's equations. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{c}\,j_\mu\,A^\mu\ . \tag{15.197}$$

Here, $A^\mu = (\phi\,, \boldsymbol{A})$ is the *electromagnetic 4-potential*, which combines the scalar field ϕ and the vector field $\boldsymbol{A}$ into a single 4-vector. The quantity $F_{\mu\nu}$ is the *electromagnetic field strength tensor* and is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ . \tag{15.198}$$

Note that as defined $F_{\mu\nu} = -F_{\nu\mu}$ is antisymmetric. Note that, if $i = 1, 2, 3$ is a spatial index, then

$$F_{0i} = -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial A^0}{\partial x^i} = E_i \tag{15.199}$$

$$F_{ij} = \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} = -\,\epsilon_{ijk}\,B_k \ . \tag{15.200}$$

Here we have used $A^\mu = (A^0\,,\,\boldsymbol{A})$ and $A_\mu = (A^0\,,\,-\boldsymbol{A})$, as well as $\partial_\mu = (c^{-1}\partial_t\,,\boldsymbol{\nabla})$.

IMPORTANT : Since the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are not part of a 4-vector, we do not use covariant / contravariant notation for their components. Thus, E_i is the i^{th} component of the vector $\boldsymbol{E}$. We will not write E^i with a raised index, but if we did, we'd mean the same thing: $E^i = E_i$. By contrast, for the spatial components of a four-vector like A^μ, we have $A_i = -A^i$.

Explicitly, then, we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{15.201}$$

where $F^{\mu\nu} = g^{\mu\alpha}\,g^{\nu\beta}\,F_{\alpha\beta}$. Note that when comparing $F^{\mu\nu}$ and $F_{\mu\nu}$, the components with one space and one time index differ by a minus sign. Thus,

$$-\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} = \frac{\boldsymbol{E}^2 - \boldsymbol{B}^2}{8\pi} \ , \tag{15.202}$$

which is the electromagnetic Lagrangian density. The $j \cdot A$ term accounts for the interaction between matter and electromagnetic degrees of freedom. We have

$$\frac{1}{c}\,j_\mu A^\mu = \varrho\,\phi - \frac{1}{c}\,\boldsymbol{j}\cdot\boldsymbol{A} \ , \tag{15.203}$$

where

$$j^\mu = \begin{pmatrix} c\varrho \\ \boldsymbol{j} \end{pmatrix} \quad , \qquad A^\mu = \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} \ , \tag{15.204}$$

where ϱ is the charge density and $\boldsymbol{j}$ is the current density. Charge conservation requires

$$\partial_\mu\, j^\mu = \frac{\partial\varrho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \ . \tag{15.205}$$

We shall have more to say about this further on below.

Let us now derive the Euler-Lagrange equations for the action functional,

$$S = -c^{-1} \int d^4\!x \left(\frac{1}{16\pi} \, F_{\mu\nu} \, F^{\mu\nu} + c^{-1} \, j_\mu \, A^\mu \right) . \tag{15.206}$$

We first vary with respect to A_μ. Clearly

$$\delta F_{\mu\nu} = \partial_\mu \, \delta A_\nu - \partial_\nu \, \delta A_\mu \ . \tag{15.207}$$

We then have

$$\delta\mathcal{L} = \left(\frac{1}{4\pi} \, \partial_\mu F^{\mu\nu} - c^{-1} j^\nu \right) \delta A_\nu - \partial_\mu \left(\frac{1}{4\pi} \, F^{\mu\nu} \, \delta A_\nu \right) . \tag{15.208}$$

Ignoring the boundary term, we obtain Maxwell's equations,

$$\partial_\mu \, F^{\mu\nu} = 4\pi c^{-1} \, j^\nu \tag{15.209}$$

The $\nu = k$ component of these equations yields

$$\partial_0 \, F^{0k} + \partial_i \, F^{jk} = -\partial_0 \, E_k - \epsilon_{jkl} \, \partial_j \, B_l = 4\pi c^{-1} \, j^k \ , \tag{15.210}$$

which is the k component of the Maxwell-Ampère law,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c} \, \boldsymbol{j} + \frac{1}{c} \, \frac{\partial \boldsymbol{E}}{\partial t} \ . \tag{15.211}$$

The $\nu = 0$ component reads

$$\partial_i \, F^{i0} = \frac{4\pi}{c} \, j^0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \varrho \quad , \tag{15.212}$$

which is Gauss's law. The remaining two Maxwell equations come 'for free' from the very definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\boldsymbol{E} = -\boldsymbol{\nabla} A^0 - \frac{1}{c} \, \frac{\partial \boldsymbol{A}}{\partial t} \tag{15.213}$$

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad , \tag{15.214}$$

which imply

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c} \, \frac{\partial \boldsymbol{B}}{\partial t} \tag{15.215}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \quad . \tag{15.216}$$

15.8.1 Lorentz force law

This has already been worked out in chapter 7. Here we reiterate our earlier derivation. The 4-current may be written as

$$j^\mu(\boldsymbol{x}, t) = c \sum_n q_n \int d\tau \, \frac{dX_n^\mu}{d\tau} \, \delta^{(4)}(x - X) \ . \tag{15.217}$$

Thus, writing $X_n^\mu = \big(ct\,,\boldsymbol{X}_n(t)\big)$, we have

$$j^0(\boldsymbol{x},t) = \sum_n q_n\, c\,\delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big) \tag{15.218}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = \sum_n q_n\, \dot{\boldsymbol{X}}_n(t)\,\delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big)\ . \tag{15.219}$$

The Lagrangian for the matter-field interaction term is then

$$\begin{aligned} L &= -c^{-1}\!\int\! d^3\!x\,\big(j^0 A^0 - \boldsymbol{j}\cdot\boldsymbol{A}\big) \\ &= -\sum_n \bigg[q_n\,\phi(\boldsymbol{X}_n,t) - \frac{q_n}{c}\,\boldsymbol{A}(\boldsymbol{X}_n,t)\cdot\dot{\boldsymbol{X}}_n \bigg]\ , \end{aligned} \tag{15.220}$$

where $\phi = A^0$. For each charge q_n, this is equivalent to a particle with velocity-dependent potential energy

$$U(\boldsymbol{x},t) = q\,\phi(\boldsymbol{x},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{x}}\ , \tag{15.221}$$

where $\boldsymbol{x} = \boldsymbol{X}_n$.

Let's work out the equations of motion. We assume a kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{x}}^2$ for the charge. We then have

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot{\boldsymbol{x}}}\bigg) = \frac{\partial L}{\partial\boldsymbol{x}} \tag{15.222}$$

with $L = T - U$, which gives

$$m\,\ddot{\boldsymbol{x}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{x}})\ , \tag{15.223}$$

or, in component notation,

$$m\,\ddot{x}^i + \frac{q}{c}\,\frac{\partial A^i}{\partial x^j}\,\dot{x}^j + \frac{q}{c}\,\frac{\partial A^i}{\partial t} = -q\,\frac{\partial\phi}{\partial x^i} + \frac{q}{c}\,\frac{\partial A^j}{\partial x^i}\,\dot{x}^j\ , \tag{15.224}$$

which is to say

$$m\,\ddot{x}^i = -q\,\frac{\partial\phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\bigg(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\bigg)\,\dot{x}^j\ . \tag{15.225}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A^k}{\partial x^j}\ , \tag{15.226}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}\ , \tag{15.227}$$

we have $\epsilon_{ijk}\,B_i = \partial^j A^k - \partial^k A^j$, and

$$m\,\ddot{x}^i = -q\,\frac{\partial\phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}^j\,B_k\ , \tag{15.228}$$

Figure 15.11: Homer celebrates the manifest gauge invariance of classical electromagnetic theory.

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{x}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times \boldsymbol{B} \ , \end{aligned} \tag{15.229}$$

which is, of course, the Lorentz force law.

15.8.2 Gauge invariance

The action $S = c^{-1}\!\int\! d^4\!x\,\mathcal{L}$ admits a *gauge invariance*. Let $A^\mu \to A^\mu + \partial^\mu \Lambda$, where $\Lambda(\boldsymbol{x}, t)$ is an arbitrary scalar function of spacetime coordinates. Clearly

$$F_{\mu\nu} \to F_{\mu\nu} + \big(\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda\big) = F_{\mu\nu} \ , \tag{15.230}$$

and hence the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ remain *invariant* under the gauge transformation, even though the 4-potential itself changes. What about the matter term? Clearly

$$\begin{aligned} -c^{-1}\,j^\mu\,A_\mu \to &-c^{-1}\,j^\mu\,A_\mu - c^{-1}\,j^\mu\,\partial_\mu \Lambda \\ &= -c^{-1}\,j^\mu\,A_\mu + c^{-1}\Lambda\,\partial_\mu\,j^\mu - \partial_\mu\big(c^{-1}\Lambda\,j^\mu\big) \ . \end{aligned} \tag{15.231}$$

Once again we ignore the boundary term. We may now invoke charge conservation to write $\partial_\mu\,j^\mu = 0$, and we conclude that the action is invariant! Woo hoo! Note also the very deep connection

$$\text{gauge invariance} \quad \longleftrightarrow \quad \text{charge conservation} \ . \tag{15.232}$$

15.8.3 Transformations of fields

One last detail remains, and that is to exhibit explicitly the Lorentz transformation properties of the electromagnetic field. For the case of vectors like A^μ, we have

$$A^\mu = L^\mu_{\ \nu}\, A'^{\,\nu} \ . \tag{15.233}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, however, appear as elements in the field strength tensor $F^{\mu\nu}$. Clearly this must transform as a tensor:

$$F^{\mu\nu} = L^\mu_{\ \alpha}\, L^\nu_{\ \beta}\, F'^{\,\alpha\beta} = L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} \ . \tag{15.234}$$

We can write a general Lorentz transformation as a product of a rotation $L_{\rm rot}$ and a boost $L_{\rm boost}$. Let's first see how rotations act on the field strength tensor. We take

$$L = L_{\rm rot} = \begin{pmatrix} 1_{1\times 1} & 0_{1\times 3} \\ 0_{3\times 1} & R_{3\times 3} \end{pmatrix} \ , \tag{15.235}$$

where $R^{\rm t} R = \mathbb{I}$, *i.e.* $R \in \mathrm{O}(3)$ is an orthogonal matrix. We must compute

$$\begin{aligned} L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix} \begin{pmatrix} 0 & -E'_k \\ E'_j & -\epsilon_{jkm}\, B'_m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R^{\rm t}_{kl} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E'_k\, R^{\rm t}_{kl} \\ R_{ij}\, E'_j & -\epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m \end{pmatrix} \ . \end{aligned} \tag{15.236}$$

Thus, we conclude

$$E_l = R_{lk}\, E'_k \tag{15.237}$$

$$\epsilon_{iln}\, B_n = \epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m \ . \tag{15.238}$$

Now for any 3×3 matrix R we have

$$\epsilon_{jks}\, R_{ij}\, R_{lk}\, R_{rs} = \det(R)\, \epsilon_{ilr} \ , \tag{15.239}$$

and therefore

$$\begin{aligned} \epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m &= \epsilon_{jkm}\, R_{ij}\, R_{lk}\, R_{nm}\, R_{ns}\, B'_s \\ &= \det(R)\, \epsilon_{iln}\, R_{ns}\, B'_s \ , \end{aligned} \tag{15.240}$$

Therefore,

$$E_i = R_{ij}\, E'_j \qquad , \qquad B_i = \det(R) \cdot R_{ij}\, B'_j \ . \tag{15.241}$$

For any orthogonal matrix, $R^{\rm t} R = \mathbb{I}$ gives that $\det(R) = \pm 1$. The extra factor of $\det(R)$ in the transformation properties of $\boldsymbol{B}$ is due to the fact that the electric field transforms as a *vector*, while the magnetic field transforms as a *pseudovector*. Under space inversion, for example, where $R = -\mathbb{I}$, the electric field is *odd* under this transformation ($\boldsymbol{E} \to -\boldsymbol{E}$) while the magnetic field is *even* ($\boldsymbol{B} \to +\boldsymbol{B}$). Similar considerations hold in particle mechanics for the linear momentum, $\boldsymbol{p}$ (a vector) and the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ (a pseudovector). The analogy is not complete, however, because while both $\boldsymbol{p}$ and $\boldsymbol{L}$ are odd under the operation of time-reversal, $\boldsymbol{E}$ is even while $\boldsymbol{B}$ is odd.

OK, so how about boosts? We can write the general boost, from eqn. 15.37, as

$$L = \begin{pmatrix} \gamma & \gamma\hat{\boldsymbol{\beta}} \\ \gamma\hat{\boldsymbol{\beta}} & \mathrm{I} + (\gamma-1)\mathrm{P}^{\beta} \end{pmatrix} \tag{15.242}$$

where $\mathrm{P}^{\boldsymbol{\beta}}_{ij} = \hat{\beta}_i\,\hat{\beta}_j$ is the projector onto the direction of $\boldsymbol{\beta}$. We now compute

$$L^{\mu}_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I}+(\gamma-1)\mathrm{P} \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{E}'^{\,\mathrm{t}} \\ \boldsymbol{E}' & -\epsilon_{jkm}\, B'_m \end{pmatrix} \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I}+(\gamma-1)\mathrm{P} \end{pmatrix} . \tag{15.243}$$

Carrying out the matrix multiplications, we obtain

$$\boldsymbol{E} = \gamma(\boldsymbol{E}' - \boldsymbol{\beta}\times\boldsymbol{B}') - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{E}')\hat{\boldsymbol{\beta}} \tag{15.244}$$

$$\boldsymbol{B} = \gamma(\boldsymbol{B}' + \boldsymbol{\beta}\times\boldsymbol{E}') - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{B}')\hat{\boldsymbol{\beta}}\ . \tag{15.245}$$

Expressed in terms of the components $E_{\|}$, $\boldsymbol{E}_{\perp}$, $B_{\|}$, and $\boldsymbol{B}_{\perp}$, one has

$$E_{\|} = E'_{\|} \qquad , \qquad \boldsymbol{E}_{\perp} = \gamma\big(\boldsymbol{E}'_{\perp} - \boldsymbol{\beta}\times\boldsymbol{B}'_{\perp}\big) \tag{15.246}$$

$$B_{\|} = B'_{\|} \qquad , \qquad \boldsymbol{B}_{\perp} = \gamma\big(\boldsymbol{B}'_{\perp} + \boldsymbol{\beta}\times\boldsymbol{E}'_{\perp}\big)\ . \tag{15.247}$$

Recall that for any vector $\boldsymbol{\xi}$, we write

$$\xi_{\|} = \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi} \tag{15.248}$$

$$\boldsymbol{\xi}_{\perp} = \boldsymbol{\xi} - (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi})\,\hat{\boldsymbol{\beta}}\ , \tag{15.249}$$

so that $\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi}_{\perp} = 0$.

15.8.4 Invariance *versus* covariance

We saw that the laws of electromagnetism were *gauge* ***in****variant*. That is, the solutions to the field equations did not change under a gauge transformation $A^{\mu} \to A^{\mu} + \partial^{\mu}\Lambda$. With respect to Lorentz transformations, however, the theory is *Lorentz* ***co****variant*. This means that Maxwell's equations in different inertial frames take the exact same form, $\partial_{\mu}F^{\mu\nu} = 4\pi c^{-1} j^{\nu}$, but that both the fields and the sources transform appropriately under a change in reference frames. The sources are described by the current 4-vector $j^{\mu} = (c\varrho\,,\,\boldsymbol{j})$ and transform as

$$c\varrho = \gamma c\varrho' + \gamma\beta j'_{\|} \tag{15.250}$$

$$j_{\|} = \gamma\beta c\varrho' + \gamma j'_{\|} \tag{15.251}$$

$$\boldsymbol{j}_{\perp} = \boldsymbol{j}'_{\perp} \quad . \tag{15.252}$$

The fields transform according to eqns. 15.246 and 15.247.

Consider, for example, a static point charge q located at the origin in the frame K', which moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to K. An observer in K' measures a charge density $\varrho'(\boldsymbol{x}', t') = q\,\delta(\boldsymbol{x}')$. The electric and magnetic fields in the K' frame are then $\boldsymbol{E}' = q\,\hat{\boldsymbol{r}}'/r'^{\,2}$ and $\boldsymbol{B}' = 0$. For an observer in the K frame, the coordinates transform as

$$ct = \gamma ct' + \gamma\beta x' \qquad\qquad ct' = \gamma ct - \gamma\beta x \tag{15.253}$$

$$x = \gamma\beta ct' + \gamma x' \qquad\qquad x' = -\gamma\beta ct + \gamma x \ , \tag{15.254}$$

as well as $y = y'$ and $z = z'$. The observer in the K frame sees instead a charge at $x^\mu = (ct\,,\,ut\,,\,0\,,\,0)$ and both a charge density as well as a current density:

$$\varrho(\boldsymbol{x}, t) = \gamma\varrho(\boldsymbol{x}', t') = q\,\delta(x - ut)\,\delta(y)\,\delta(z) \tag{15.255}$$

$$\boldsymbol{j}(\boldsymbol{x}, t) = \gamma\beta c\,\varrho(\boldsymbol{x}', t')\,\hat{\boldsymbol{x}} = u\,q\,\delta(x - ut)\,\delta(y)\,\delta(z)\,\hat{\boldsymbol{x}} \ . \tag{15.256}$$

OK, so much for the sources. How about the fields? Expressed in terms of Cartesian coordinates, the electric field in K' is given by

$$\boldsymbol{E}'(\boldsymbol{x}', t') = q\,\frac{x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}}}{\big(x'^{\,2} + y'^{\,2} + z'^{\,2}\big)^{3/2}} \ . \tag{15.257}$$

From eqns. 15.246 and 15.247, we have $E_x = E'_x$ and $B_x = B'_x = 0$. Furthermore, we have $E_y = \gamma E'_y$, $E_z = \gamma E'_z$, $B_y = -\gamma\beta E'_z$, and $B_z = \gamma\beta E'_y$. Thus,

$$\boldsymbol{E}(\boldsymbol{x}, t) = \gamma q\,\frac{(x - ut)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\big[\gamma^2(x - ut)^2 + y^2 + z^2\big]^{3/2}} \tag{15.258}$$

$$\boldsymbol{B}(\boldsymbol{x}, t) = \frac{\gamma u}{c}\,q\,\frac{y\hat{\boldsymbol{z}} - z\hat{\boldsymbol{y}}}{\big[\gamma^2(x - ut)^2 + y^2 + z^2\big]^{3/2}} \ . \tag{15.259}$$

Let us define

$$\boldsymbol{R}(t) = (x - ut)\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} \ . \tag{15.260}$$

We further define the angle $\theta \equiv \cos^{-1}\!\big(\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{R}}\big)$. We may then write

$$\boldsymbol{E}(x, t) = \frac{q\,\boldsymbol{R}}{R^3} \cdot \frac{1 - \beta^2}{\big(1 - \beta^2\sin^2\!\theta\big)^{3/2}}$$

$$\boldsymbol{B}(x, t) = \frac{q\,\hat{\boldsymbol{\beta}} \times \boldsymbol{R}}{R^3} \cdot \frac{1 - \beta^2}{\big(1 - \beta^2\sin^2\!\theta\big)^{3/2}} \ . \tag{15.261}$$

The fields are therefore enhanced in the transverse directions: $E_\perp/E_\parallel = \gamma^3$.

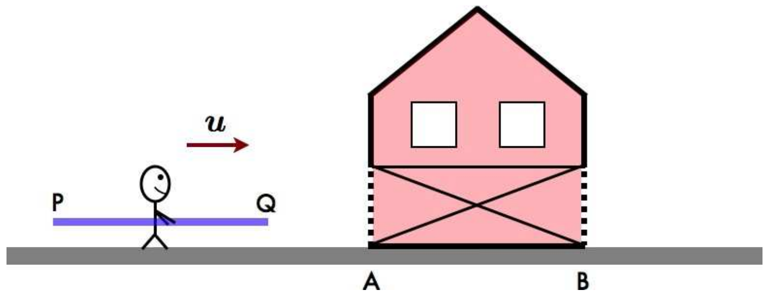


Figure 15.12: A relativistic runner carries a pole of proper length ℓ and runs into a barn of proper length ℓ.

15.9 Appendix I : The Pole, the Barn, and *Rashoman*

Akira Kurosawa's 1950 cinematic masterpiece, *Rashoman*, describes a rape, murder, and battle from four different and often contradictory points of view. It poses deep questions regarding the nature of truth. Psychologists sometimes refer to problems of subjective perception as the *Rashoman effect*. In literature, William Faulkner's 1929 novel, *The Sound and the Fury*, which describes the tormented incestuous life of a Mississippi family, also is told from four points of view. Perhaps Faulkner would be a more apt comparison with Einstein, since time plays an essential role in his novel. For example, Quentin's watch, given to him by his father, represents time and the sweep of life's arc (*"Quentin, I give you the mausoleum of all hope and desire..."*). By breaking the watch, Quentin symbolically attempts to escape time and fate. One could draw an analogy to Einstein, inheriting a watch from those who came before him, which he too broke – and refashioned. Did Faulkner know of Einstein? But I digress.

Consider a relativistic runner carrying a pole of proper length ℓ, as depicted in fig. 15.12. He runs toward a barn of proper length ℓ at velocity $u = c\beta$. Let the frame of the barn be K and the frame of the runner be K'. Recall that the Lorentz transformations between frames K and K' are given by

$$ct = \gamma ct' + \gamma x' \qquad\qquad ct' = \gamma ct - \gamma\beta x \tag{15.262}$$

$$x = \gamma\beta ct' + \gamma x' \qquad\qquad x' = -\gamma\beta ct + \gamma x \; . \tag{15.263}$$

We define the following points. Let A denote the left door of the barn and B the right door. Furthermore, let P denote the left end of the pole and Q its right end. The spacetime coordinates for these points in the two frames are clearly .

$$A = (ct\,,\,0) \qquad\qquad P' = (ct'\,,\,0) \tag{15.264}$$

$$B = (ct\,,\,\ell) \qquad\qquad Q' = (ct'\,,\,\ell) \tag{15.265}$$

We now compute A' and B' in frame K', as well as P and Q in frame K:

$$A' = (\gamma ct\,,\,-\gamma\beta ct) \qquad\qquad B' = (\gamma ct - \gamma\beta\ell\,,\,-\gamma\beta ct + \gamma\ell) \tag{15.266}$$

$$\equiv (ct'\,,\,-\beta ct') \qquad\qquad \equiv (ct'\,,\,-\beta ct' + \gamma^{-1}\ell) \; . \tag{15.267}$$

Similarly,

$$P = (\gamma ct' \, , \, \gamma\beta ct') \qquad Q = (\gamma ct' + \gamma\beta\ell \, , \, \gamma\beta ct' + \gamma\ell) \tag{15.268}$$

$$\equiv (ct \, , \, \beta ct) \qquad \equiv (ct \, , \, \beta ct + \gamma^{-1}\ell) \; . \tag{15.269}$$

We now define four events, by the coincidences of A and B with P and Q:

- Event I : The right end of the pole enters the left door of the barn. This is described by $Q = A$ in frame K and by $Q' = A'$ in frame K'.
- Event II : The right end of the pole exits the right door of the barn. This is described by $Q = B$ in frame K and by $Q' = B'$ in frame K'.
- Event III : The left end of the pole enters the left door of the barn. This is described by $P = A$ in frame K and by $P' = A'$ in frame K'.
- Event IV : The left end of the pole exits the right door of the barn. This is described by $P = B$ in frame K and by $P' = B'$ in frame K'.

Mathematically, we have in frame K that

$$\text{I}: \quad Q = A \qquad \Rightarrow \qquad t_{\text{I}} = -\frac{\ell}{\gamma u} \tag{15.270}$$

$$\text{II}: \quad Q = B \qquad \Rightarrow \qquad t_{\text{II}} = (\gamma - 1)\,\frac{\ell}{\gamma u} \tag{15.271}$$

$$\text{III}: \quad P = A \qquad \Rightarrow \qquad t_{\text{III}} = 0 \tag{15.272}$$

$$\text{IV}: \quad P = B \qquad \Rightarrow \qquad t_{\text{IV}} = \frac{\ell}{u} \tag{15.273}$$

In frame K', however

$$\text{I}: \quad Q' = A' \qquad \Rightarrow \qquad t'_{\text{I}} = -\frac{\ell}{u} \tag{15.274}$$

$$\text{II}: \quad Q' = B' \qquad \Rightarrow \qquad t'_{\text{II}} = -(\gamma - 1)\,\frac{\ell}{\gamma u} \tag{15.275}$$

$$\text{III}: \quad P' = A' \qquad \Rightarrow \qquad t'_{\text{III}} = 0 \tag{15.276}$$

$$\text{IV}: \quad P' = B' \qquad \Rightarrow \qquad t'_{\text{IV}} = \frac{\ell}{\gamma u} \tag{15.277}$$

Thus, to an observer in frame K, the order of events is I, III, II, and IV, because

$$t_{\text{I}} \; < \; t_{\text{III}} \; < \; t_{\text{II}} \; < \; t_{\text{IV}} \; . \tag{15.278}$$

For $t_{\text{III}} < t < t_{\text{II}}$, he observes that *the pole is entirely in the barn*. Indeed, the right door can start shut and the left door open, and sensors can automatically and, for the purposes of argument, instantaneously trigger the closing of the left door immediately following event

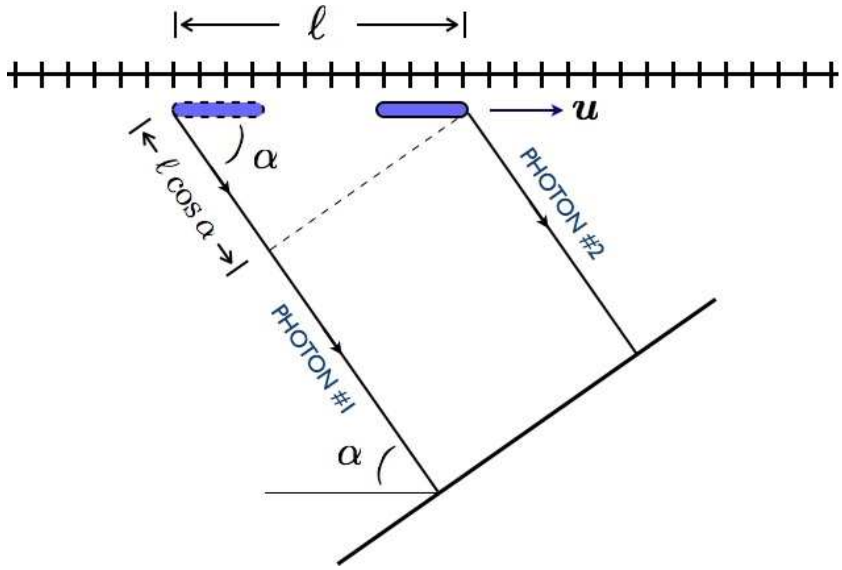


Figure 15.13: An object of proper length ℓ and moving with velocity $\boldsymbol{u}$, when photographed from an angle α, appears to have a length $\tilde{\ell}$.

III and the opening of the right door immediately prior to event II. So the pole can be inside the barn with both doors shut!

But now for the *Rashoman effect*: according to the runner, the order of events is I, II, III, and IV, because

$$t'_{\text{I}} \ < \ t'_{\text{II}} \ < \ t'_{\text{III}} \ < \ t'_{\text{IV}} \ . \tag{15.279}$$

At no time does the runner observe the pole to be entirely within the barn. Indeed, for $t'_{\text{II}} < t' < t'_{\text{III}}$, both ends of the pole are sticking outside of the barn!

15.10 Appendix II : Photographing a Moving Pole

What is the length ℓ of a moving pole of proper length ℓ_0 as measured by an observer at rest? The answer would appear to be $\gamma^{-1}\ell_0$, as we computed in eqn. 15.71. However, we should be more precise when we we speak of 'length'. The relation $\ell(\beta) = \gamma^{-1}\ell_0$ tells us the *instantaneous end-to-end distance as measured in the observer's rest frame* K. But an actual experiment might not measure this quantity.

For example, suppose a relativistic runner carrying a pole of proper length ℓ_0 runs past a measuring rod which is at rest in the rest frame K of an observer. The observer *takes a photograph* of the moving pole as it passes by. Suppose further that the angle between the observer's line of sight and the velocity $\boldsymbol{u}$ of the pole is α, as shown in fig. 15.13. What is the apparent length $\ell(\alpha, u)$ of the pole as observed in the photograph? (*I.e.* the pole will appear to cover a portion of the measuring rod which is of length ℓ.)

The point here is that the shutter of the camera is very fast (otherwise the image will appear blurry). In our analysis we will assume the shutter opens and closes instantaneously. Let's define two events:

- Event 1 : photon γ_1 is emitted by the rear end of the pole.
- Event 2 : photon γ_2 is emitted by the front end of the pole.

Both photons must arrive at the camera's lens simultaneously. Since, as shown in the figure, the path of photon #1 is longer by a distance $\ell \cos\alpha$, where ℓ is the apparent length of the pole, γ_2 must be emitted a time $\Delta t = c^{-1}\ell\cos\alpha$ after γ_1. Now if we Lorentz transform from frame K to frame K', we have

$$\Delta x' = \gamma\Delta x - \gamma\beta\Delta t \ . \tag{15.280}$$

But $\Delta x' = \ell_0$ is the proper length of the pole, and $\Delta x = \ell$ is the apparent length. With $c\Delta t = \ell\cos\alpha$, then, we have

$$\ell = \frac{\gamma^{-1}\,\ell_0}{1 - \beta\cos\alpha} \ . \tag{15.281}$$

When $\alpha = 90^\circ$, we recover the familiar Lorentz-Fitzgerald contraction $\ell(\beta) = \gamma^{-1}\,\ell_0$. This is because the photons γ_1 and γ_2 are then emitted simultaneously, and the photograph measures the instantaneous end-to-end distance of the pole as measured in the observer's rest frame K. When $\cos\alpha \neq 0$, however, the two photons are not emitted simultaneously, and the apparent length is given by eqn. 15.281.

Chapter 16

Hamiltonian Mechanics

16.1 The Hamiltonian

Recall that $L = L(q, \dot{q}, t)$, and

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \ . \tag{16.1}$$

The Hamiltonian, $H(q,p)$ is obtained by a Legendre transformation,

$$H(q,p) = \sum_{\sigma=1}^{n} p_\sigma \, \dot{q}_\sigma - L \ . \tag{16.2}$$

Note that

$$\begin{aligned} dH &= \sum_{\sigma=1}^{n} \bigg(p_\sigma \, d\dot{q}_\sigma + \dot{q}_\sigma \, dp_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma} \, d\dot{q}_\sigma \bigg) - \frac{\partial L}{\partial t} \, dt \\ &= \sum_{\sigma=1}^{n} \bigg(\dot{q}_\sigma \, dp_\sigma - \frac{\partial L}{\partial q_\sigma} \, dq_\sigma \bigg) - \frac{\partial L}{\partial t} \, dt \ . \end{aligned} \tag{16.3}$$

Thus, we obtain Hamilton's equations of motion,

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma \quad , \quad \frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -\dot{p}_\sigma \tag{16.4}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \ . \tag{16.5}$$

Some remarks:

- As an example, consider a particle moving in three dimensions, described by spherical polar coordinates (r, θ, ϕ). Then

$$L = \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \, \dot{\theta}^2 + r^2 \sin^2\!\theta \, \dot{\phi}^2 \right) - U(r, \theta, \phi) \ . \tag{16.6}$$

We have

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\,\dot{\theta} \quad , \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\,\sin^2\!\theta\,\dot{\phi} \ , \tag{16.7}$$

and thus

$$\begin{aligned} H &= p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\phi\,\dot{\phi} - L \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2\sin^2\!\theta} + U(r,\theta,\phi) \ . \end{aligned} \tag{16.8}$$

Note that H is time-independent, hence $\frac{\partial H}{\partial t} = \frac{dH}{dt} = 0$, and therefore H is a constant of the motion.

- In order to obtain $H(q,p)$ we must invert the relation $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = p_\sigma(q,\dot{q})$ to obtain $\dot{q}_\sigma(q,p)$. This is possible if the Hessian,

$$\frac{\partial p_\alpha}{\partial \dot{q}_\beta} = \frac{\partial^2 L}{\partial \dot{q}_\alpha\,\partial \dot{q}_\beta} \tag{16.9}$$

is nonsingular. This is the content of the 'inverse function theorem' of multivariable calculus.

- Define the rank $2n$ vector, ξ, by its components,

$$\xi_i = \begin{cases} q_i & \text{if } 1 \le i \le n \\ p_{i-n} & \text{if } n < i \le 2n \ . \end{cases} \tag{16.10}$$

Then we may write Hamilton's equations compactly as

$$\dot{\xi}_i = J_{ij}\,\frac{\partial H}{\partial \xi_j} \ , \tag{16.11}$$

where

$$J = \begin{pmatrix} \mathbb{O}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix} \tag{16.12}$$

is a rank $2n$ matrix. Note that $J^{\rm t} = -J$, *i.e.* J is antisymmetric, and that $J^2 = -\mathbb{I}_{2n\times 2n}$. We shall utilize this 'symplectic structure' to Hamilton's equations shortly.

16.2 Modified Hamilton's Principle

We have that

$$
\begin{aligned}
0 = \delta \int_{t_a}^{t_b} dt\, L = \delta \int_{t_a}^{t_b} dt\, \big(p_\sigma\, \dot{q}_\sigma - H\big) \qquad (16.13) \\
= \int_{t_a}^{t_b} dt \left\{ p_\sigma\, \delta\dot{q}_\sigma + \dot{q}_\sigma\, \delta p_\sigma - \frac{\partial H}{\partial q_\sigma}\, \delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\, \delta p_\sigma \right\} \\
= \int_{t_a}^{t_b} dt \left\{ -\left(\dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma}\right) \delta q_\sigma + \left(\dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma}\right) \delta p_\sigma \right\} + \big(p_\sigma\, \delta q_\sigma\big)\Big|_{t_a}^{t_b} \ ,
\end{aligned}
$$

assuming $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. Setting the coefficients of δq_σ and δp_σ to zero, we recover Hamilton's equations.

16.3 Phase Flow is Incompressible

A flow for which $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ is *incompressible* – we shall see why in a moment. Let's check that the divergence of the phase space velocity does indeed vanish:

$$
\begin{aligned}
\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\xi}} &= \sum_{\sigma=1}^{n} \left\{ \frac{\partial \dot{q}_\sigma}{\partial q_\sigma} + \frac{\partial \dot{p}_\sigma}{\partial p_\sigma} \right\} \\
&= \sum_{i=1}^{2n} \frac{\partial \dot{\xi}_i}{\partial \xi_i} = \sum_{i,j} J_{ij}\, \frac{\partial^2 H}{\partial \xi_i\, \partial \xi_j} = 0 \ . \qquad (16.14)
\end{aligned}
$$

Now let $\rho(\boldsymbol{\xi}, t)$ be a distribution on phase space. Continuity implies

$$
\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \dot{\boldsymbol{\xi}}) = 0 \ . \qquad (16.15)
$$

Invoking $\boldsymbol{\nabla} \cdot \dot{\boldsymbol{\xi}} = 0$, we have that

$$
\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \dot{\boldsymbol{\xi}} \cdot \boldsymbol{\nabla} \rho = 0 \ , \qquad (16.16)
$$

where $D\rho/Dt$ is sometimes called the *convective derivative* – it is the total derivative of the function $\rho\big(\boldsymbol{\xi}(t), t\big)$, evaluated at a point $\xi(t)$ in phase space which moves according to the dynamics. This says that the density in the "comoving frame" is locally constant.

16.4 Poincaré Recurrence Theorem

Let g_τ be the 'τ-advance mapping' which evolves points in phase space according to Hamilton's equations

$$\dot{q}_i = +\frac{\partial H}{\partial p_i} \quad , \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{16.17}$$

for a time interval $\varDelta t = \tau$. Consider a region $\varOmega$ in phase space. Define $g_\tau^n \varOmega$ to be the n^{th} image of $\varOmega$ under the mapping g_τ. Clearly g_τ is invertible; the inverse is obtained by integrating the equations of motion backward in time. We denote the inverse of g_τ by g_τ^{-1}. By Liouville's theorem, g_τ is volume preserving when acting on regions in phase space, since the evolution of any given point is Hamiltonian. This follows from the continuity equation for the phase space density,

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\boldsymbol{u}\varrho) = 0 \tag{16.18}$$

where $\boldsymbol{u} = \{\dot{\boldsymbol{q}}, \dot{\boldsymbol{p}}\}$ is the velocity vector in phase space, and Hamilton's equations, which say that the phase flow is incompressible, *i.e.* $\nabla \cdot \boldsymbol{u} = 0$:

$$\begin{aligned} \nabla \cdot \boldsymbol{u} &= \sum_{i=1}^n \left\{ \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right\} \\ &= \sum_{i=1}^n \left\{ \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) \right\} = 0 \ . \end{aligned} \tag{16.19}$$

Thus, we have that the convective derivative vanishes, *viz.*

$$\frac{D\varrho}{Dt} \equiv \frac{\partial \varrho}{\partial t} + \boldsymbol{u} \cdot \nabla \varrho = 0 \ , \tag{16.20}$$

which guarantees that the density remains constant in a frame moving with the flow.

The proof of the recurrence theorem is simple. Assume that g_τ is invertible and volume-preserving, as is the case for Hamiltonian flow. Further assume that phase space volume is finite. Since the energy is preserved in the case of time-independent Hamiltonians, we simply ask that the volume of phase space *at fixed total energy* E be finite, *i.e.*

$$\int d\mu \, \delta\big(E - H(\boldsymbol{q}, \boldsymbol{p})\big) < \infty \ , \tag{16.21}$$

where $d\mu = d\boldsymbol{q}\, d\boldsymbol{p}$ is the phase space uniform integration measure.

Theorem: In any finite neighborhood Ω of phase space there exists a point $\boldsymbol{\varphi}_0$ which will return to Ω after n applications of g_τ, where n is finite.

Proof: Assume the theorem fails; we will show this assumption results in a contradiction. Consider the set Υ formed from the union of all sets $g_\tau^m\, \Omega$ for all m:

$$\Upsilon = \bigcup_{m=0}^{\infty} g_\tau^m\, \Omega \tag{16.22}$$

We assume that the set $\{g_\tau^m \,\Omega \,|\, m \in \mathbb{Z}\,, m \geq 0\}$ is disjoint. The volume of a union of disjoint sets is the sum of the individual volumes. Thus,

$$\begin{aligned}\mathrm{vol}(\Upsilon) &= \sum_{m=0}^{\infty} \mathrm{vol}(g_\tau^m \,\Omega) \\ &= \mathrm{vol}(\Omega) \cdot \sum_{m=1}^{\infty} 1 = \infty \ ,\end{aligned} \tag{16.23}$$

since $\mathrm{vol}(g_\tau^m \,\Omega) = \mathrm{vol}(\Omega)$ from volume preservation. But clearly Υ is a subset of the entire phase space, hence we have a contradiction, because by assumption phase space is of finite volume.

Thus, the assumption that the set $\{g_\tau^m \,\Omega \,|\, m \in \mathbb{Z}\,, m \geq 0\}$ is disjoint fails. This means that there exists some pair of integers k and l, with $k \neq l$, such that $g_\tau^k \,\Omega \cap g_\tau^l \,\Omega \neq \emptyset$. Without loss of generality we may assume $k > l$. Apply the inverse g_τ^{-1} to this relation l times to get $g_\tau^{k-l}\,\Omega \cap \Omega \neq \emptyset$. Now choose any point $\boldsymbol{\varphi} \in g_\tau^n \,\Omega \cap \Omega$, where $n = k - l$, and define $\boldsymbol{\varphi}_0 = g_\tau^{-n} \boldsymbol{\varphi}$. Then by construction both $\boldsymbol{\varphi}_0$ and $g_\tau^n \,\boldsymbol{\varphi}_0$ lie within Ω and the theorem is proven.

Each of the two central assumptions – invertibility and volume preservation – is crucial. Without either of them, the proof fails. Consider, for example, a volume-preserving map which is not invertible. An example might be a mapping $f\colon \mathbb{R} \to \mathbb{R}$ which takes any real number to its fractional part. Thus, $f(\pi) = 0.14159265\ldots$. Let us restrict our attention to intervals of width less than unity. Clearly f is then volume preserving. The action of f on the interval $[2, 3)$ is to map it to the interval $[0, 1)$. But $[0, 1)$ remains fixed under the action of f, so no point within the interval $[2, 3)$ will ever return under repeated iterations of f. Thus, f does not exhibit Poincaré recurrence.

Consider next the case of the damped harmonic oscillator. In this case, phase space volumes contract. For a one-dimensional oscillator obeying $\ddot{x} + 2\beta\dot{x} + \Omega_0^2 x = 0$ one has $\nabla \cdot \boldsymbol{u} = -2\beta < 0$ ($\beta > 0$ for damping). Thus the convective derivative obeys $D_t \varrho = -(\nabla \cdot \boldsymbol{u})\varrho = +2\beta\varrho$ which says that the density increases exponentially in the comoving frame, as $\varrho(t) = e^{2\beta t}\,\varrho(0)$. Thus, phase space volumes collapse, and are not preserved by the dynamics. In this case, it is possible for the set Υ to be of finite volume, even if it is the union of an infinite number of sets $g_\tau^n \,\Omega$, because the volumes of these component sets themselves decrease exponentially, as $\mathrm{vol}(g_\tau^n \,\Omega) = e^{-2n\beta\tau}\,\mathrm{vol}(\Omega)$. A damped pendulum, released from rest at some small angle θ_0, will not return arbitrarily close to these initial conditions.

16.5 Poisson Brackets

The time evolution of any function $F(q, p)$ over phase space is given by

$$\begin{aligned}\frac{d}{dt}\,F\big(q(t), p(t), t\big) &= \frac{\partial F}{\partial t} + \sum_{\sigma=1}^{n} \left\{ \frac{\partial F}{\partial q_\sigma}\,\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\,\dot{p}_\sigma \right\} \\ &\equiv \frac{\partial F}{\partial t} + \big\{ F, H \big\} \ ,\end{aligned} \tag{16.24}$$

where the *Poisson bracket* $\{\cdot\,,\cdot\}$ is given by

$$\big\{A,B\big\} \equiv \sum_{\sigma=1}^{n} \left(\frac{\partial A}{\partial q_\sigma}\,\frac{\partial B}{\partial p_\sigma} - \frac{\partial A}{\partial p_\sigma}\,\frac{\partial B}{\partial q_\sigma}\right) \tag{16.25}$$

$$= \sum_{i,j=1}^{2n} J_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j}\ . \tag{16.26}$$

Properties of the Poisson bracket:

- Antisymmetry:
$$\big\{f,g\big\} = -\big\{g,f\big\}\ . \tag{16.27}$$

- Bilinearity: if λ is a constant, and f, g, and h are functions on phase space, then
$$\big\{f+\lambda\, g,h\big\} = \big\{f,h\big\} + \lambda\big\{g,h\big\}\ . \tag{16.28}$$
Linearity in the second argument follows from this and the antisymmetry condition.

- Associativity:
$$\big\{fg,h\big\} = f\big\{g,h\big\} + g\big\{f,h\big\}\ . \tag{16.29}$$

- Jacobi identity:
$$\big\{f,\{g,h\}\big\} + \big\{g,\{h,f\}\big\} + \big\{h,\{f,g\}\big\} = 0\ . \tag{16.30}$$

Some other useful properties:

- If $\{A,H\}=0$ and $\frac{\partial A}{\partial t}=0$, then $\frac{dA}{dt}=0$, *i.e.* $A(q,p)$ is a constant of the motion.
- If $\{A,H\}=0$ and $\{B,H\}=0$, then $\big\{\{A,B\},H\big\}=0$. If in addition A and B have no explicit time dependence, we conclude that $\{A,B\}$ is a constant of the motion.
- It is easily established that
$$\{q_\alpha,q_\beta\}=0 \quad , \quad \{p_\alpha,p_\beta\}=0 \quad , \quad \{q_\alpha,p_\beta\}=\delta_{\alpha\beta}\ . \tag{16.31}$$

16.6 Canonical Transformations

16.6.1 Point transformations in Lagrangian mechanics

In Lagrangian mechanics, we are free to redefine our generalized coordinates, *viz.*

$$Q_\sigma = Q_\sigma(q_1,\ldots,q_n,t)\ . \tag{16.32}$$

This is called a "point transformation." The transformation is invertible if

$$\det\left(\frac{\partial Q_\alpha}{\partial q_\beta}\right) \neq 0 \ . \tag{16.33}$$

The transformed Lagrangian, $\tilde{L}$, written as a function of the new coordinates Q and velocities $\dot{Q}$, is

$$\tilde{L}\big(Q, \dot{Q}, t\big) = L\big(q(Q,t), \dot{q}(Q, \dot{Q}, t)\big) \ . \tag{16.34}$$

Finally, Hamilton's principle,

$$\delta \int\limits_{t_1}^{t_b} dt\, \tilde{L}(Q, \dot{Q}, t) = 0 \tag{16.35}$$

with $\delta Q_\sigma(t_a) = \delta Q_\sigma(t_b) = 0$, still holds, and the form of the Euler-Lagrange equations remains unchanged:

$$\frac{\partial \tilde{L}}{\partial Q_\sigma} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = 0 \ . \tag{16.36}$$

The invariance of the equations of motion under a point transformation may be verified explicitly. We first evaluate

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma}\right) , \tag{16.37}$$

where the relation

$$\frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_\sigma} = \frac{\partial q_\alpha}{\partial Q_\sigma} \tag{16.38}$$

follows from

$$\dot{q}_\alpha = \frac{\partial q_\alpha}{\partial Q_\sigma}\,\dot{Q}_\sigma + \frac{\partial q_\alpha}{\partial t} \ . \tag{16.39}$$

Now we compute

$$\begin{aligned}
\frac{\partial \tilde{L}}{\partial Q_\sigma} &= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial \dot{q}_\alpha}{\partial Q_\sigma} \\
&= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial Q_{\sigma'}}\,\dot{Q}_{\sigma'} + \frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial t}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{d}{dt}\left(\frac{\partial q_\alpha}{\partial Q_\sigma}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial q_\alpha}{\partial Q_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) ,
\end{aligned} \tag{16.40}$$

where the last equality is what we obtained earlier in eqn. 16.37.

16.6.2 Canonical transformations in Hamiltonian mechanics

In Hamiltonian mechanics, we will deal with a much broader class of transformations – ones which mix all the $q's$ and $p's$. The general form for a canonical transformation (CT) is

$$q_\sigma = q_\sigma\big(Q_1, \ldots, Q_n; P_1, \ldots, P_n; t\big) \tag{16.41}$$

$$p_\sigma = p_\sigma\big(Q_1, \ldots, Q_n; P_1, \ldots, P_n; t\big) \ , \tag{16.42}$$

with $\sigma \in \{1, \ldots, n\}$. We may also write

$$\xi_i = \xi_i\big(\Xi_1, \ldots, \Xi_{2n}; t\big) \ , \tag{16.43}$$

with $i \in \{1, \ldots, 2n\}$. The transformed Hamiltonian is $\tilde{H}(Q, P, t)$.

What sorts of transformations are allowed? Well, if Hamilton's equations are to remain invariant, then

$$\dot{Q}_\sigma = \frac{\partial \tilde{H}}{\partial P_\sigma} \quad , \quad \dot{P}_\sigma = -\frac{\partial \tilde{H}}{\partial Q_\sigma} \ , \tag{16.44}$$

which gives

$$\frac{\partial \dot{Q}_\sigma}{\partial Q_\sigma} + \frac{\partial \dot{P}_\sigma}{\partial P_\sigma} = 0 = \frac{\partial \dot{\Xi}_i}{\partial \Xi_i} \ . \tag{16.45}$$

I.e. the flow remains incompressible in the new (Q, P) variables. We will also require that phase space volumes are preserved by the transformation, *i.e.*

$$\det\!\left(\frac{\partial \Xi_i}{\partial \xi_j}\right) = \left|\left|\frac{\partial(Q, P)}{\partial(q, p)}\right|\right| = 1 \ . \tag{16.46}$$

Additional conditions will be discussed below.

16.6.3 Hamiltonian evolution

Hamiltonian evolution itself defines a canonical transformation. Let $\xi_i = \xi_i(t)$ and $\xi_i' = \xi_i(t + dt)$. Then from the dynamics $\dot{\xi}_i = J_{ij} \frac{\partial H}{\partial \xi_j}$, we have

$$\xi_i(t + dt) = \xi_i(t) + J_{ij} \frac{\partial H}{\partial \xi_j} \, dt + \mathcal{O}\big(dt^2\big) \ . \tag{16.47}$$

Thus,

$$\begin{aligned} \frac{\partial \xi_i'}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\bigg(\xi_i + J_{ik} \frac{\partial H}{\partial \xi_k} \, dt + \mathcal{O}\big(dt^2\big)\bigg) \\ &= \delta_{ij} + J_{ik} \frac{\partial^2 H}{\partial \xi_j \, \partial \xi_k} \, dt + \mathcal{O}\big(dt^2\big) \ . \end{aligned} \tag{16.48}$$

Now, using the result

$$\det\big(1 + \epsilon M\big) = 1 + \epsilon \, \mathrm{Tr}\, M + \mathcal{O}(\epsilon^2) \ , \tag{16.49}$$

we have

$$\left\| \frac{\partial \xi'_i}{\partial \xi_j} \right\| = 1 + J_{jk} \, \frac{\partial^2 H}{\partial \xi_j \, \partial \xi_k} \, dt + \mathcal{O}\big(dt^2\big) \tag{16.50}$$

$$= 1 + \mathcal{O}\big(dt^2\big) \ . \tag{16.51}$$

16.6.4 Symplectic structure

We have that

$$\dot{\xi}_i = J_{ij} \, \frac{\partial H}{\partial \xi_j} \ . \tag{16.52}$$

Suppose we make a time-independent canonical transformation to new phase space coordinates, $\Xi_a = \Xi_a(\xi)$. We then have

$$\dot{\Xi}_a = \frac{\partial \Xi_a}{\partial \xi_j} \, \dot{\xi}_j = \frac{\partial \Xi_a}{\partial \xi_j} \, J_{jk} \, \frac{\partial H}{\partial \xi_k} \ . \tag{16.53}$$

But if the transformation is canonical, then the equations of motion are preserved, and we also have

$$\dot{\Xi}_a = J_{ab} \, \frac{\partial \tilde{H}}{\partial \Xi_b} = J_{ab} \, \frac{\partial \xi_k}{\partial \Xi_b} \, \frac{\partial H}{\partial \xi_k} \ . \tag{16.54}$$

Equating these two expressions, we have

$$M_{aj} \, J_{jk} \, \frac{\partial H}{\partial \xi_k} = J_{ab} \, M^{-1}_{kb} \, \frac{\partial H}{\partial \xi_k} \ , \tag{16.55}$$

where

$$M_{aj} \equiv \frac{\partial \Xi_a}{\partial \xi_j} \tag{16.56}$$

is the Jacobian of the transformation. Since the equality must hold for all ξ, we conclude

$$MJ = J\big(M^{\rm t}\big)^{-1} \quad \Longrightarrow \quad MJM^{\rm t} = J \ . \tag{16.57}$$

A matrix M satisfying $MM^{\rm t} = \mathbb{I}$ is of course an *orthogonal* matrix. A matrix M satisfying $MJM^{\rm t} = J$ is called *symplectic*. We write $M \in {\rm Sp}(2n)$, *i.e.* M is an element of the group of *symplectic matrices*[1] of rank $2n$.

The symplectic property of M guarantees that the Poisson brackets are preserved under a

[1]Note that the rank of a symplectic matrix is always even. Note also $MJM^{\rm t} = J$ implies $M^{\rm t}JM = J$.

canonical transformation:

$$
\begin{aligned}
\{A,B\}_{\xi} &= J_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j} \\
&= J_{ij}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial \Xi_a}{\partial \xi_i}\,\frac{\partial B}{\partial \Xi_b}\,\frac{\partial \Xi_b}{\partial \xi_j} \\
&= \big(M_{ai}\,J_{ij}\,M^{\rm t}_{jb}\big)\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} \\
&= J_{ab}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} \\
&= \{A,B\}_{\Xi}\ . \qquad (16.58)
\end{aligned}
$$

16.6.5 Generating functions for canonical transformations

For a transformation to be canonical, we require

$$\delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{p_\sigma\,\dot q_\sigma - H(q,p,t)\Big\} = 0 = \delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{P_\sigma\,\dot Q_\sigma - \tilde H(Q,P,t)\Big\}\ . \qquad (16.59)$$

This is satisfied provided

$$\Big\{p_\sigma\,\dot q_\sigma - H(q,p,t)\Big\} = \lambda\,\bigg\{P_\sigma\,\dot Q_\sigma - \tilde H(Q,P,t) + \frac{dF}{dt}\bigg\}\ , \qquad (16.60)$$

where λ is a constant. For canonical transformations, $\lambda=1$.[2] Thus,

$$
\begin{aligned}
\tilde H(Q,P,t) = H(q,p,t) + P_\sigma\,\dot Q_\sigma - p_\sigma\,\dot q_\sigma &+ \frac{\partial F}{\partial q_\sigma}\,\dot q_\sigma + \frac{\partial F}{\partial Q_\sigma}\,\dot Q_\sigma \\
&+ \frac{\partial F}{\partial p_\sigma}\,\dot p_\sigma + \frac{\partial F}{\partial P_\sigma}\,\dot P_\sigma + \frac{\partial F}{\partial t}\ . \qquad (16.61)
\end{aligned}
$$

Thus, we require

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \quad,\quad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad,\quad \frac{\partial F}{\partial p_\sigma} = 0 \quad,\quad \frac{\partial F}{\partial P_\sigma} = 0\ . \qquad (16.62)$$

The transformed Hamiltonian is

$$\tilde H(Q,P,t) = H(q,p,t) + \frac{\partial F}{\partial t}\ . \qquad (16.63)$$

[2]Solutions of eqn. 16.60 with $\lambda \neq 1$ are known as *extended* canonical transformations. We can always rescale coordinates and/or momenta to achieve $\lambda = 1$.

There are four possibilities, corresponding to the freedom to make Legendre transformations with respect to each of the arguments of $F(q,Q)$:

$$F(q,Q,t) = \begin{cases} F_1(q,Q,t) & ; \quad p_\sigma = +\frac{\partial F_1}{\partial q_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \quad \text{(type I)} \\ \\ F_2(q,P,t) - P_\sigma\, Q_\sigma & ; \quad p_\sigma = +\frac{\partial F_2}{\partial q_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_2}{\partial P_\sigma} \quad \text{(type II)} \\ \\ F_3(p,Q,t) + p_\sigma\, q_\sigma & ; \quad q_\sigma = -\frac{\partial F_3}{\partial p_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_3}{\partial Q_\sigma} \quad \text{(type III)} \\ \\ F_4(p,P,t) + p_\sigma\, q_\sigma - P_\sigma\, Q_\sigma & ; \quad q_\sigma = -\frac{\partial F_4}{\partial p_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_4}{\partial P_\sigma} \quad \text{(type IV)} \end{cases}$$

In each case ($\gamma = 1,2,3,4$), we have

$$\tilde{H}(Q,P,t) = H(q,p,t) + \frac{\partial F_\gamma}{\partial t} \ . \tag{16.64}$$

Let's work out some examples:

- Consider the type-II transformation generated by

$$F_2(q,P) = A_\sigma(q)\, P_\sigma \ , \tag{16.65}$$

where $A_\sigma(q)$ is an arbitrary function of the $\{q_\sigma\}$. We then have

$$Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} = A_\sigma(q) \qquad , \qquad p_\sigma = \frac{\partial F_2}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma}\, P_\alpha \ . \tag{16.66}$$

Thus,

$$Q_\sigma = A_\sigma(q) \qquad , \qquad P_\sigma = \frac{\partial q_\alpha}{\partial Q_\sigma}\, p_\alpha \ . \tag{16.67}$$

This is a general point transformation of the kind discussed in eqn. 16.32. For a general linear point transformation, $Q_\alpha = M_{\alpha\beta}\, q_\beta$, we have $P_\alpha = p_\beta\, M^{-1}_{\beta\alpha}$, *i.e.* $Q = Mq$, $P = p\, M^{-1}$. If $M_{\alpha\beta} = \delta_{\alpha\beta}$, this is the identity transformation. $F_2 = q_1 P_3 + q_3 P_1$ interchanges labels 1 and 3, *etc.*

- Consider the type-I transformation generated by

$$F_1(q,Q) = A_\sigma(q)\, Q_\sigma \ . \tag{16.68}$$

We then have

$$p_\sigma = \frac{\partial F_1}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma}\, Q_\alpha \tag{16.69}$$

$$P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} = -A_\sigma(q) \ . \tag{16.70}$$

Note that $A_\sigma(q) = q_\sigma$ generates the transformation

$$\begin{pmatrix} q \\ p \end{pmatrix} \longrightarrow \begin{pmatrix} -P \\ +Q \end{pmatrix} \ . \tag{16.71}$$

- A mixed transformation is also permitted. For example,

$$F(q,Q) = q_1\,Q_1 + (q_3 - Q_2)\,P_2 + (q_2 - Q_3)\,P_3 \tag{16.72}$$

is of type-I with respect to index $\sigma = 1$ and type-II with respect to indices $\sigma = 2,3$. The transformation effected is

$$Q_1 = p_1 \qquad Q_2 = q_3 \qquad Q_3 = q_2 \tag{16.73}$$

$$P_1 = -q_1 \qquad P_2 = p_3 \qquad P_3 = p_2\ . \tag{16.74}$$

- Consider the harmonic oscillator,

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2\ . \tag{16.75}$$

If we could find a time-independent canonical transformation such that

$$p = \sqrt{2mf(P)}\,\cos Q \quad , \quad q = \sqrt{\frac{2\,f(P)}{k}}\,\sin Q\ , \tag{16.76}$$

where $f(P)$ is some function of P, then we'd have $\tilde{H}(Q,P) = f(P)$, which is cyclic in Q. To find this transformation, we take the ratio of p and q to obtain

$$p = \sqrt{mk}\,q\ \text{ctn}\,Q\ , \tag{16.77}$$

which suggests the type-I transformation

$$F_1(q,Q) = \tfrac{1}{2}\sqrt{mk}\,q^2\ \text{ctn}\,Q\ . \tag{16.78}$$

This leads to

$$p = \frac{\partial F_1}{\partial q} = \sqrt{mk}\,q\ \text{ctn}\,Q \quad , \quad P = -\frac{\partial F_1}{\partial Q} = \frac{\sqrt{mk}\,q^2}{2\,\sin^2\!Q}\ . \tag{16.79}$$

Thus,

$$q = \frac{\sqrt{2P}}{\sqrt[4]{mk}}\,\sin Q \quad \Longrightarrow \quad f(P) = \sqrt{\frac{k}{m}}\,P = \omega P\ , \tag{16.80}$$

where $\omega = \sqrt{k/m}$ is the oscillation frequency. We therefore have

$$\tilde{H}(Q,P) = \omega P\ , \tag{16.81}$$

whence $P = E/\omega$. The equations of motion are

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \quad , \quad \dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \omega\ , \tag{16.82}$$

which yields

$$Q(t) = \omega t + \varphi_0 \quad , \quad q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin\big(\omega t + \varphi_0\big)\ . \tag{16.83}$$

16.7 Hamilton-Jacobi Theory

We've stressed the great freedom involved in making canonical transformations. Coordinates and momenta, for example, may be interchanged – the distinction between them is purely a matter of convention! We now ask: is there any specially preferred canonical transformation? In this regard, one obvious goal is to make the Hamiltonian $\tilde{H}(Q,P,t)$ and the corresponding equations of motion as simple as possible.

Recall the general form of the canonical transformation:

$$\tilde{H}(Q,P) = H(q,p) + \frac{\partial F}{\partial t} \ , \tag{16.84}$$

with

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \qquad\qquad \frac{\partial F}{\partial p_\sigma} = 0 \tag{16.85}$$

$$\frac{\partial F}{\partial Q_\sigma} = -P_\sigma \qquad\qquad \frac{\partial F}{\partial P_\sigma} = 0 \ . \tag{16.86}$$

We now demand that this transformation result in the simplest Hamiltonian possible, that is, $\tilde{H}(Q,P,t) = 0$. This requires we find a function F such that

$$\frac{\partial F}{\partial t} = -H \quad , \quad \frac{\partial F}{\partial q_\sigma} = p_\sigma \ . \tag{16.87}$$

The remaining functional dependence may be taken to be either on Q (type I) or on P (type II). As it turns out, the generating function F we seek is in fact the action, S, which is the integral of L with respect to time, expressed as a function of its endpoint values.

16.7.1 The action as a function of coordinates and time

We have seen how the action $S[\eta(\tau)]$ is a *functional* of the path $\eta(\tau)$ and a *function* of the endpoint values $\{q_a, t_a\}$ and $\{q_b, t_b\}$. Let us define the action *function* $S(q,t)$ as

$$S(q,t) = \int\limits_{t_a}^{t} d\tau \, L\big(\eta, \dot{\eta}, \tau\big) \ , \tag{16.88}$$

where $\eta(\tau)$ starts at (q_a, t_a) and ends at (q,t). We also require that $\eta(\tau)$ satisfy the Euler-Lagrange equations,

$$\frac{\partial L}{\partial \eta_\sigma} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right) = 0 \tag{16.89}$$

Let us now consider a new path, $\tilde{\eta}(\tau)$, also starting at (q_a, t_a), but ending at $(q + dq, t + dt)$, and also satisfying the equations of motion. The differential of S is

$$\begin{aligned}
dS &= S\big[\tilde{\eta}(\tau)\big] - S\big[\eta(\tau)\big] \\
&= \int_{t_a}^{t+dt} d\tau \, L(\tilde{\eta}, \dot{\tilde{\eta}}, \tau) - \int_{t_a}^{t} d\tau \, L\big(\eta, \dot{\eta}, \tau\big) \qquad (16.90) \\
&= \int_{t_a}^{t} d\tau \left\{ \frac{\partial L}{\partial \eta_\sigma} \Big[\tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau)\Big] + \frac{\partial L}{\partial \dot{\eta}_\sigma} \Big[\dot{\tilde{\eta}}_\sigma(\tau) - \dot{\eta}_\sigma(\tau)\Big] \right\} + L\big(\tilde{\eta}(t), \dot{\tilde{\eta}}(t), t\big)\, dt \\
&= \int_{t_a}^{t} d\tau \left\{ \frac{\partial L}{\partial \eta_\sigma} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{\eta}_\sigma}\right) \right\} \Big[\tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau)\Big] \\
&\qquad + \frac{\partial L}{\partial \dot{\eta}_\sigma}\bigg|_t \Big[\tilde{\eta}_\sigma(t) - \eta_\sigma(t)\Big] + L\big(\tilde{\eta}(t), \dot{\tilde{\eta}}(t), t\big)\, dt \\
&= 0 + \pi_\sigma(t)\, \delta\eta_\sigma(t) + L\big(\eta(t), \dot{\eta}(t), t\big)\, dt + \mathcal{O}(\delta q \cdot dt) \ , \qquad (16.91)
\end{aligned}$$

where we have defined

$$\pi_\sigma = \frac{\partial L}{\partial \dot{\eta}_\sigma} \ , \qquad (16.92)$$

and

$$\delta\eta_\sigma(\tau) \equiv \tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau) \ . \qquad (16.93)$$

Note that the differential dq_σ is given by

$$\begin{aligned}
dq_\sigma &= \tilde{\eta}_\sigma(t + dt) - \eta_\sigma(t) \qquad (16.94) \\
&= \tilde{\eta}_\sigma(t + dt) - \tilde{\eta}_\sigma(t) + \tilde{\eta}_\sigma(t) - \eta_\sigma(t) \\
&= \dot{\tilde{\eta}}_\sigma(t)\, dt + \delta\eta_\sigma(t) \\
&= \dot{q}_\sigma(t)\, dt + \delta\eta_\sigma(t) + \mathcal{O}(\delta q \cdot dt) \ . \qquad (16.95)
\end{aligned}$$

Thus, with $\pi_\sigma(t) \equiv p_\sigma$, we have

$$\begin{aligned}
dS &= p_\sigma \, dq_\sigma + \big(L - p_\sigma \, \dot{q}_\sigma\big)\, dt \\
&= p_\sigma \, dq_\sigma - H\, dt \ . \qquad (16.96)
\end{aligned}$$

We therefore obtain

$$\frac{\partial S}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial S}{\partial t} = -H \quad , \quad \frac{dS}{dt} = L \ . \qquad (16.97)$$

What about the lower limit at t_a? Clearly there are $n+1$ constants associated with this limit: $\big\{q_1(t_a), \ldots, q_n(t_a); t_a\big\}$. Thus, we may write

$$S = S(q_1, \ldots, q_n; \Lambda_1, \ldots, \Lambda_n, t) + \Lambda_{n+1} \ , \qquad (16.98)$$

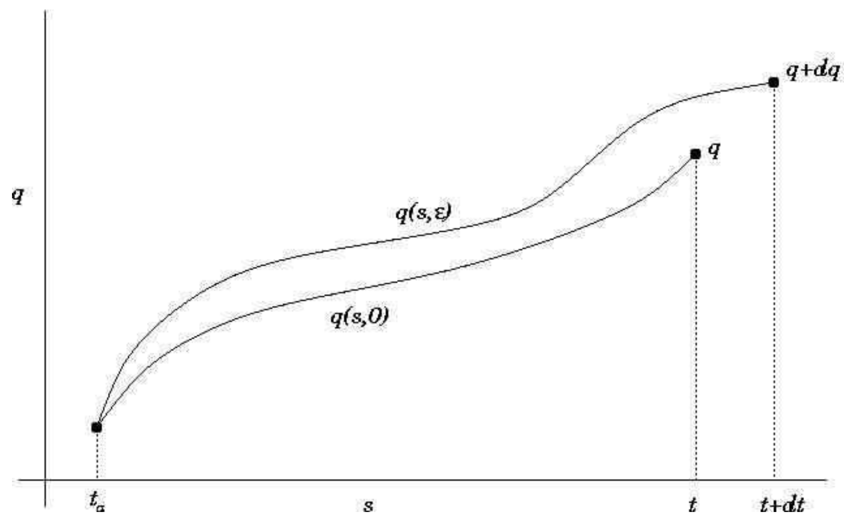


Figure 16.1: A one-parameter family of paths $q(s;\epsilon)$.

where our $n+1$ constants are $\{\Lambda_1,\ldots,\Lambda_{n+1}\}$. If we regard S as a mixed generator, which is type-I in some variables and type-II in others, then each Λ_σ for $1\le\sigma\le n$ may be chosen to be either Q_σ or P_σ. We will define

$$\Gamma_\sigma = \frac{\partial S}{\partial\Lambda_\sigma} = \begin{cases} +Q_\sigma & \text{if } \Lambda_\sigma = P_\sigma \\ -P_\sigma & \text{if } \Lambda_\sigma = Q_\sigma \end{cases} \tag{16.99}$$

For each σ, the two possibilities $\Lambda_\sigma = Q_\sigma$ or $\Lambda_\sigma = P_\sigma$ are of course rendered equivalent by a canonical transformation $(Q_\sigma, P_\sigma) \to (P_\sigma, -Q_\sigma)$.

16.7.2 The Hamilton-Jacobi equation

Since the action $S(q,\Lambda,t)$ has been shown to generate a canonical transformation for which $\tilde{H}(Q,P)=0$. This requirement may be written as

$$H\Big(q_1,\ldots,q_n,\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n},t\Big) + \frac{\partial S}{\partial t} = 0\ . \tag{16.100}$$

This is the *Hamilton-Jacobi equation* (HJE). It is a first order partial differential equation in $n+1$ variables, and in general is nonlinear (since kinetic energy is generally a quadratic function of momenta). Since $\tilde{H}(Q,P,t)=0$, the equations of motion are trivial, and

$$Q_\sigma(t) = \text{const.} \quad , \quad P_\sigma(t) = \text{const.} \tag{16.101}$$

Once the HJE is solved, one must invert the relations $\Gamma_\sigma = \partial S(q,\Lambda,t)/\partial\Lambda_\sigma$ to obtain $q(Q,P,t)$. This is possible only if

$$\det\!\left(\frac{\partial^2 S}{\partial q_\alpha\,\partial\Lambda_\beta}\right) \neq 0\ , \tag{16.102}$$

which is known as the *Hessian condition*.

It is worth noting that the HJE may have several solutions. For example, consider the case of the free particle, with $H(q,p) = p^2/2m$. The HJE is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0 \ . \tag{16.103}$$

One solution of the HJE is

$$S(q,\Lambda,t) = \frac{m\,(q-\Lambda)^2}{2t} \ . \tag{16.104}$$

For this we find

$$\Gamma = \frac{\partial S}{\partial \Lambda} = -\frac{m}{t}\,(q-\Lambda) \quad \Rightarrow \quad q(t) = \Lambda - \frac{\Gamma}{m}\,t \ . \tag{16.105}$$

Here $\Lambda = q(0)$ is the initial value of q, and $\Gamma = -p$ is minus the momentum.

Another equally valid solution to the HJE is

$$S(q,\Lambda,t) = q\,\sqrt{2m\Lambda} - \Lambda\,t \ . \tag{16.106}$$

This yields

$$\Gamma = \frac{\partial S}{\partial \Lambda} = q\,\sqrt{\frac{2m}{\Lambda}} - t \quad \Rightarrow \quad q(t) = \sqrt{\frac{\Lambda}{2m}}\,(t+\Gamma) \ . \tag{16.107}$$

For this solution, Λ is the energy and Γ may be related to the initial value of $q(t) = \Gamma\sqrt{\Lambda/2m}$.

16.7.3 Time-independent Hamiltonians

When H has no explicit time dependence, we may reduce the order of the HJE by one, writing

$$S(q,\Lambda,t) = W(q,\Lambda) + T(\Lambda,t) \ . \tag{16.108}$$

The HJE becomes

$$H\left(q, \frac{\partial W}{\partial q}\right) = -\frac{\partial T}{\partial t} \ . \tag{16.109}$$

Note that the LHS of the above equation is independent of t, and the RHS is independent of q. Therefore, each side must only depend on the constants Λ, which is to say that each side must be a constant, which, without loss of generality, we take to be Λ_1. Therefore

$$S(q,\Lambda,t) = W(q,\Lambda) - \Lambda_1 t \ . \tag{16.110}$$

The function $W(q,\Lambda)$ is called *Hamilton's characteristic function*. The HJE now takes the form

$$H\left(q_1,\ldots,q_n,\frac{\partial W}{\partial q_1},\ldots,\frac{\partial W}{\partial q_n}\right) = \Lambda_1 \ . \tag{16.111}$$

Note that adding an arbitrary constant C to S generates the same equation, and simply shifts the last constant $\Lambda_{n+1} \to \Lambda_{n+1} + C$. This is equivalent to replacing t by $t - t_0$ with $t_0 = C/\Lambda_1$, *i.e.* it just redefines the zero of the time variable.

16.7.4 Example: one-dimensional motion

As an example of the method, consider the one-dimensional system,

$$H(q,p) = \frac{p^2}{2m} + U(q) \ . \tag{16.112}$$

The HJE is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + U(q) = \Lambda \ . \tag{16.113}$$

which may be recast as

$$\frac{\partial S}{\partial q} = \sqrt{2m\big[\Lambda - U(q)\big]} \ , \tag{16.114}$$

with solution

$$S(q,\Lambda,t) = \sqrt{2m}\int^{q} \! dq' \, \sqrt{\Lambda - U(q')} - \Lambda\, t \ . \tag{16.115}$$

We now have

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\big[\Lambda - U(q)\big]} \ , \tag{16.116}$$

as well as

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \sqrt{\frac{m}{2}}\int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q')}} - t \ . \tag{16.117}$$

Thus, the motion $q(t)$ is given by quadrature:

$$\Gamma + t = \sqrt{\frac{m}{2}}\int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q')}} \ , \tag{16.118}$$

where Λ and Γ are constants. The lower limit on the integral is arbitrary and merely shifts t by another constant. Note that Λ is the total energy.

16.7.5 Separation of variables

It is convenient to first work an example before discussing the general theory. Consider the following Hamiltonian, written in spherical polar coordinates:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\!\theta}\right) + \overbrace{A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\!\theta}}^{\text{potential } U(r,\theta,\phi)} \ . \tag{16.119}$$

We seek a solution with the characteristic function

$$W(r,\theta,\phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi) \ . \tag{16.120}$$

The HJE is then

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{2mr^2\sin^2\!\theta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2\sin^2\!\theta} = \Lambda_1 = E \ . \tag{16.121}$$

Multiply through by $r^2\sin^2\!\theta$ to obtain

$$\frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) = -\sin^2\!\theta\left\{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta)\right\} - r^2\sin^2\!\theta\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\} \ . \tag{16.122}$$

The LHS is independent of (r,θ), and the RHS is independent of ϕ. Therefore, we may set

$$\frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) = \Lambda_2 \ . \tag{16.123}$$

Proceeding, we replace the LHS in eqn. 16.122 with Λ_2, arriving at

$$\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\!\theta} = -r^2\left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\} \ . \tag{16.124}$$

The LHS of this equation is independent of r, and the RHS is independent of θ. Therefore,

$$\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\!\theta} = \Lambda_3 \ . \tag{16.125}$$

We're left with

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) + \frac{\Lambda_3}{r^2} = \Lambda_1 \ . \tag{16.126}$$

The full solution is therefore

$$S(q,\Lambda,t) = \sqrt{2m}\int^{r}\!dr'\,\sqrt{\Lambda_1 - A(r') - \frac{\Lambda_3}{r'^2}} \tag{16.127}$$

$$+ \sqrt{2m}\int^{\theta}\!d\theta'\,\sqrt{\Lambda_3 - B(\theta') - \frac{\Lambda_2}{\sin^2\!\theta'}} + \sqrt{2m}\int^{\phi}\!d\phi'\,\sqrt{\Lambda_2 - C(\phi')} - \Lambda_1 t \ . \tag{16.128}$$

We then have

$$\Gamma_1 = \frac{\partial S}{\partial \Lambda_1} = \int^{r(t)} \frac{\sqrt{\frac{m}{2}}\, dr'}{\sqrt{\Lambda_1 - A(r') - \Lambda_3\, r'^{-2}}} - t \tag{16.129}$$

$$\Gamma_2 = \frac{\partial S}{\partial \Lambda_2} = -\int^{\theta(t)} \frac{\sqrt{\frac{m}{2}}\, d\theta'}{\sin^2\theta' \sqrt{\Lambda_3 - B(\theta') - \Lambda_2 \csc^2\theta'}} + \int^{\phi(t)} \frac{\sqrt{\frac{m}{2}}\, d\phi'}{\sqrt{\Lambda_2 - C(\phi')}} \tag{16.130}$$

$$\Gamma_3 = \frac{\partial S}{\partial \Lambda_3} = -\int^{r(t)} \frac{\sqrt{\frac{m}{2}}\, dr'}{r'^2 \sqrt{\Lambda_1 - A(r') - \Lambda_3\, r'^{-2}}} + \int^{\theta(t)} \frac{\sqrt{\frac{m}{2}}\, d\theta'}{\sqrt{\Lambda_3 - B(\theta') - \Lambda_2 \csc^2\theta'}} \ . \tag{16.131}$$

The game plan here is as follows. The first of the above trio of equations is inverted to yield $r(t)$ in terms of t and constants. This solution is then invoked in the last equation (the upper limit on the first integral on the RHS) in order to obtain an implicit equation for $\theta(t)$, which is invoked in the second equation to yield an implicit equation for $\phi(t)$. The net result is the motion of the system in terms of time t and the six constants $(\Lambda_1, \Lambda_2, \Lambda_3, \Gamma_1, \Gamma_2, \Gamma_3)$. A seventh constant, associated with an overall shift of the zero of t, arises due to the arbitrary lower limits of the integrals.

In general, the separation of variables method begins with[3]

$$W(q, \Lambda) = \sum_{\sigma=1}^{n} W_\sigma(q_\sigma, \Lambda) \ . \tag{16.132}$$

Each $W_\sigma(q_\sigma, \Lambda)$ may be regarded as a function of the single variable q_σ, and is obtained by satisfying an ODE of the form[4]

$$H_\sigma\left(q_\sigma, \frac{dW_\sigma}{dq_\sigma}\right) = \Lambda_\sigma \ . \tag{16.133}$$

We then have

$$p_\sigma = \frac{\partial W_\sigma}{\partial q_\sigma} \quad , \quad \Gamma_\sigma = \frac{\partial W}{\partial \Lambda_\sigma} + \delta_{\sigma,1}\, t \ . \tag{16.134}$$

Note that while each W_σ depends on only a single q_σ, it may depend on several of the Λ_σ.

16.7.6 Example #2 : point charge plus electric field

Consider a potential of the form

$$U(r) = \frac{k}{r} - Fz \ , \tag{16.135}$$

which corresponds to a charge in the presence of an external point charge plus an external electric field. This problem is amenable to separation in parabolic coordinates, (ξ, η, φ):

$$x = \sqrt{\xi\eta}\, \cos\varphi \quad , \quad y = \sqrt{\xi\eta}\, \sin\varphi \quad , \quad z = \tfrac{1}{2}(\xi - \eta) \ . \tag{16.136}$$

[3]Here we assume *complete separability*. A given system may only be *partially* separable.

[4]$H_\sigma(q_\sigma, p_\sigma)$ may also depend on several of the Λ_α . See *e.g.* eqn. 16.126, which is of the form $H_r\big(r, \partial_r W_r, \Lambda_3\big) = \Lambda_1$.

Note that

$$\begin{aligned}\rho &\equiv \sqrt{x^2+y^2} = \sqrt{\xi\eta} && (16.137)\\ r &= \sqrt{\rho^2+z^2} = \tfrac{1}{2}(\xi+\eta)\ . && (16.138)\end{aligned}$$

The kinetic energy is

$$\begin{aligned}T &= \tfrac{1}{2}m\big(\dot\rho^2+\rho^2\,\dot\varphi^2+\dot z^2\big)\\ &= \tfrac{1}{8}m\,(\xi+\eta)\bigg(\frac{\dot\xi^2}{\xi}+\frac{\dot\eta^2}{\eta}\bigg)+\tfrac{1}{2}m\,\xi\eta\,\dot\varphi^2\ , && (16.139)\end{aligned}$$

and hence the Lagrangian is

$$L = \tfrac{1}{8}m\,(\xi+\eta)\bigg(\frac{\dot\xi^2}{\xi}+\frac{\dot\eta^2}{\eta}\bigg)+\tfrac{1}{2}m\,\xi\eta\,\dot\varphi^2-\frac{2k}{\xi+\eta}+\tfrac{1}{2}F\,(\xi-\eta)\ . \qquad (16.140)$$

Thus, the conjugate momenta are

$$\begin{aligned}p_\xi &= \frac{\partial L}{\partial\dot\xi} = \tfrac{1}{4}m\,(\xi+\eta)\,\frac{\dot\xi}{\xi} && (16.141)\\ p_\eta &= \frac{\partial L}{\partial\dot\eta} = \tfrac{1}{4}m\,(\xi+\eta)\,\frac{\dot\eta}{\eta} && (16.142)\\ p_\varphi &= \frac{\partial L}{\partial\dot\varphi} = m\,\xi\eta\,\dot\varphi\ , && (16.143)\end{aligned}$$

and the Hamiltonian is

$$\begin{aligned}H &= p_\xi\,\dot\xi+p_\eta\,\dot\eta+p_\varphi\,\dot\varphi && (16.144)\\ &= \frac{2}{m}\bigg(\frac{\xi\,p_\xi^2+\eta\,p_\eta^2}{\xi+\eta}\bigg)+\frac{p_\varphi^2}{2m\xi\eta}+\frac{2k}{\xi+\eta}-\tfrac{1}{2}F\,(\xi-\eta)\ . && (16.145)\end{aligned}$$

Notice that $\partial H/\partial t=0$, which means $dH/dt=0$, *i.e.* $H=E\equiv\Lambda_1$ is a constant of the motion. Also, φ is cyclic in H, so its conjugate momentum p_φ is a constant of the motion.

We write

$$\begin{aligned}S(q,\Lambda) &= W(q,\Lambda)-Et && (16.146)\\ &= W_\xi(\xi,\Lambda)+W_\eta(\eta,\Lambda)+W_\varphi(\varphi,\Lambda)-Et\ . && (16.147)\end{aligned}$$

with $E=\Lambda_1$. Clearly we may take

$$W_\varphi(\varphi,\Lambda) = P_\varphi\,\varphi\ , \qquad (16.148)$$

where $P_\varphi = \Lambda_2$. Multiplying the Hamilton-Jacobi equation by $\frac{1}{2}m\,(\xi+\eta)$ then gives

$$\xi\bigg(\frac{dW_\xi}{d\xi}\bigg)^2 + \frac{P_\varphi^2}{4\xi} + mk - \tfrac{1}{4}F\xi^2 - \tfrac{1}{2}mE\xi$$
$$= -\eta\bigg(\frac{dW_\eta}{d\eta}\bigg)^2 - \frac{P_\varphi^2}{4\eta} - \tfrac{1}{4}F\eta^2 + \tfrac{1}{2}mE\eta \equiv \Upsilon \ , \tag{16.149}$$

where $\Upsilon = \Lambda_3$ is the third constant: $\Lambda = (E, P_\varphi, \Upsilon)$. Thus,

$$S\big(\overbrace{\xi,\eta,\varphi}^{q};\underbrace{E,P_\varphi,\Upsilon}_{\Lambda}\big) = \int^\xi d\xi'\sqrt{\tfrac{1}{2}mE + \frac{\Upsilon - mk}{\xi'} + \tfrac{1}{4}mF\xi' - \frac{P_\varphi^2}{4\xi'^2}}$$
$$+\int^\eta d\eta'\sqrt{\tfrac{1}{2}mE - \frac{\Upsilon}{\eta'} - \tfrac{1}{4}mF\eta' - \frac{P_\varphi^2}{4\eta'^2}}$$
$$+ P_\varphi\,\varphi - Et \ . \tag{16.150}$$

16.7.7 Example #3 : Charged Particle in a Magnetic Field

The Hamiltonian is

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\Big)^2 \ . \tag{16.151}$$

We choose the gauge $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$, and we write

$$S(x,y,P_1,P_2) = W_x(x,P_1,P_2) + W_y(y,P_1,P_2) - P_1\,t \ . \tag{16.152}$$

Note that here we will consider S to be a function of $\{q_\sigma\}$ and $\{P_\sigma\}$.

The Hamilton-Jacobi equation is then

$$\bigg(\frac{\partial W_x}{\partial x}\bigg)^2 + \bigg(\frac{\partial W_y}{\partial y} - \frac{eBx}{c}\bigg)^2 = 2mP_1 \ . \tag{16.153}$$

We solve by writing

$$W_y = P_2\,y \qquad \Rightarrow \qquad \bigg(\frac{dW_x}{dx}\bigg)^2 + \bigg(P_2 - \frac{eBx}{c}\bigg)^2 = 2mP_1 \ . \tag{16.154}$$

This equation suggests the substitution

$$x = \frac{cP_2}{eB} + \frac{c}{eB}\sqrt{2mP_1}\,\sin\theta \ . \tag{16.155}$$

in which case

$$\frac{\partial x}{\partial\theta} = \frac{c}{eB}\sqrt{2mP_1}\,\cos\theta \tag{16.156}$$

and

$$\frac{\partial W_x}{\partial x} = \frac{\partial W_x}{\partial \theta} \cdot \frac{\partial \theta}{\partial x} = \frac{eB}{c\sqrt{2mP_1}} \, \frac{1}{\cos\theta} \, \frac{\partial W_x}{\partial \theta} \ . \tag{16.157}$$

Substitution this into eqn. 16.154, we have

$$\frac{\partial W_x}{\partial \theta} = \frac{2mcP_1}{eB} \, \cos^2\!\theta \ , \tag{16.158}$$

with solution

$$W_x = \frac{mcP_1}{eB} \, \theta + \frac{mcP_1}{2eB} \, \sin(2\theta) \ . \tag{16.159}$$

We then have

$$p_x = \frac{\partial W_x}{\partial x} = \frac{\partial W_x}{\partial \theta} \bigg/ \frac{\partial x}{\partial \theta} = \sqrt{2mP_1} \, \cos\theta \tag{16.160}$$

and

$$p_y = \frac{\partial W_y}{\partial y} = P_2 \ . \tag{16.161}$$

The type-II generator we seek is then

$$S(q,P,t) = \frac{mcP_1}{eB} \, \theta + \frac{mcP_1}{2eB} \, \sin(2\theta) + P_2 \, y - P_1 \, t \ , \tag{16.162}$$

where

$$\theta = \frac{eB}{c\sqrt{2mP_1}} \, \sin^{-1}\!\bigg(x - \frac{cP_2}{eB}\bigg) \ . \tag{16.163}$$

Note that, from eqn. 16.155, we may write

$$dx = \frac{c}{eB} \, dP_2 + \frac{mc}{eB} \, \frac{1}{\sqrt{2mP_1}} \, \sin\theta \, dP_1 + \frac{c}{eB} \, \sqrt{2mP_1} \, \cos\theta \, d\theta \ , \tag{16.164}$$

from which we derive

$$\frac{\partial \theta}{\partial P_1} = -\frac{\tan\theta}{2P_1} \qquad , \qquad \frac{\partial \theta}{\partial P_2} = -\frac{1}{\sqrt{2mP_1} \, \cos\theta} \ . \tag{16.165}$$

These results are useful in the calculation of Q_1 and Q_2:

$$\begin{aligned} Q_1 &= \frac{\partial S}{\partial P_1} \\ &= \frac{mc}{eB} \, \theta + \frac{mcP_1}{eB} \, \frac{\partial \theta}{\partial P_1} + \frac{mc}{2eB} \, \sin(2\theta) + \frac{mcP_1}{eB} \, \cos(2\theta) \, \frac{\partial \theta}{\partial P_1} - t \\ &= \frac{mc}{eB} \, \theta - t \end{aligned} \tag{16.166}$$

and

$$\begin{aligned} Q_2 &= \frac{\partial S}{\partial P_2} \\ &= y + \frac{mcP_1}{eB} \big[1 + \cos(2\theta) \big] \, \frac{\partial \theta}{\partial P_2} \\ &= y - \frac{c}{eB} \, \sqrt{2mP_1} \, \cos\theta \ . \end{aligned} \tag{16.167}$$

Now since $\tilde{H}(P,Q)=0$, we have that $\dot{Q}_\sigma=0$, which means that each Q_σ is a constant. We therefore have the following solution:

$$x(t) = x_0 + A\,\sin(\omega_c t + \delta) \tag{16.168}$$

$$y(t) = y_0 + A\,\cos(\omega_c t + \delta)\ , \tag{16.169}$$

where $\omega_c = eB/mc$ is the 'cyclotron frequency', and

$$x_0 = \frac{cP_2}{eB} \quad , \quad y_0 = Q_2 \quad , \quad \delta \equiv \omega_c\, Q_1 \quad , \quad A = \frac{c}{eB}\,\sqrt{2mP_1}\ . \tag{16.170}$$

16.8 Action-Angle Variables

16.8.1 Circular Phase Orbits: Librations and Rotations

In a completely integrable system, the Hamilton-Jacobi equation may be solved by separation of variables. Each momentum p_σ is a function of only its corresponding coordinate q_σ plus constants – no other coordinates enter:

$$p_\sigma = \frac{\partial W_\sigma}{\partial q_\sigma} = p_\sigma(q_\sigma, \Lambda)\ . \tag{16.171}$$

The motion satisfies

$$H_\sigma(q_\sigma, p_\sigma) = \Lambda_\sigma\ . \tag{16.172}$$

The level sets of H_σ are curves $\mathcal{C}_\sigma$. In general, these curves each depend on all of the constants Λ, so we write $\mathcal{C}_\sigma = \mathcal{C}_\sigma(\Lambda)$. The curves $\mathcal{C}_\sigma$ are the *projections* of the full motion onto the (q_σ, p_σ) plane. In general we will assume the motion, and hence the curves $\mathcal{C}_\sigma$, is *bounded*. In this case, two types of projected motion are possible: librations and rotations. Librations are periodic oscillations about an equilibrium position. Rotations involve the advancement of an angular variable by 2π during a cycle. This is most conveniently illustrated in the case of the simple pendulum, for which

$$H(p_\phi, \phi) = \frac{p_\phi^2}{2I} + \tfrac{1}{2}I\omega^2\left(1 - \cos\phi\right)\ . \tag{16.173}$$

- When $E < I\,\omega^2$, the momentum p_ϕ vanishes at $\phi = \pm\cos^{-1}(2E/I\omega^2)$. The system executes librations between these extreme values of the angle ϕ.
- When $E > I\,\omega^2$, the kinetic energy is always positive, and the angle advances monotonically, executing rotations.

In a completely integrable system, each $\mathcal{C}_\sigma$ is either a libration or a rotation[5]. Both librations and rotations are closed curves. Thus, each $\mathcal{C}_\sigma$ is in general homotopic to (= "can be

[5] $\mathcal{C}_\sigma$ may correspond to a separatrix, but this is a nongeneric state of affairs.

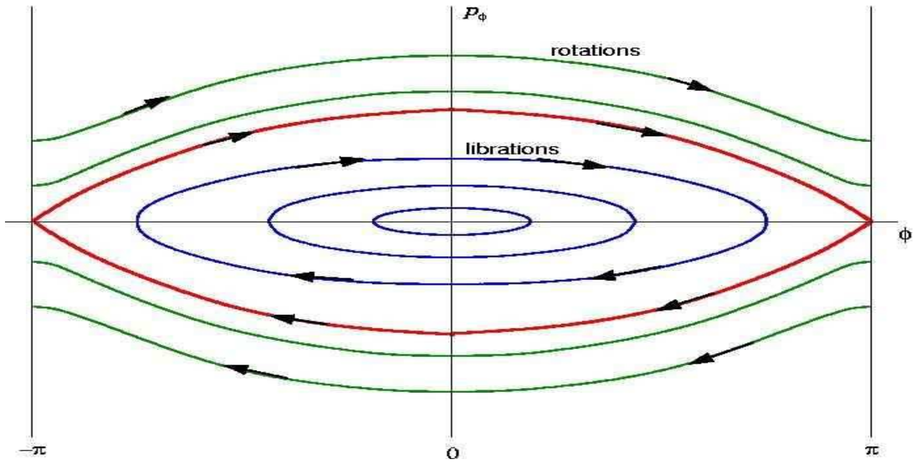


Figure 16.2: Phase curves for the simple pendulum, showing librations (in blue), rotations (in green), and the separatrix (in red). This phase flow is most correctly viewed as taking place on a cylinder, obtained from the above sketch by identifying the lines $\phi = \pi$ and $\phi = -\pi$.

continuously distorted to yield") a circle, $\mathbb{S}^1$. For n freedoms, the motion is therefore confined to an n-torus, $\mathbb{T}^n$:

$$\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}} . \tag{16.174}$$

These are called *invariant tori* (or *invariant manifolds*). There are many such tori, as there are many $\mathcal{C}_\sigma$ curves in each of the n two-dimensional submanifolds.

Invariant tori never intersect! This is ruled out by the uniqueness of the solution to the dynamical system, expressed as a set of coupled ordinary differential equations.

Note also that phase space is of dimension $2n$, while the invariant tori are of dimension n. Phase space is 'covered' by the invariant tori, but it is in general difficult to conceive of how this happens. Perhaps the most accessible analogy is the $n = 1$ case, where the '1-tori' are just circles. Two-dimensional phase space is covered noninteracting circular orbits. (The orbits are *topologically* equivalent to circles, although *geometrically* they may be distorted.) It is challenging to think about the $n = 2$ case, where a four-dimensional phase space is filled by nonintersecting 2-tori.

16.8.2 Action-Angle Variables

For a completely integrable system, one can transform canonically from (q, p) to new coordinates (ϕ, J) which specify a particular n-torus $\mathbb{T}^n$ as well as the location on the torus, which is specified by n angle variables. The $\{J_\sigma\}$ are 'momentum' variables which specify the torus itself; they are constants of the motion since the tori are invariant. They are called *action variables*. Since $\dot{J}_\sigma = 0$, we must have

$$\dot{J}_\sigma = -\frac{\partial H}{\partial \phi_\sigma} = 0 \quad \Longrightarrow \quad H = H(J) \ . \tag{16.175}$$

The $\{\phi_\sigma\}$ are the *angle variables*.

The coordinate ϕ_σ describes the projected motion along $\mathcal{C}_\sigma$, and is normalized by

$$\oint\limits_{\mathcal{C}_\sigma} d\phi_\sigma = 2\pi \qquad (\text{once around } \mathcal{C}_\sigma) \ . \tag{16.176}$$

The dynamics of the angle variables are given by

$$\dot{\phi}_\sigma = \frac{\partial H}{\partial J_\sigma} \equiv \nu_\sigma(J) \ . \tag{16.177}$$

Thus,

$$\phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(J)\, t \ . \tag{16.178}$$

The $\big\{\nu_\sigma(J)\big\}$ are *frequencies* describing the rate at which the $\mathcal{C}_\sigma$ are traversed; $T_\sigma(J) = 2\pi/\nu_\sigma(J)$ is the period.

16.8.3 Canonical Transformation to Action-Angle Variables

The $\{J_\sigma\}$ determine the $\{\mathcal{C}_\sigma\}$; each q_σ determines a point on $\mathcal{C}_\sigma$. This suggests a type-II transformation, with generator $F_2(q,J)$:

$$p_\sigma = \frac{\partial F_2}{\partial q_\sigma} \quad , \quad \phi_\sigma = \frac{\partial F_2}{\partial J_\sigma} \ . \tag{16.179}$$

Note that[6]

$$2\pi = \oint\limits_{\mathcal{C}_\sigma} d\phi_\sigma = \oint\limits_{\mathcal{C}_\sigma} d\bigg(\frac{\partial F_2}{\partial J_\sigma}\bigg) = \oint\limits_{\mathcal{C}_\sigma} \frac{\partial^2 F_2}{\partial J_\sigma\,\partial q_\sigma}\, dq_\sigma = \frac{\partial}{\partial J_\sigma} \oint\limits_{\mathcal{C}_\sigma} p_\sigma\, dq_\sigma \ , \tag{16.180}$$

which suggests the definition

$$J_\sigma = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} p_\sigma\, dq_\sigma \ . \tag{16.181}$$

I.e. J_σ is $(2\pi)^{-1}$ times the area enclosed by $\mathcal{C}_\sigma$.

If, separating variables,

$$W(q,\Lambda) = \sum_\sigma W_\sigma(q_\sigma,\Lambda) \tag{16.182}$$

[6] In general, we should write $d\big(\frac{\partial F_2}{\partial J_\sigma}\big) = \frac{\partial^2 F_2}{\partial J_\sigma\,\partial q_\alpha}\, dq_\alpha$ with a sum over α. However, in eqn. 16.180 all coordinates and momenta other than q_σ and p_σ are held fixed. Thus, $\alpha = \sigma$ is the only term in the sum which contributes.

is Hamilton's characteristic function for the transformation $(q,p) \to (Q,P)$, then

$$J_\sigma = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} \frac{\partial W_\sigma}{\partial q_\sigma}\, dq_\sigma = J_\sigma(\Lambda) \tag{16.183}$$

is a function only of the $\{\Lambda_\alpha\}$ and not the $\{\Gamma_\alpha\}$. We then invert this relation to obtain $\Lambda(J)$, to finally obtain

$$F_2(q,J) = W\big(q, \Lambda(J)\big) = \sum_\sigma W_\sigma\big(q_\sigma, \Lambda(J)\big) \ . \tag{16.184}$$

Thus, the recipe for canonically transforming to action-angle variable is as follows:

(1) Separate and solve the Hamilton-Jacobi equation for $W(q,\Lambda) = \sum_\sigma W_\sigma(q_\sigma, \Lambda)$.

(2) Find the orbits $\mathcal{C}_\sigma$ – the level sets of satisfying $H_\sigma(q_\sigma, p_\sigma) = \Lambda_\sigma$.

(3) Invert the relation $J_\sigma(\Lambda) = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} \frac{\partial W_\sigma}{\partial q_\sigma}\, dq_\sigma$ to obtain $\Lambda(J)$.

(4) $F_2(q,J) = \sum_\sigma W_\sigma\big(q_\sigma, \Lambda(J)\big)$ is the desired type-II generator[7].

16.8.4 Example : Harmonic Oscillator

The Hamiltonian is

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega_0^2 q^2 \ , \tag{16.185}$$

hence the Hamilton-Jacobi equation is

$$\left(\frac{dW}{dq}\right)^2 + m^2\omega_0^2 q^2 = 2m\Lambda \ . \tag{16.186}$$

Thus,

$$p = \frac{dW}{dq} = \pm\sqrt{2m\Lambda - m^2\omega_0^2 q^2} \ . \tag{16.187}$$

We now define

$$q \equiv \left(\frac{2\Lambda}{m\omega_0^2}\right)^{1/2} \sin\theta \quad \Rightarrow \quad p = \sqrt{2m\Lambda}\,\cos\theta \ , \tag{16.188}$$

in which case

$$J = \frac{1}{2\pi} \oint p\, dq = \frac{1}{2\pi} \cdot \frac{2\Lambda}{\omega_0} \cdot \int\limits_0^{2\pi} d\theta\, \cos^2\!\theta = \frac{\Lambda}{\omega_0} \ . \tag{16.189}$$

[7] Note that $F_2(q,J)$ is time-independent. *I.e.* we are not transforming to $\tilde{H} = 0$, but rather to $\tilde{H} = \tilde{H}(J)$.

Solving the HJE, we write

$$\frac{dW}{d\theta} = \frac{\partial q}{\partial \theta} \cdot \frac{dW}{dq} = 2J\cos^2\!\theta \ . \tag{16.190}$$

Integrating,

$$W = J\theta + \tfrac{1}{2} J \sin 2\theta \ , \tag{16.191}$$

up to an irrelevant constant. We then have

$$\phi = \frac{\partial W}{\partial J}\bigg|_q = \theta + \tfrac{1}{2}\sin 2\theta + J\big(1+\cos 2\theta\big)\,\frac{\partial \theta}{\partial J}\bigg|_q \ . \tag{16.192}$$

To find $(\partial\theta/\partial J)_q$, we differentiate $q = \sqrt{2J/m\omega_0}\,\sin\theta$:

$$dq = \frac{\sin\theta}{\sqrt{2m\omega_0 J}}\,dJ + \sqrt{\frac{2J}{m\omega_0}}\cos\theta\,d\theta \quad \Rightarrow \quad \frac{\partial \theta}{\partial J}\bigg|_q = -\frac{1}{2J}\,\tan\theta \ . \tag{16.193}$$

Plugging this result into eqn. 16.192, we obtain $\phi = \theta$. Thus, the full transformation is

$$q = \bigg(\frac{2J}{m\omega_0}\bigg)^{1/2} \sin\phi \quad , \quad p = \sqrt{2m\omega_0 J}\,\cos\phi \ . \tag{16.194}$$

The Hamiltonian is

$$H = \omega_0\,J \ , \tag{16.195}$$

hence $\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0$ and $\dot{J} = -\frac{\partial H}{\partial \phi} = 0$, with solution $\phi(t) = \phi(0) + \omega_0\,t$ and $J(t) = J(0)$.

16.8.5 Example : Particle in a Box

Consider a particle in an open box of dimensions $L_x \times L_y$ moving under the influence of gravity. The bottom of the box lies at $z = 0$. The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz \ . \tag{16.196}$$

Step one is to solve the Hamilton-Jacobi equation via separation of variables. The Hamilton-Jacobi equation is written

$$\frac{1}{2m}\bigg(\frac{\partial W_x}{\partial x}\bigg)^2 + \frac{1}{2m}\bigg(\frac{\partial W_y}{\partial y}\bigg)^2 + \frac{1}{2m}\bigg(\frac{\partial W_z}{\partial z}\bigg)^2 + mgz = E \equiv \Lambda_z \ . \tag{16.197}$$

We can solve for $W_{x,y}$ by inspection:

$$W_x(x) = \sqrt{2m\Lambda_x}\,x \quad , \quad W_y(y) = \sqrt{2m\Lambda_y}\,y \ . \tag{16.198}$$

We then have[8]

$$W_z'(z) = -\sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \tag{16.199}$$

$$W_z(z) = \frac{2\sqrt{2}}{3\sqrt{m}\,g}\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)^{3/2} \ . \tag{16.200}$$

[8]Our choice of signs in taking the square roots for W_x', W_y', and W_z' is discussed below.

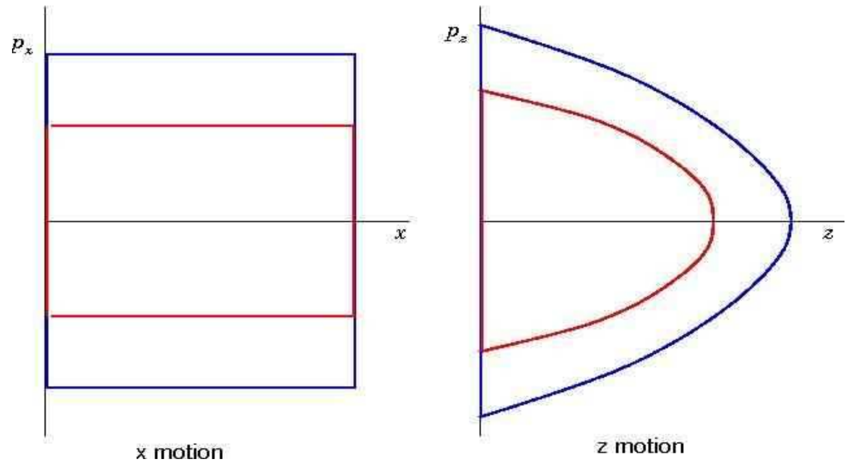


Figure 16.3: The librations $\mathcal{C}_z$ and $\mathcal{C}_x$. Not shown is $\mathcal{C}_y$, which is of the same shape as $\mathcal{C}_x$.

Step two is to find the $\mathcal{C}_\sigma$. Clearly $p_{x,y} = \sqrt{2m\Lambda_{x,y}}$. For fixed p_x, the x motion proceeds from $x = 0$ to $x = L_x$ and back, with corresponding motion for y. For x, we have

$$p_z(z) = W_z'(z) = \sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \ , \tag{16.201}$$

and thus $\mathcal{C}_z$ is a truncated parabola, with $z_{\text{max}} = (\Lambda_z - \Lambda_x - \Lambda_y)/mg$.

Step three is to compute $J(\Lambda)$ and invert to obtain $\Lambda(J)$. We have

$$J_x = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_x} p_x \, dx = \frac{1}{\pi} \int\limits_0^{L_x} dx \sqrt{2m\Lambda_x} = \frac{L_x}{\pi} \sqrt{2m\Lambda_x} \tag{16.202}$$

$$J_y = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_y} p_y \, dy = \frac{1}{\pi} \int\limits_0^{L_y} dy \sqrt{2m\Lambda_y} = \frac{L_y}{\pi} \sqrt{2m\Lambda_y} \tag{16.203}$$

and

$$\begin{aligned} J_z &= \frac{1}{2\pi} \oint\limits_{\mathcal{C}_z} p_z \, dz = \frac{1}{\pi} \int\limits_0^{z_{\text{max}}} dx \, \sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \\ &= \frac{2\sqrt{2}}{3\pi\sqrt{m}\, g} \big(\Lambda_z - \Lambda_x - \Lambda_y\big)^{3/2} \ . \end{aligned} \tag{16.204}$$

We now invert to obtain

$$\Lambda_x = \frac{\pi^2}{2mL_x^2} \, J_x^2 \qquad , \qquad \Lambda_y = \frac{\pi^2}{2mL_y^2} \, J_y^2 \tag{16.205}$$

$$\Lambda_z = \left(\frac{3\pi\sqrt{m}\, g}{2\sqrt{2}} \right)^{2/3} J_z^{2/3} + \frac{\pi^2}{2mL_x^2} \, J_x^2 + \frac{\pi^2}{2mL_y^2} \, J_y^2 \ . \tag{16.206}$$

$$F_2\big(x,y,z,J_x,J_y,J_z\big) = \frac{\pi x}{L_x}\,J_x + \frac{\pi y}{L_y}\,J_y + \pi\bigg(J_z^{2/3} - \frac{2m^{2/3}g^{1/3}z}{(3\pi)^{2/3}}\bigg)^{3/2} \ . \tag{16.207}$$

We now find

$$\phi_x = \frac{\partial F_2}{\partial J_x} = \frac{\pi x}{L_x} \quad , \quad \phi_y = \frac{\partial F_2}{\partial J_y} = \frac{\pi y}{L_y} \tag{16.208}$$

and

$$\phi_z = \frac{\partial F_2}{\partial J_z} = \pi\sqrt{1 - \frac{2m^{2/3}g^{1/3}z}{(3\pi J_z)^{2/3}}} = \pi\sqrt{1 - \frac{z}{z_{\rm max}}} \ , \tag{16.209}$$

where

$$z_{\rm max}(J_z) = \frac{(3\pi J_z)^{2/3}}{2m^{2/3}g^{1/3}} \ . \tag{16.210}$$

The momenta are

$$p_x = \frac{\partial F_2}{\partial x} = \frac{\pi J_x}{L_x} \quad , \quad p_y = \frac{\partial F_2}{\partial y} = \frac{\pi J_y}{L_y} \tag{16.211}$$

and

$$p_z = \frac{\partial F_2}{\partial z} = -\sqrt{2m}\,\Bigg(\bigg(\frac{3\pi\sqrt{m}\,g}{2\sqrt{2}}\bigg)^{2/3} J_z^{2/3} - mgz\Bigg)^{1/2} \ . \tag{16.212}$$

We note that the angle variables $\phi_{x,y,z}$ seem to be restricted to the range $[0,\pi]$, which seems to be at odds with eqn. 16.180. Similarly, the momenta $p_{x,y,z}$ all seem to be positive, whereas we know the momenta reverse sign when the particle bounces off a wall. The origin of the apparent discrepancy is that when we solved for the functions $W_{x,y,z}$, we had to take a square root in each case, and we chose a particular branch of the square root. So rather than $W_x(x) = \sqrt{2m\Lambda_x}\,x$, we should have taken

$$W_x(x) = \begin{cases} \sqrt{2m\Lambda_x}\,x & \text{if } p_x > 0 \\ \sqrt{2m\Lambda_x}\,(2L_x - x) & \text{if } p_x < 0 \ . \end{cases} \tag{16.213}$$

The relation $J_x = (L_x/\pi)\sqrt{2m\Lambda_x}$ is unchanged, hence

$$W_x(x) = \begin{cases} (\pi x/L_x)\,J_x & \text{if } p_x > 0 \\ 2\pi J_x - (\pi x/L_x)\,J_x & \text{if } p_x < 0 \ . \end{cases} \tag{16.214}$$

and

$$\phi_x = \begin{cases} \pi x/L_x & \text{if } p_x > 0 \\ \pi(2L_x - x)/L_x & \text{if } p_x < 0 \ . \end{cases} \tag{16.215}$$

Now the angle variable ϕ_x advances by 2π during the cycle $\mathcal{C}_x$. Similar considerations apply to the y and z sectors.

16.8.6 Kepler Problem in Action-Angle Variables

This is discussed in detail in standard texts, such as Goldstein. The potential is $V(r) = -k/r$, and the problem is separable. We write[9]

$$W(r,\theta,\varphi) = W_r(r) + W_\theta(\theta) + W_\varphi(\varphi) \ , \tag{16.216}$$

hence

$$\frac{1}{2m}\bigg(\frac{\partial W_r}{\partial r}\bigg)^2 + \frac{1}{2mr^2}\bigg(\frac{\partial W_\theta}{\partial \theta}\bigg)^2 + \frac{1}{2mr^2\sin^2\!\theta}\bigg(\frac{\partial W_\varphi}{\partial \varphi}\bigg)^2 + V(r) = E \equiv \Lambda_r \ . \tag{16.217}$$

Separating, we have

$$\frac{1}{2m}\bigg(\frac{dW_\varphi}{d\varphi}\bigg)^2 = \Lambda_\varphi \quad \Rightarrow \quad J_\varphi = \oint\limits_{\mathcal{C}_\varphi} d\varphi\, \frac{dW_\varphi}{d\varphi} = 2\pi\sqrt{2m\Lambda_\varphi} \ . \tag{16.218}$$

Next we deal with the θ coordinate:

$$\begin{aligned}
\frac{1}{2m}\bigg(\frac{dW_\theta}{d\theta}\bigg)^2 = \Lambda_\theta - \frac{\Lambda_\varphi}{\sin^2\!\theta} \qquad &\Rightarrow \\
J_\theta = 4\sqrt{2m\Lambda_\theta}\int\limits_0^{\theta_0}\! d\theta\, &\sqrt{1 - \big(\Lambda_\varphi/\Lambda_\theta\big)\csc^2\!\theta} \\
= 2\pi\sqrt{2m}&\Big(\sqrt{\Lambda_\theta} - \sqrt{\Lambda_\varphi}\Big) \ ,
\end{aligned} \tag{16.219}$$

where $\theta_0 = \sin^{-1}(\Lambda_\varphi/\Lambda_\theta)$. Finally, we have[10]

$$\begin{aligned}
\frac{1}{2m}\bigg(\frac{dW_r}{dr}\bigg)^2 = E + \frac{k}{r} - \frac{\Lambda_\theta}{r^2} \quad &\Rightarrow \\
J_r = \oint\limits_{\mathcal{C}_r} dr\, &\sqrt{2m\bigg(E + \frac{k}{r} - \frac{\Lambda_\theta}{r^2}\bigg)} \\
= -(J_\theta + J_\varphi) &+ \pi k\sqrt{\frac{2m}{|E|}} \ ,
\end{aligned} \tag{16.220}$$

where we've assumed $E < 0$, *i.e.* bound motion.

Thus, we find

$$H = E = -\frac{2\pi^2 m k^2}{\big(J_r + J_\theta + J_\varphi\big)^2} \ . \tag{16.221}$$

Note that the frequencies are completely degenerate:

$$\nu \equiv \nu_{r,\theta,\varphi} = \frac{\partial H}{\partial J_{r,\theta,\varphi}} = \frac{4\pi^2 m k^2}{\big(J_r + J_\theta + J_\varphi\big)^3} = \bigg(\frac{\pi^2 m k^2}{2|E|^3}\bigg)^{1/2} \ . \tag{16.222}$$

[9]We denote the azimuthal angle by φ to distinguish it from the AA variable ϕ.

[10]The details of performing the integral around $\mathcal{C}_r$ are discussed in *e.g.* Goldstein.

This threefold degeneracy may be removed by a transformation to new AA variables,

$$\Big\{(\phi_r, J_r),\ (\phi_\theta, J_\theta),\ (\phi_\varphi, J_\varphi)\Big\} \longrightarrow \Big\{(\phi_1, J_1),\ (\phi_2, J_2),\ (\phi_3, J_3)\Big\}\ , \tag{16.223}$$

using the type-II generator

$$F_2(\phi_r, \phi_\theta, \phi_\varphi; J_1, J_2, J_3) = (\phi_\varphi - \phi_\theta)\, J_1 + (\phi_\theta - \phi_r)\, J_2 + \phi_r\, J_3\ , \tag{16.224}$$

which results in

$$\phi_1 = \frac{\partial F_2}{\partial J_1} = \phi_\varphi - \phi_\theta \qquad\qquad J_r = \frac{\partial F_2}{\partial \phi_r} = J_3 - J_2 \tag{16.225}$$

$$\phi_2 = \frac{\partial F_2}{\partial J_2} = \phi_\theta - \phi_r \qquad\qquad J_\theta = \frac{\partial F_2}{\partial \phi_\theta} = J_2 - J_1 \tag{16.226}$$

$$\phi_3 = \frac{\partial F_2}{\partial J_3} = \phi_r \qquad\qquad J_\varphi = \frac{\partial F_2}{\partial \phi_\varphi} = J_1\ . \tag{16.227}$$

The new Hamiltonian is

$$H(J_1, J_2, J_3) = -\frac{2\pi^2 m k^2}{J_3^2}\ , \tag{16.228}$$

whence $\nu_1 = \nu_2 = 0$ and $\nu_3 = \nu$.

16.8.7 Charged Particle in a Magnetic Field

For the case of the charged particle in a magnetic field, studied above in section 16.7.7, we found

$$x = \frac{cP_2}{eB} + \frac{c}{eB}\sqrt{2mP_1}\,\sin\theta \tag{16.229}$$

and

$$p_x = \sqrt{2mP_1}\,\cos\theta \qquad , \qquad p_y = P_2\ . \tag{16.230}$$

The action variable J is then

$$J = \oint p_x\, dx = \frac{2mcP_1}{eB}\int\limits_0^{2\pi} d\theta\, \cos^2\!\theta = \frac{mcP_1}{eB}\ . \tag{16.231}$$

We then have

$$W = J\theta + \tfrac{1}{2} J \sin(2\theta) + Py\ , \tag{16.232}$$

where $P \equiv P_2$. Thus,

$$\begin{aligned}\phi &= \frac{\partial W}{\partial J} \\ &= \theta + \tfrac{1}{2}\sin(2\theta) + J\big[1 + \cos(2\theta)\big]\,\frac{\partial\theta}{\partial J} \\ &= \theta + \tfrac{1}{2}\sin(2\theta) + 2J\cos^2\!\theta \cdot \left(-\frac{\tan\theta}{2J}\right) \\ &= \theta\ .\end{aligned} \tag{16.233}$$

The other canonical pair is (Q, P), where

$$Q = \frac{\partial W}{\partial P} = y - \sqrt{\frac{2cJ}{eB}}\,\cos\phi \ . \tag{16.234}$$

Therefore, we have

$$x = \frac{cP}{eB} + \sqrt{\frac{2cJ}{eB}}\,\sin\phi \qquad , \qquad y = Q + \sqrt{\frac{2cJ}{eB}}\,\cos\phi \tag{16.235}$$

and

$$p_x = \sqrt{\frac{2eBJ}{c}}\,\cos\phi \qquad , \qquad p_y = P \ . \tag{16.236}$$

The Hamiltonian is

$$\begin{aligned} H &= \frac{p_x^2}{2m} + \frac{1}{2m}\bigg(p_y - \frac{eBx}{c}\bigg)^2 \\ &= \frac{eBJ}{mc}\,\cos^2\!\phi + \frac{eBJ}{mc}\,\sin^2\!\phi \\ &= \omega_c\, J \ , \end{aligned} \tag{16.237}$$

where $\omega_c = eB/mc$. The equations of motion are

$$\dot\phi = \frac{\partial H}{\partial J} = \omega_c \qquad , \qquad \dot J = -\frac{\partial H}{\partial \phi} = 0 \tag{16.238}$$

and

$$\dot Q = \frac{\partial H}{\partial P} = 0 \qquad , \qquad \dot P = -\frac{\partial H}{\partial Q} = 0 \ . \tag{16.239}$$

Thus, Q, P, and J are constants, and $\phi(t) = \phi_0 + \omega_c\, t$.

16.8.8 Motion on Invariant Tori

The angle variables evolve as

$$\phi_\sigma(t) = \nu_\sigma(J)\, t + \phi_\sigma(0) \ . \tag{16.240}$$

Thus, they wind around the invariant torus, specified by $\{J_\sigma\}$ at constant rates. In general, while each ϕ_σ executed periodic motion around a circle, the motion of the system as a whole is not periodic, since the frequencies $\nu_\sigma(J)$ are not, in general, commensurate. In order for the motion to be periodic, there must exist a set of integers, $\{l_\sigma\}$, such that

$$\sum_{\sigma=1}^n l_\sigma\, \nu_\sigma(J) = 0 \ . \tag{16.241}$$

This means that the ratio of any two frequencies ν_σ/ν_α must be a rational number. On a given torus, there are several possible orbits, depending on initial conditions $\phi(0)$. However, since the frequencies are determined by the action variables, which specify the tori, on a given torus either all orbits are periodic, or none are.

In terms of the original coordinates q, there are two possibilities:

$$\begin{aligned} q_\sigma(t) &= \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_n=-\infty}^{\infty} A^{(\sigma)}_{\ell_1\ell_2\cdots\ell_n}\, e^{i\ell_1\phi_1(t)} \cdots e^{i\ell_n\phi_n(t)} \\ &\equiv \sum_{\boldsymbol{\ell}} A^\sigma_{\boldsymbol{\ell}}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}(t)} \qquad \text{(libration)} \end{aligned} \tag{16.242}$$

or

$$q_\sigma(t) = q^\circ_\sigma\, \phi_\sigma(t) + \sum_{\boldsymbol{\ell}} B^\sigma_{\boldsymbol{\ell}}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}(t)} \qquad \text{(rotation)}\ . \tag{16.243}$$

For rotations, the variable $q_\sigma(t)$ increased by $\Delta q_\sigma = 2\pi\, q^\circ_\sigma$. R

16.9 Canonical Perturbation Theory

16.9.1 Canonical Transformations and Perturbation Theory

Suppose we have a Hamiltonian

$$H(\xi,t) = H_0(\xi,t) + \epsilon\, H_1(\xi,t)\ , \tag{16.244}$$

where ϵ is a small dimensionless parameter. Let's implement a type-II transformation, generated by $S(q,P,t)$:[11]

$$\tilde H(Q,P,t) = H(q,p,t) + \frac{\partial}{\partial t} S(q,P,t)\ . \tag{16.245}$$

Let's expand everything in powers of ϵ:

$$q_\sigma = Q_\sigma + \epsilon\, q_{1,\sigma} + \epsilon^2\, q_{2,\sigma} + \ldots \tag{16.246}$$

$$p_\sigma = P_\sigma + \epsilon\, p_{1,\sigma} + \epsilon^2\, p_{2,\sigma} + \ldots \tag{16.247}$$

$$\tilde H = \tilde H_0 + \epsilon\, \tilde H_1 + \epsilon^2\, \tilde H_2 + \ldots \tag{16.248}$$

$$S = \underbrace{q_\sigma\, P_\sigma}_{\substack{\text{identity}\\ \text{transformation}}} + \epsilon\, S_1 + \epsilon^2\, S_2 + \ldots\ . \tag{16.249}$$

Then

$$\begin{aligned} Q_\sigma = \frac{\partial S}{\partial P_\sigma} &= q_\sigma + \epsilon\, \frac{\partial S_1}{\partial P_\sigma} + \epsilon^2\, \frac{\partial S_2}{\partial P_\sigma} + \ldots \\ &= Q_\sigma + \left(q_{1,\sigma} + \frac{\partial S_1}{\partial P_\sigma}\right)\epsilon + \left(q_{2,\sigma} + \frac{\partial S_2}{\partial P_\sigma}\right)\epsilon^2 + \ldots \end{aligned} \tag{16.250}$$

[11] $Here, S(q,P,t)$ is not meant to signify Hamilton's principal function.

and

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + \epsilon\, \frac{\partial S_1}{\partial q_\sigma} + \epsilon^2\, \frac{\partial S_2}{\partial q_\sigma} + \ldots \tag{16.251}$$

$$= P_\sigma + \epsilon\, p_{1,\sigma} + \epsilon^2\, p_{2,\sigma} + \ldots\ . \tag{16.252}$$

We therefore conclude, order by order in ϵ,

$$q_{k,\sigma} = -\frac{\partial S_k}{\partial P_\sigma} \quad , \quad p_{k,\sigma} = +\frac{\partial S_k}{\partial q_\sigma}\ . \tag{16.253}$$

Now let's expand the Hamiltonian:

$$\tilde{H}(Q,P,t) = H_0(q,p,t) + \epsilon\, H_1(q,p,t) + \frac{\partial S}{\partial t} \tag{16.254}$$

$$\begin{aligned} &= H_0(Q,P,t) + \frac{\partial H_0}{\partial Q_\sigma}\,(q_\sigma - Q_\sigma) + \frac{\partial H_0}{\partial P_\sigma}\,(p_\sigma - P_\sigma) \\ &\qquad + \epsilon H_1(Q,P,t) + \epsilon\,\frac{\partial}{\partial t}\,S_1(Q,P,t) + \mathcal{O}(\epsilon^2) \\ &= H_0(Q,P,t) + \bigg(-\frac{\partial H_0}{\partial Q_\sigma}\,\frac{\partial S_1}{\partial P_\sigma} + \frac{\partial H_0}{\partial P_\sigma}\,\frac{\partial S_1}{\partial Q_\sigma} + \frac{\partial S_1}{\partial t} + H_1\bigg)\,\epsilon + \mathcal{O}(\epsilon^2) \\ &= H_0(Q,P,t) + \bigg(H_1 + \big\{S_1, H_0\big\} + \frac{\partial S_1}{\partial t}\bigg)\,\epsilon + \mathcal{O}(\epsilon^2)\ . \end{aligned} \tag{16.255}$$

In the above expression, we evaluate $H_k(q,p,t)$ and $S_k(q,P,t)$ at $q=Q$ and $p=P$ and expand in the differences $q-Q$ and $p-P$. Thus, we have derived the relation

$$\tilde{H}(Q,P,t) = \tilde{H}_0(Q,P,t) + \epsilon \tilde{H}_1(Q,P,t) + \ldots \tag{16.256}$$

with

$$\tilde{H}_0(Q,P,t) = H_0(Q,P,t) \tag{16.257}$$

$$\tilde{H}_1(Q,P,t) = H_1 + \big\{S_1, H_0\big\} + \frac{\partial S_1}{\partial t}\ . \tag{16.258}$$

The problem, though, is this: we have one equation, eqn, 16.258, for the two unknowns $\tilde{H}_1$ and S_1. Thus, the problem is underdetermined. Of course, we could choose $\tilde{H}_1 = 0$, which basically recapitulates standard Hamilton-Jacobi theory. But we might just as well demand that $\tilde{H}_1$ satisfy some other requirement, such as that $\tilde{H}_0 + \epsilon\, \tilde{H}_1$ being integrable.

Incidentally, this treatment is paralleled by one in quantum mechanics, where a unitary transformation may be implemented to eliminate a perturbation to lowest order in a small parameter. Consider the Schrödinger equation,

$$i\hbar \frac{\partial \psi}{\partial t} = (\mathcal{H}_0 + \epsilon\, \mathcal{H}_1)\, \psi\ , \tag{16.259}$$

and define χ by

$$\psi \equiv e^{iS/\hbar}\,\chi\ , \tag{16.260}$$

with

$$S = \epsilon\, S_1 + \epsilon^2\, S_2 + \ldots\ . \tag{16.261}$$

As before, the transformation $U \equiv \exp(iS/\hbar)$ collapses to the identity in the $\epsilon \to 0$ limit. Now let's write the Schrödinger equation for χ. Expanding in powers of ϵ, one finds

$$i\hbar \frac{\partial\chi}{\partial t} = \mathcal{H}_0\,\chi + \epsilon\left(\mathcal{H}_1 + \frac{1}{i\hbar}\big[S_1,\mathcal{H}_0\big] + \frac{\partial S_1}{\partial t}\right)\chi + \ldots \equiv \tilde{\mathcal{H}}\,\chi\ , \tag{16.262}$$

where $[A,B] = AB - BA$ is the commutator. Note the classical-quantum correspondence,

$$\{A,B\} \longleftrightarrow \frac{1}{i\hbar}\,[A,B]\ . \tag{16.263}$$

Again, what should we choose for S_1? Usually the choice is made to make the $\mathcal{O}(\epsilon)$ term in $\tilde{\mathcal{H}}$ vanish. But this is not the only possible simplifying choice.

16.9.2 Canonical Perturbation Theory for $n = 1$ Systems

Henceforth we shall assume $H(\xi,t) = H(\xi)$ is time-independent, and we write the perturbed Hamiltonian as

$$H(\xi) = H_0(\xi) + \epsilon H_1(\xi)\ . \tag{16.264}$$

Let (ϕ_0, J_0) be the action-angle variables for H_0. Then

$$\tilde{H}_0(\phi_0,J_0) = H_0\big(q(\phi_0,J_0),p(\phi_0,J_0)\big) = \tilde{H}_0(J_0)\ . \tag{16.265}$$

We define

$$\tilde{H}_1(\phi_0,J_0) = H_1\big(q(\phi_0,J_0),p(\phi_0,J_0)\big)\ . \tag{16.266}$$

We assume that $\tilde{H} = \tilde{H}_0 + \epsilon\,\tilde{H}_1$ is integrable[12], so it, too, possesses action-angle variables, which we denote by (ϕ, J)[13]. Thus, there must be a canonical transformation taking $(\phi_0, J_0) \to (\phi, J)$, with

$$\tilde{H}\big(\phi_0(\phi,J),J_0(\phi,J)\big) \equiv K(J) = E(J)\ . \tag{16.267}$$

We solve via a type-II canonical transformation:

$$S(\phi_0,J) = \phi_0 J + \epsilon\, S_1(\phi_0,J) + \epsilon^2\, S_2(\phi_0,J) + \ldots\ , \tag{16.268}$$

where $\phi_0 J$ is the identity transformation. Then

$$J_0 = \frac{\partial S}{\partial\phi_0} = J + \epsilon\,\frac{\partial S_1}{\partial\phi_0} + \epsilon^2\,\frac{\partial S_2}{\partial\phi_0} + \ldots \tag{16.269}$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \epsilon\,\frac{\partial S_1}{\partial J} + \epsilon^2\,\frac{\partial S_2}{\partial J} + \ldots\ , \tag{16.270}$$

[12]This is always true, in fact, for $n = 1$.
[13]We assume the motion is bounded, so action-angle variables may be used.

and

$$\begin{aligned} E(J) &= E_0(J) + \epsilon\, E_1(J) + \epsilon^2\, E_2(J) + \ldots && (16.271) \\ &= \tilde{H}_0(\phi_0, J_0) + \tilde{H}_1(\phi_0, J_0) \ . && (16.272) \end{aligned}$$

We now expand $\tilde{H}(\phi_0, J_0)$ in powers of $J_0 - J$:

$$\begin{aligned} \tilde{H}(\phi_0, J_0) &= \tilde{H}_0(\phi_0, J_0) + \epsilon\, \tilde{H}_1(\phi_0, J_0) && (16.273) \\ &= \tilde{H}_0(J) + \frac{\partial \tilde{H}_0}{\partial J}\,(J_0 - J) + \tfrac{1}{2}\,\frac{\partial^2 \tilde{H}_0}{\partial J^2}\,(J_0 - J)^2 + \ldots \\ &\qquad + \epsilon\, \tilde{H}_1(\phi_0, J_0) + \epsilon\, \frac{\partial \tilde{H}_1}{\partial J}\,(J_0 - J) + \ldots \\ &= \tilde{H}_0(J) + \left(\tilde{H}_1(\phi_0, J_0) + \frac{\partial \tilde{H}_0}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} \right) \epsilon && (16.274) \\ &\qquad + \left(\frac{\partial \tilde{H}_0}{\partial J}\,\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\,\frac{\partial^2 \tilde{H}_0}{\partial J^2} \left(\frac{\partial S_1}{\partial \phi_0} \right)^2 + \frac{\partial \tilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} \right) \epsilon^2 + \ldots \ . \end{aligned}$$

Equating terms, then,

$$\begin{aligned} E_0(J) &= \tilde{H}_0(J) && (16.275) \\ E_1(J) &= \tilde{H}_1(\phi_0, J) + \frac{\partial \tilde{H}_0}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} && (16.276) \\ E_2(J) &= \frac{\partial \tilde{H}_0}{\partial J}\,\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\,\frac{\partial^2 \tilde{H}_0}{\partial J^2} \left(\frac{\partial S_1}{\partial \phi_0} \right)^2 + \frac{\partial \tilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} \ . && (16.277) \end{aligned}$$

How, one might ask, can we be sure that the LHS of each equation in the above hierarchy depends only on J when each RHS seems to depend on ϕ_0 as well? The answer is that we use the freedom to choose each S_k to make this so. We demand each RHS be independent of ϕ_0, which means it must be equal to its average, $\langle\, \text{RHS}(\phi_0)\,\rangle$, where

$$\big\langle f\big(\phi_0\big)\big\rangle = \int\limits_0^{2\pi} \frac{d\phi_0}{2\pi}\, f\big(\phi_0\big) \ . \qquad (16.278)$$

The average is performed *at fixed* J and *not* at fixed J_0. In this regard, we note that holding J constant and increasing ϕ_0 by 2π also returns us to the same starting point. Therefore, J is a periodic function of ϕ_0. We must then be able to write

$$S_k(\phi_0, J) = \sum_{m=-\infty}^{\infty} S_k(J; m)\, e^{im\phi_0} \qquad (16.279)$$

for each $k > 0$, in which case

$$\left\langle \frac{\partial S_k}{\partial \phi_0} \right\rangle = \frac{1}{2\pi} \big[S_k(2\pi) - S_k(0) \big] = 0 \ . \qquad (16.280)$$

Let's see how this averaging works to the first two orders of the hierarchy. Since $\tilde{H}_0(J)$ is independent of ϕ_0 and since $\partial S_1/\partial\phi_0$ is periodic, we have

$$E_1(J) = \big\langle \tilde{H}_1(\phi_0, J)\big\rangle + \frac{\partial \tilde{H}_0}{\partial J} \overbrace{\left\langle \frac{\partial S_1}{\partial \phi_0}\right\rangle}^{\text{this vanishes!}} \tag{16.281}$$

and hence S_1 must satisfy

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0(J)} \ , \tag{16.282}$$

where $\nu_0(J) = \partial\tilde{H}_0/\partial J$. Clearly the RHS of eqn. 16.282 has zero average, and must be a periodic function of ϕ_0. The solution is $S_1 = S_1(\phi_0, J) + g(J)$, where $g(J)$ is an arbitrary function of J. However, $g(J)$ affects only the difference $\phi - \phi_0$, changing it by a constant value $g'(J)$. So there is no harm in taking $g(J) = 0$.

Next, let's go to second order in ϵ. We have

$$E_2(J) = \left\langle \frac{\partial \tilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial\phi_0}\right\rangle + \tfrac{1}{2}\frac{\partial\nu_0}{\partial J}\left\langle \left(\frac{\partial S_1}{\partial \phi_1}\right)^2\right\rangle + \nu_0(J)\overbrace{\left\langle\frac{\partial S_2}{\partial\phi_0}\right\rangle}^{\text{this vanishes!}} \quad . \tag{16.283}$$

The equation for S_2 is then

$$\begin{aligned}\frac{\partial S_2}{\partial\phi_0} = \frac{1}{\nu_0^2(J)}\Bigg\{&\left\langle\frac{\partial\tilde{H}_1}{\partial J}\right\rangle\langle\tilde{H}_0\rangle - \left\langle\frac{\partial\tilde{H}_1}{\partial J}\,\tilde{H}_0\right\rangle - \frac{\partial\tilde{H}_1}{\partial J}\,\langle\tilde{H}_1\rangle + \frac{\partial\tilde{H}_1}{\partial J}\,\tilde{H}_1\\ &+\frac{1}{2}\,\frac{\partial\ln\nu_0}{\partial J}\left(\langle\tilde{H}_1^2\rangle - 2\langle\tilde{H}_1\rangle^2 + 2\langle\tilde{H}_1\rangle - \tilde{H}_1^2\right)\Bigg\} \ .\end{aligned} \tag{16.284}$$

The expansion for the energy $E(J)$ is then

$$\begin{aligned}E(J) = \tilde{H}_0(J) + \epsilon\langle\tilde{H}_1\rangle + \frac{\epsilon^2}{\nu_0(J)}\Bigg\{&\left\langle\frac{\partial\tilde{H}_1}{\partial J}\right\rangle\langle\tilde{H}_1\rangle - \left\langle\frac{\partial\tilde{H}_1}{\partial J}\,\tilde{H}_1\right\rangle\\ &+\frac{1}{2}\,\frac{\partial\ln\nu_0}{\partial J}\left(\langle\tilde{H}_1^2 - \langle\tilde{H}_1\rangle^2\right)\Bigg\} + \mathcal{O}(\epsilon^3) \ .\end{aligned} \tag{16.285}$$

Note that we don't need S to find $E(J)$! The perturbed frequencies are

$$\nu(J) = \frac{\partial E}{\partial J} \ . \tag{16.286}$$

Sometimes the frequencies are all that is desired. However, we can of course obtain the full motion of the system via the succession of canonical transformations,

$$(\phi, J) \longrightarrow (\phi_0, J_0) \longrightarrow (q, p) \ . \tag{16.287}$$

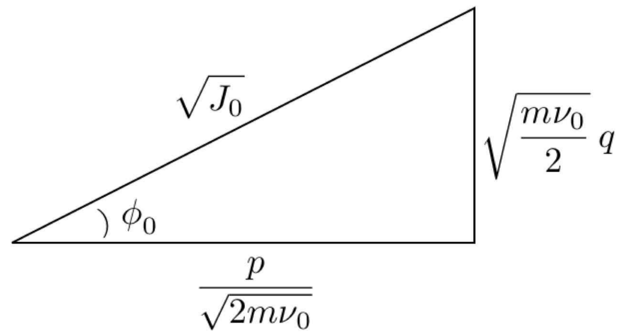


Figure 16.4: Action-angle variables for the harmonic oscillator.

16.9.3 Example : Nonlinear Oscillator

Consider the nonlinear oscillator with Hamiltonian

$$H(q,p) = \overbrace{\frac{p^2}{2m} + \tfrac{1}{2}m\nu_0^2 q^2}^{H_0} + \tfrac{1}{4}\epsilon\alpha q^4 \ . \tag{16.288}$$

The action-angle variables for the harmonic oscillator Hamiltonian H_0 are

$$\phi_0 = \tan^{-1}\big(mvq/p\big) \quad , \quad J_0 = \frac{p^2}{2m\nu_0} + \tfrac{1}{2}m\nu_0 q^2 \ , \tag{16.289}$$

and the relation between (ϕ_0, J_0) and (q,p) is further depicted in fig. 16.4. Note $H_0 = \nu_0\, J_0$. For the full Hamiltonian, we have

$$\begin{aligned}\tilde{H}(\phi_0, J_0) &= \nu_0 J_0 + \tfrac{1}{4}\epsilon\,\alpha\left(\sqrt{\frac{2J_0}{m\nu_0}}\,\sin\phi_0\right)^4 \\ &= \nu_0 J_0 + \frac{\epsilon\alpha}{m^2\nu_0^2}\, J_0^2\,\sin^4\!\phi_0 \ .\end{aligned} \tag{16.290}$$

We may now evaluate

$$E_1(J) = \big\langle \tilde{H}_1 \big\rangle = \frac{\alpha J^2}{m^2\nu_0^2}\int_0^{2\pi}\frac{d\phi_0}{2\pi}\,\sin^4\!\phi_0 = \frac{3\alpha J^2}{8m^2\nu_0^2} \ . \tag{16.291}$$

The frequency, to order ϵ, is

$$\nu(J) = \nu_0 + \frac{3\,\epsilon\,\alpha J}{4m^2\nu_0^2} \ . \tag{16.292}$$

Now to lowest order in ϵ, we may replace J by $J_0 = \frac{1}{2}m\nu_0 A^2$, where A is the amplitude of the q motion. Thus,

$$\nu(A) = \nu_0 + \frac{3\epsilon\alpha}{8m\nu_0} \ . \tag{16.293}$$

This result agrees with that obtained via heavier lifting, using the Poincaré-Lindstedt method.

Next, let's evaluate the canonical transformation $(\phi_0, J_0) \to (\phi, J)$. We have

$$\nu_0 \, \frac{\partial S_1}{\partial \phi_0} = \frac{\alpha J^2}{m^2 \nu_0^2} \Big(\tfrac{3}{8} - \sin^4\!\phi_0 \Big) \qquad \Rightarrow$$

$$S(\phi_0, J) = \phi_0 \, J + \frac{\epsilon \alpha J^2}{8m^2 \nu_0^3} \big(3 + 2 \sin^2\!\phi_0 \big) \, \sin\phi_0 \, \cos\phi_0 + \mathcal{O}(\epsilon^2) \ . \tag{16.294}$$

Thus,

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \frac{\epsilon \alpha J}{4m^2 \nu_0^3} \big(3 + 2 \sin^2\!\phi_0 \big) \, \sin\phi_0 \, \cos\phi_0 + \mathcal{O}(\epsilon^2) \tag{16.295}$$

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \frac{\epsilon \alpha J^2}{8m^2 \nu_0^3} \big(4 \cos 2\phi_0 - \cos 4\phi_0 \big) + \mathcal{O}(\epsilon^2) \ . \tag{16.296}$$

Again, to lowest order, we may replace J by J_0 in the above, whence

$$J = J_0 - \frac{\epsilon \alpha J_0^2}{8m^2 \nu_0^3} \big(4 \cos 2\phi_0 - \cos 4\phi_0 \big) + \mathcal{O}(\epsilon^2) \tag{16.297}$$

$$\phi = \phi_0 + \frac{\epsilon \alpha J_0}{8m^2 \nu_0^3} \big(3 + 2 \sin^2\!\phi_0 \big) \, \sin 2\phi_0 + \mathcal{O}(\epsilon^2) \ . \tag{16.298}$$

To obtain (q,p) in terms of (ϕ, J) is not analytically tractable – the relations cannot be analytically inverted.

16.9.4 $n > 1$ Systems : Degeneracies and Resonances

Generalizing the procedure we derived for $n = 1$, we obtain

$$J_0^\alpha = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \epsilon \, \frac{\partial S_1}{\partial \phi_0^\alpha} + \epsilon^2 \, \frac{\partial S_2}{\partial \phi_0^\alpha} + \ldots \tag{16.299}$$

$$\phi^\alpha = \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \epsilon \, \frac{\partial S_1}{\partial J^\alpha} + \epsilon^2 \, \frac{\partial S_2}{\partial J^\alpha} + \ldots \tag{16.300}$$

and

$$E_0(\boldsymbol{J}) = \tilde{H}_0(\boldsymbol{J}) \tag{16.301}$$

$$E_1(\boldsymbol{J}) = \tilde{H}_0(\boldsymbol{\phi}_0, \boldsymbol{J}) + \nu_0^\alpha(\boldsymbol{J}) \, \frac{\partial S_1}{\partial \phi_0^\alpha} \tag{16.302}$$

$$E_2(\boldsymbol{J}) = \frac{\partial \tilde{H}_0}{\partial J_\alpha} \, \frac{\partial S_2}{\partial \phi_0^\alpha} + \frac{1}{2} \, \frac{\partial \nu_0^\alpha}{\partial J^\beta} \, \frac{\partial S_1}{\partial \phi_0^\alpha} \, \frac{\partial S_1}{\partial \phi_0^\beta} + \nu_0^\alpha \, \frac{\partial S_1}{\partial \phi_0^\alpha} \ . \tag{16.303}$$

We now implement the averaging procedure, with

$$\big\langle f(J^1, \ldots, J^n) \big\rangle = \int\limits_0^{2\pi} \! \frac{d\phi_0^1}{2\pi} \, \cdots \int\limits_0^{2\pi} \! \frac{d\phi_0^n}{2\pi} \, f\big(\phi_0^1, \ldots, \phi_0^n, J^1, \ldots, J^n\big) \ . \tag{16.304}$$

The equation for S_1 is

$$\nu_0^\alpha \, \frac{\partial S_1}{\partial \phi_0^\alpha} = \big\langle \tilde{H}_1 \big\rangle - \tilde{H}_1 \equiv -{\sum_{\boldsymbol{\ell}}}' V_{\boldsymbol{\ell}} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \ , \tag{16.305}$$

where $\boldsymbol{\ell} = \{\ell^1, \ell^2, \ldots, \ell^n\}$, with each ℓ^σ an integer, and with $\boldsymbol{\ell} \neq 0$. The solution is

$$S_1(\boldsymbol{\phi}_0, \boldsymbol{J}) = i \, {\sum_{l}}' \frac{V_{\boldsymbol{\ell}}}{\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \ . \tag{16.306}$$

where $\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 = l^\alpha \nu_0^\alpha$. When two or more of the frequencies $\nu_\alpha(J)$ are *commensurate*, there exists a set of integers l such that the denominator of $D(l)$ vanishes. But even when the frequencies are not rationally related, one can approximate the ratios $\nu_0^\alpha/\nu_0^{\alpha'}$ by rational numbers, and for large enough l the denominator can become arbitrarily small.

Periodic time-dependent perturbations present a similar problem. Consider the system

$$H(\boldsymbol{\phi}, \boldsymbol{J}, t) = H_0(\boldsymbol{J}) + \epsilon \, V(\boldsymbol{\phi}, \boldsymbol{J}, t) \ , \tag{16.307}$$

where $V(t+T) = V(t)$. This means we may write

$$V(\boldsymbol{\phi}, \boldsymbol{J}, t) = \sum_k V_k(\boldsymbol{\phi}, \boldsymbol{J}) \, e^{-ik\Omega t} \tag{16.308}$$

$$= \sum_{\boldsymbol{k}} \sum_{\boldsymbol{\ell}} \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}) \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \, e^{-ik\Omega t} \ . \tag{16.309}$$

by Fourier transforming from both time and angle variables; here $\Omega = 2\pi/T$. Note that $V(\boldsymbol{\phi}, \boldsymbol{J}, t)$ is real if $V^*_{k,\boldsymbol{\ell}} = V_{-\boldsymbol{k},-l}$. The equations of motion are

$$\dot{J}^\alpha = -\frac{\partial H}{\partial \phi^\alpha} = -i\epsilon \sum_{k,\boldsymbol{\ell}} l^\alpha \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}) \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \, e^{-ik\Omega t} \tag{16.310}$$

$$\dot{\phi}^\alpha = +\frac{\partial H}{\partial J^\alpha} = \nu_0^\alpha(\boldsymbol{J}) + \epsilon \sum_{k,\boldsymbol{\ell}} \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J})}{\partial J^\alpha} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \, e^{-ik\Omega t} \ . \tag{16.311}$$

We now expand in ϵ:

$$\phi^\alpha = \phi_0^\alpha + \epsilon \, \phi_1^\alpha + \epsilon^2 \, \phi_2^\alpha + \ldots \tag{16.312}$$

$$J^\alpha = J_0^\alpha + \epsilon \, J_1^\alpha + \epsilon^2 \, J_2^\alpha + \ldots \ . \tag{16.313}$$

To order ϵ^0, $J^\alpha = J_0^\alpha$ and $\phi_0^\alpha = \nu_0^\alpha \, t + \beta_0^\alpha$. To order ϵ^1,

$$\dot{J}_1^\alpha = -i \sum_{k,l} l^\alpha \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}_0) \, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \tag{16.314}$$

and

$$\dot{\phi}_1^\alpha = \frac{\partial \nu_0^\alpha}{\partial J^\beta} \, J_1^\beta + \sum_{k,\boldsymbol{\ell}} \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J})}{\partial J^\alpha} \, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \ , \tag{16.315}$$

where derivatives are evaluated at $\boldsymbol{J} = \boldsymbol{J}_0$. The solution is:

$$J_1^\alpha = \sum_{k,\boldsymbol{\ell}} \frac{l^\alpha\, \hat{V}_{k,\boldsymbol{\ell}}(J_0)}{k\Omega - \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0}\, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)t}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \tag{16.316}$$

$$\phi_1^\alpha = \left\{ \frac{\partial \nu_0^\alpha}{\partial J^\beta}\, \frac{l^\beta\, \hat{V}_{k,\boldsymbol{\ell}}(J_0)}{(k\Omega - \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0)^2} + \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(J)}{\partial J^\alpha}\, \frac{1}{k\Omega - \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0} \right\} e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)t}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \ . \tag{16.317}$$

When the resonance condition,

$$k\Omega = \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0(\boldsymbol{J}_0) \ , \tag{16.318}$$

holds, the denominators vanish, and the perturbation theory breaks down.

16.9.5 Particle-Wave Interaction

Consider a particle of charge e moving in the presence of a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and a space- and time-varying electric field $\boldsymbol{E}(\boldsymbol{x}, t)$, described by the Hamiltonian

$$H = \frac{1}{2m}\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big)^2 + \epsilon\, eV_0\, \cos(k_\perp x + k_z z - \omega t) \ , \tag{16.319}$$

where ϵ is a dimensionless expansion parameter. Working in the gauge $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$, from our earlier discussions in section 16.7.7, we may write

$$H = \omega_c J + \frac{p_z^2}{2m} + \epsilon\, eV_0\, \cos\left(k_z z + \frac{k_\perp P}{m\omega_c} + k_\perp \sqrt{\frac{2J}{m\omega_c}}\, \sin\phi - \omega t\right) . \tag{16.320}$$

Here,

$$x = \frac{P}{m\omega_c} + \sqrt{\frac{2J}{m\omega_c}}\, \sin\phi \qquad , \qquad y = Q + \sqrt{\frac{2J}{m\omega_c}}\, \cos\phi \ , \tag{16.321}$$

with $\omega_c = eB/mc$, the cyclotron frequency. We now make a mixed canonical transformation, generated by

$$F = \phi J' + \Big(k_z z + \frac{k_\perp P}{m\omega_c} - \omega t\Big) K' - PQ' \ , \tag{16.322}$$

where the new sets of conjugate variables are $\big\{(\phi', J')\,,\, (Q', P')\,,\, (\psi', K')\big\}$. We then have

$$\phi' = \frac{\partial F}{\partial J'} = \phi \qquad\qquad J = \frac{\partial F}{\partial \phi} = J' \tag{16.323}$$

$$Q = -\frac{\partial F}{\partial P} = -\frac{k_\perp K'}{m\omega_c} + Q' \qquad\qquad P' = -\frac{\partial F}{\partial Q'} = P \tag{16.324}$$

$$\psi' = \frac{\partial F}{\partial K'} = k_z z + \frac{k_\perp P}{m\omega_c} - \omega t \qquad\qquad p_z = \frac{\partial F}{\partial z} = k_z K' \ . \tag{16.325}$$

The transformed Hamiltonian is

$$\begin{aligned} H' &= H + \frac{\partial F}{\partial t} \\ &= \omega_c J' + \frac{k_z^2}{2m}\, K'^2 - \omega K' + \epsilon\, eV_0\, \cos\left(\psi' + k_\perp \sqrt{\frac{2J'}{m\omega_c}}\, \sin\phi'\right) . \end{aligned} \tag{16.326}$$

We will now drop primes and simply write $H = H_0 + \epsilon\, H_1$, with

$$H_0 = \omega_c J + \frac{k_z^2}{2m}\, K^2 - \omega K \tag{16.327}$$

$$H_1 = eV_0\, \cos\bigg(\psi + k_\perp\sqrt{\frac{2J}{m\omega_c}}\, \sin\phi\bigg) \ . \tag{16.328}$$

When $\epsilon = 0$, the frequencies associated with the ϕ and ψ motion are

$$\omega^0_\phi = \frac{\partial H_0}{\partial \phi} = \omega_c \qquad , \qquad \omega^0_\psi = \frac{\partial H_0}{\partial \psi} = \frac{k_z^2 K}{m} - \omega = k_z v_z - \omega \ , \tag{16.329}$$

where $v_z = p_z/m$ is the z-component of the particle's velocity. Now let us solve eqn. 16.305:

$$\omega^0_\phi\, \frac{\partial S_1}{\partial \phi} + \omega^0_\psi\, \frac{\partial S_1}{\partial \psi} = \langle\, H_1 \,\rangle - H_1 \ . \tag{16.330}$$

This yields

$$\begin{aligned} \omega_c\, \frac{\partial S_1}{\partial \phi} + \bigg(\frac{k_z^2 K}{m} - \omega\bigg)\frac{\partial S_1}{\partial \psi} &= -eA_0 \cos\bigg(\psi + k_\perp\sqrt{\frac{2J}{m\omega_c}}\, \sin\phi\bigg) \\ &= -eA_0 \sum_{n=-\infty}^{\infty} J_n\bigg(k_\perp\sqrt{\frac{2J}{m\omega_c}}\bigg)\, \cos(\psi + n\phi) \ , \end{aligned} \tag{16.331}$$

where we have used the result

$$e^{iz\sin\theta} = \sum_{n=-\infty}^{\infty} J_n(z)\, e^{in\theta} \ . \tag{16.332}$$

The solution for S_1 is

$$S_1 = \sum_n \frac{eV_0}{\omega - n\omega_c - k_z^2 \bar{K}/m}\, J_n\bigg(k_\perp\sqrt{\frac{2\bar{J}}{m\omega_c}}\bigg)\, \sin(\psi + n\phi) \ . \tag{16.333}$$

We then have new action variables $\bar{J}$ and $\bar{K}$, where

$$J = \bar{J} + \epsilon\, \frac{\partial S_1}{\partial \phi} + \mathcal{O}(\epsilon^2) \tag{16.334}$$

$$K = \bar{K} + \epsilon\, \frac{\partial S_1}{\partial \psi} + \mathcal{O}(\epsilon^2) \ . \tag{16.335}$$

Defining the dimensionless variable

$$\lambda \equiv k_\perp\sqrt{\frac{2J}{m\omega_c}} \ , \tag{16.336}$$

we obtain the result

$$\bigg(\frac{m\omega_c^2}{2eV_0 k_\perp^2}\bigg)\bar{\lambda}^2 = \bigg(\frac{m\omega_c^2}{2eV_0 k_\perp^2}\bigg)\lambda^2 - \epsilon\sum_n \frac{nJ_n(\lambda)\cos(\psi + n\phi)}{\frac{\omega}{\omega_c} - n - \frac{k_z^2 K}{m\omega_c}} + \mathcal{O}(\epsilon^2) \ , \tag{16.337}$$

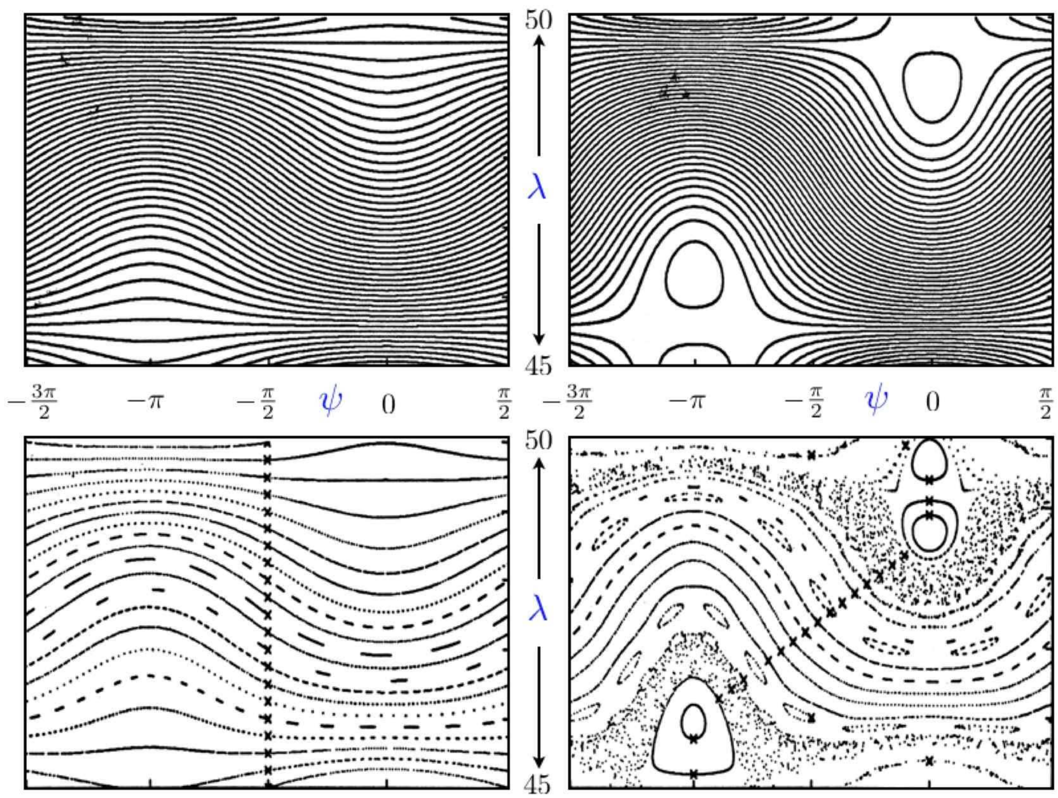


Figure 16.5: Plot of λ *versus* ψ for $\phi = 0$ (Poincaré section) for $\omega = 30.11\,\omega_c$ Top panels are nonresonant invariant curves calculated to first order. Bottom panels are exact numerical dynamics, with x symbols marking the initial conditions. Left panels: weak amplitude (no trapping). Right panels: stronger amplitude (shows trapping). From Lichtenberg and Lieberman (1983).

where $\bar{\lambda} = k_\perp \sqrt{2\bar{J}/m\omega_c}$.[14]

We see that resonances occur whenever

$$\frac{\omega}{\omega_c} - \frac{k_z^2 K}{m\omega_c} = n \ , \tag{16.338}$$

for any integer n. Let us consider the case $k_z = 0$, in which the resonance condition is $\omega = n\omega_c$. We then have

$$\frac{\bar{\lambda}^2}{2\alpha} = \frac{\lambda^2}{2\alpha} - \sum_n \frac{n\, J_n(\lambda) \cos(\psi + n\phi)}{\frac{\omega}{\omega_c} - n} \ , \tag{16.339}$$

where

$$\alpha = \frac{E_0}{B} \cdot \frac{ck_\perp}{\omega_c} \tag{16.340}$$

[14]Note that the argument of J_n in eqn. 16.337 is λ and not $\bar{\lambda}$. This arises because we are computing the new action $\bar{J}$ in terms of the old variables (ϕ, J) and (ψ, K).

is a dimensionless measure of the strength of the perturbation, with $E_0 \equiv k_\perp V_0$. In Fig. 16.5 we plot the level sets for the RHS of the above equation $\lambda(\psi)$ for $\phi = 0$, for two different values of the dimensionless amplitude α, for $\omega/\omega_c = 30.11$ (*i.e.* off resonance). Thus, when the amplitude is small, the level sets are far from a primary resonance, and the analytical and numerical results are very similar (left panels). When the amplitude is larger, resonances may occur which are not found in the lowest order perturbation treatment. However, as is apparent from the plots, the gross features of the phase diagram are reproduced by perturbation theory. What is missing is the existence of 'chaotic islands' which initially emerge in the vicinity of the trapping regions.

16.10 Adiabatic Invariants

Adiabatic perturbations are slow, smooth, time-dependent perturbations to a dynamical system. A classic example: a pendulum with a slowly varying length $l(t)$. Suppose $\lambda(t)$ is the adiabatic parameter. We write $H = H\big(q, p; \lambda(t)\big)$. All explicit time-dependence to H comes through $\lambda(t)$. Typically, a dimensionless parameter ϵ may be associated with the perturbation:

$$\epsilon = \frac{1}{\omega_0}\left|\frac{d\ln\lambda}{dt}\right| , \tag{16.341}$$

where ω_0 is the natural frequency of the system when λ is constant. We require $\epsilon \ll 1$ for adiabaticity. In adiabatic processes, the action variables are conserved to a high degree of accuracy. These are the *adiabatic invariants*. For example, for the harmonix oscillator, the action is $J = E/\nu$. While E and ν may vary considerably during the adiabatic process, their ratio is very nearly fixed. As a consequence, assuming small oscillations,

$$E = \nu J = \tfrac{1}{2} mgl\,\theta_0^2 \quad \Rightarrow \quad \theta_0(l) \approx \frac{2J}{m\sqrt{g}\,l^{3/2}} , \tag{16.342}$$

so $\theta_0(\ell) \propto l^{-3/4}$.

Suppose that for fixed λ the Hamiltonian is transformed to action-angle variables via the generator $S(q, J; \lambda)$. The transformed Hamiltonian is

$$\tilde{H}(\phi, J, t) = H(\phi, J; \lambda) + \frac{\partial S}{\partial\lambda}\,\frac{d\lambda}{dt} , \tag{16.343}$$

where

$$H(\phi, J; \lambda) = H\big(q(\phi, J; \lambda), p(\phi, J; \lambda); \lambda\big) . \tag{16.344}$$

We assume $n = 1$ here. Hamilton's equations are now

$$\dot{\phi} = +\frac{\partial\tilde{H}}{\partial J} = \nu(J;\lambda) + \frac{\partial^2 S}{\partial\lambda\,\partial J}\,\frac{d\lambda}{dt} \tag{16.345}$$

$$\dot{J} = -\frac{\partial\tilde{H}}{\partial\phi} = -\frac{\partial^2 S}{\partial\lambda\,\partial\phi}\,\frac{d\lambda}{dt} . \tag{16.346}$$

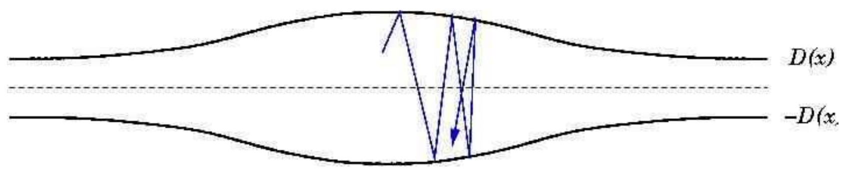


Figure 16.6: A mechanical mirror.

The second of these may be Fourier decomposed as

$$\dot{J} = -i\dot{\lambda}\sum_m m\,\frac{\partial S_m(J;\lambda)}{\partial\lambda}\,e^{im\phi}\ , \tag{16.347}$$

hence

$$\Delta J = J(t=+\infty) - J(t=-\infty) = -i\sum_m m\int\limits_{-\infty}^{\infty}\!dt\ \frac{\partial S_m(J;\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\ e^{im\phi}\ . \tag{16.348}$$

Since $\dot{\lambda}$ is small, we have $\phi(t) = \nu\,t + \beta$, to lowest order. We must therefore evaluate integrals such as

$$\mathcal{I}_m = \int\limits_{-\infty}^{\infty}\!dt\ \frac{\partial S_m(J;\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\ e^{im\nu t}\ . \tag{16.349}$$

The term in curly brackets is a smooth, slowly varying function of t. Call it $f(t)$. We presume $f(t)$ can be analytically continued off the real t axis, and that its closest singularity in the complex t plane lies at $t = \pm i\tau$, in which case $\mathcal{I}$ behaves as $\exp(-|m|\nu\tau)$. Consider, for example, the Lorentzian,

$$f(t) = \frac{\mathcal{C}}{1+(t/\tau)^2} \qquad \Rightarrow \qquad \int\limits_{-\infty}^{\infty}\!dt\, f(t)\, e^{im\nu t} = \pi\tau\, e^{-|m|\nu\tau}\ , \tag{16.350}$$

which is exponentially small in the time scale τ. Because of this, only $m = \pm 1$ need be considered. What this tells us is that the change ΔJ may be made arbitrarily small by a sufficiently slowly varying $\lambda(t)$.

16.10.1 Example: mechanical mirror

Consider a two-dimensional version of a mechanical mirror, depicted in fig. 16.6. A particle bounces between two curves, $y = \pm D(x)$, where $|D'(x)| << 1$. The bounce time is $\tau_{\mathrm{b}\perp} = 2D/v_y$. We assume $\tau \ll L/v_x$, where $v_{x,y}$ are the components of the particle's velocity, and L is the total length of the system. There are, therefore, many bounces, which means the particle gets to sample the curvature in $D(x)$.

The adiabatic invariant is the action,

$$J = \frac{1}{2\pi} \int\limits_{-D}^{D} dy \, m \, v_y + \frac{1}{2\pi} \int\limits_{D}^{-D} dy \, m \, (-v_y) = \frac{2}{\pi} \, m v_y \, D(x) \ . \tag{16.351}$$

Thus,

$$E = \tfrac{1}{2} m \big(v_x^2 + v_y^2 \big) = \tfrac{1}{2} m v_x^2 + \frac{\pi^2 J^2}{8 m D^2(x)} \ , \tag{16.352}$$

or

$$v_x^2 = \frac{2E}{m} - \left(\frac{\pi J}{2 m D(x)} \right)^2 \ . \tag{16.353}$$

The particle is reflected in the throat of the device at horizontal coordinate x^*, where

$$D(x^*) = \frac{\pi J}{\sqrt{8 m E}} \ . \tag{16.354}$$

16.10.2 Example: magnetic mirror

Consider a particle of charge e moving in the presence of a uniform magnetic field $\boldsymbol{B} = B \hat{\boldsymbol{z}}$. Recall the basic physics: velocity in the parallel direction v_z is conserved, while in the plane perpendicular to $\boldsymbol{B}$ the particle executes circular 'cyclotron orbits', satisfying

$$\frac{m v_\perp^2}{\rho} = \frac{e}{c} \, v_\perp B \quad \Rightarrow \quad \rho = \frac{m c v_\perp}{e B} \ , \tag{16.355}$$

where ρ is the radial coordinate in the plane perpendicular to $\boldsymbol{B}$. The period of the orbits is $T = 2\pi\rho . v_\perp = 2\pi m c / eB$, hence their frequency is the cyclotron frequency $\omega_c = eB/mc$.

Now assume that the magnetic field is spatially dependent. Note that a spatially varying $\boldsymbol{B}$-field cannot be unidirectional:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \boldsymbol{\nabla}_{\!\perp} \cdot \boldsymbol{B}_\perp + \frac{\partial B_z}{\partial z} = 0 \ . \tag{16.356}$$

The non-collinear nature of $\boldsymbol{B}$ results in the *drift* of the cyclotron orbits. Nevertheless, if the field $\boldsymbol{B}$ felt by the particle varies slowly on the time scale $T = 2\pi/\omega_c$, then the system possesses an adiabatic invariant:

$$J = \frac{1}{2\pi} \oint\limits_{\mathcal{C}} \boldsymbol{p} \cdot d\boldsymbol{\ell} = \frac{1}{2\pi} \oint\limits_{\mathcal{C}} \big(m \boldsymbol{v} + \tfrac{e}{c} \boldsymbol{A} \big) \cdot d\boldsymbol{\ell} \tag{16.357}$$

$$= \frac{m}{2\pi} \oint\limits_{\mathcal{C}} \boldsymbol{v} \cdot d\boldsymbol{\ell} + \frac{e}{2\pi c} \oint\limits_{\text{int}(\mathcal{C})} \boldsymbol{B} \cdot \hat{\boldsymbol{n}} \, d\Sigma \ . \tag{16.358}$$

The last two terms are of opposite sign, and one has

$$J = -\frac{m}{2\pi} \cdot \frac{\rho e B_z}{mc} \cdot 2\pi\rho + \frac{e}{2\pi c} \cdot B_z \cdot \pi \rho^2 \tag{16.359}$$

$$= -\frac{e B_z \rho^2}{2c} = -\frac{e}{2\pi c} \cdot \Phi_B(\mathcal{C}) = -\frac{m^2 v_\perp^2 c}{2 e B_z} \ , \tag{16.360}$$

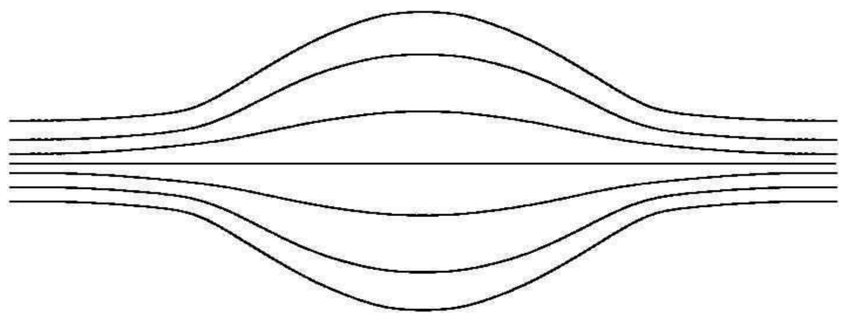


Figure 16.7: $\boldsymbol{B}$ field lines in a magnetic bottle.

where $\Phi_B(\mathcal{C})$ is the magnetic flux enclosed by $\mathcal{C}$.

The energy is

$$E = \tfrac{1}{2}mv_\perp^2 + \tfrac{1}{2}mv_z^2 \ , \tag{16.361}$$

hence we have

$$v_z = \sqrt{\frac{2}{m}\big(E - MB\big)} \ . \tag{16.362}$$

where

$$M \equiv -\frac{e}{mc}\,J = \frac{e^2}{2\pi mc^2}\,\Phi_{\rm B}(\mathcal{C}) \tag{16.363}$$

is the *magnetic moment*. Note that v_z vanishes when $B = B_{\rm max} = E/M$. When this limit is reached, the particle turns around. This is a *magnetic mirror*. A pair of magnetic mirrors may be used to confine charged particles in a *magnetic bottle*, depicted in fig. 16.7.

Let $v_{\parallel,0}$, $\boldsymbol{v}_{\perp,0}$, and $B_{\parallel,0}$ be the longitudinal particle velocity, transverse particle velocity, and longitudinal component of the magnetic field, respectively, at the point of injection. Our two conservation laws (J and E) guarantee

$$v_\parallel^2(z) + v_\perp^2(z) = v_{\parallel,0}^2 + v_{\perp,0}^2 \tag{16.364}$$

$$\frac{v_\perp(z)^2}{B_\parallel(z)} = \frac{v_{\perp,0}^2}{B_{\parallel,0}} \ . \tag{16.365}$$

This leads to reflection at a longitudinal coordinate z^*, where

$$B_\parallel(z^*) = B_{\parallel,0}\sqrt{1 + \frac{v_{\parallel,0}^2}{v_{\perp,0}^2}} \ . \tag{16.366}$$

The physics is quite similar to that of the mechanical mirror.

16.10.3 Resonances

When $n > 1$, we have

$$\dot{J}^\alpha = -i\dot{\lambda}\sum_{\boldsymbol{m}} m^\alpha \, \frac{\partial S_m(J;\lambda)}{\partial\lambda}\, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \tag{16.367}$$

$$\Delta J = -i\sum_{\boldsymbol{m}} m^\alpha \int\limits_{-\infty}^{\infty} \!dt\, \frac{\partial S_{\boldsymbol{m}}(J;\lambda)}{\partial\lambda}\,\frac{d\lambda}{dt}\; e^{i\boldsymbol{m}\cdot\boldsymbol{\nu}t}\, e^{i\boldsymbol{m}\cdot\boldsymbol{\beta}} \ . \tag{16.368}$$

Therefore, when $\boldsymbol{m}\cdot\boldsymbol{\nu}(J) = 0$ we have a resonance, and the integral grows linearly with time – a violation of the adiabatic invariance of J^α.

16.11 Appendix : Canonical Perturbation Theory

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\,\omega_0^2\, q^2 + \tfrac{1}{3}\epsilon\, m\,\omega_0^2\,\frac{q^3}{a} \ ,$$

where ϵ is a small dimensionless parameter.

(a) Show that the oscillation frequency satisfies $\nu(J) = \omega_0 + \mathcal{O}(\epsilon^2)$. That is, show that the first order (in ϵ) frequency shift vanishes.

Solution: It is good to recall the basic formulae

$$q = \sqrt{\frac{2J_0}{m\omega_0}}\,\sin\phi_0 \qquad , \qquad p = \sqrt{2m\,\omega_0\, J_0}\,\cos\phi_0 \tag{16.369}$$

as well as the results

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \epsilon\,\frac{\partial S_1}{\partial\phi_0} + \epsilon^2\,\frac{\partial S_2}{\partial\phi_0} + \ldots \tag{16.370}$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \epsilon\,\frac{\partial S_1}{\partial J} + \epsilon^2\,\frac{\partial S_2}{\partial J} + \ldots \ , \tag{16.371}$$

and

$$E_0(J) = \tilde{H}_0(J) \tag{16.372}$$

$$E_1(J) = \tilde{H}_1(\phi_0, J) + \frac{\partial\tilde{H}_0}{\partial J}\,\frac{\partial S_1}{\partial\phi_0} \tag{16.373}$$

$$E_2(J) = \frac{\partial\tilde{H}_0}{\partial J}\,\frac{\partial S_2}{\partial\phi_0} + \frac{1}{2}\,\frac{\partial^2\tilde{H}_0}{\partial J^2}\left(\frac{\partial S_1}{\partial\phi_0}\right)^{\!2} + \frac{\partial\tilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial\phi_0} \ . \tag{16.374}$$

Expressed in action-angle variables,

$$\tilde{H}_0(\phi_0, J) = \omega_0\, J \tag{16.375}$$

$$\tilde{H}_1(\phi_0, J) = \frac{2}{3}\sqrt{\frac{2\omega_0}{ma^2}}\, J^{3/2}\, \sin^3\!\phi_0\ . \tag{16.376}$$

Thus, $\nu_0 = \frac{\partial \tilde{H}_0}{\partial J} = \omega_0$.

Averaging the equation for $E_1(J)$ yields

$$E_1(J) = \big\langle \tilde{H}_1(\phi_0, J)\big\rangle = \frac{2}{3}\sqrt{\frac{2\omega_0}{ma^2}}\, J^{3/2}\, \big\langle \sin^3\!\phi_0\big\rangle = 0\ . \tag{16.377}$$

(b) Compute the frequency shift $\nu(J)$ to second order in ϵ.

Solution : From the equation for E_1, we also obtain

$$\frac{\partial S_1}{\partial \phi_0} = \frac{1}{\nu_0}\Big(\big\langle \tilde{H}_1\big\rangle - \tilde{H}_1\Big)\ . \tag{16.378}$$

Inserting this into the equation for $E_2(J)$ and averaging then yields

$$E_2(J) = \frac{1}{\nu_0}\bigg\langle \frac{\partial \tilde{H}_1}{\partial J}\,\Big(\big\langle \tilde{H}_1\big\rangle - \tilde{H}_1\Big)\bigg\rangle = -\frac{1}{\nu_0}\bigg\langle \tilde{H}_1\,\frac{\partial \tilde{H}_1}{\partial J}\bigg\rangle \tag{16.379}$$

$$= -\frac{4\nu_0 J^2}{3ma^2}\,\big\langle \sin^6\!\phi_0\big\rangle \tag{16.380}$$

In computing the average of $\sin^6\!\phi_0$, it is good to recall the binomial theorem, or the Fibonacci tree. The sixth order coefficents are easily found to be $\{1, 6, 15, 20, 15, 6, 1\}$, whence

$$\sin^6\!\phi_0 = \frac{1}{(2i)^6}\big(e^{i\phi_0} - e^{-i\phi_0}\big)^6 \tag{16.381}$$

$$= \tfrac{1}{64}\big(-2\sin 6\phi_0 + 12\sin 4\phi_0 - 30\sin 2\phi_0 + 20\big)\ . \tag{16.382}$$

Thus,

$$\big\langle \sin^6\!\phi_0\big\rangle = \tfrac{5}{16}\ , \tag{16.383}$$

whence

$$E(J) = \omega_0\, J - \tfrac{5}{12}\epsilon^2\,\frac{J^2}{ma^2} \tag{16.384}$$

and

$$\nu(J) = \frac{\partial E}{\partial J} = \omega_0 - \tfrac{5}{6}\epsilon^2\,\frac{J}{ma^2}\ . \tag{16.385}$$

(c) Find $q(t)$ to order ϵ. Your result should be finite for all times.

Solution : From the equation for $E_1(J)$, we have

$$\frac{\partial S_1}{\partial \phi_0} = -\frac{2}{3}\sqrt{\frac{2J^3}{m\omega_0 a^2}}\,\sin^3\!\phi_0\ . \tag{16.386}$$

Integrating, we obtain

$$S_1(\phi_0, J) = \frac{2}{3}\sqrt{\frac{2J^3}{m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{3}\cos^3\!\phi_0\big) \tag{16.387}$$

$$= \frac{J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{9}\cos 3\phi_0\big)\ . \tag{16.388}$$

Thus, with

$$S(\phi_0, J) = \phi_0\, J + \epsilon\, S_1(\phi_0, J) + \ldots\ , \tag{16.389}$$

we have

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \frac{3}{2}\frac{\epsilon\, J^{1/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{9}\cos 3\phi_0\big) \tag{16.390}$$

$$J_0 = \frac{\partial S}{\partial \phi_0} = J - \frac{\epsilon\, J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\sin\phi_0 - \tfrac{1}{3}\sin 3\phi_0\big)\ . \tag{16.391}$$

Inverting, we may write ϕ_0 and J_0 in terms of ϕ and J:

$$\phi_0 = \phi + \frac{3}{2}\frac{\epsilon\, J^{1/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\tfrac{1}{9}\cos 3\phi - \cos\phi\big) \tag{16.392}$$

$$J_0 = J + \frac{\epsilon\, J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\tfrac{1}{3}\sin 3\phi - \sin\phi\big)\ . \tag{16.393}$$

Thus,

$$q(t) = \sqrt{\frac{2J_0}{m\omega_0}}\ \sin\phi_0 \tag{16.394}$$

$$= \sqrt{\frac{2J}{m\omega_0}}\ \sin\phi\cdot\Big(1 + \frac{\delta J}{2J} + \ldots\Big)\Big(\sin\phi + \delta\phi\,\cos\phi + \ldots\Big) \tag{16.395}$$

$$= \sqrt{\frac{2J}{m\omega_0}}\ \sin\phi - \frac{\epsilon\, J}{m\omega_0 a}\,\big(1 + \tfrac{1}{3}\cos 2\phi\big) + \mathcal{O}\big(\epsilon^2\big)\ , \tag{16.396}$$

with

$$\phi(t) = \phi(0) + \nu(J)\, t\ . \tag{16.397}$$

Chapter 17

Physics 110A-B Exams

The following pages contain problems and solutions from midterm and final exams in Physics 110A-B.

17.1 F05 Physics 110A Midterm #1

[1] A particle of mass m moves in the one-dimensional potential

$$U(x) = U_0 \, \frac{x^2}{a^2} \, e^{-x/a} \ . \tag{17.1}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.

(b) Sketch a representative set of phase curves. Identify and classify any and all fixed points. Find the energy of each and every separatrix.

(c) Sketch all the phase curves for motions with total energy $E = \frac{2}{5}\, U_0$. Do the same for $E = U_0$. (Recall that $e = 2.71828\ldots$.)

(d) Derive and expression for the period T of the motion when $|x| \ll a$.

Solution:

(a) Clearly $U(x)$ diverges to $+\infty$ for $x \to -\infty$, and $U(x) \to 0$ for $x \to +\infty$. Setting $U'(x) = 0$, we obtain the equation

$$U'(x) = \frac{U_0}{a^2} \left(2x - \frac{x^2}{a} \right) e^{-x/a} = 0 \ , \tag{17.2}$$

with (finite x) solutions at $x = 0$ and $x = 2a$. Clearly $x = 0$ is a local minimum and $x = 2a$ a local maximum. Note $U(0) = 0$ and $U(2a) = 4\, e^{-2}\, U_0 \approx 0.541\, U_0$.

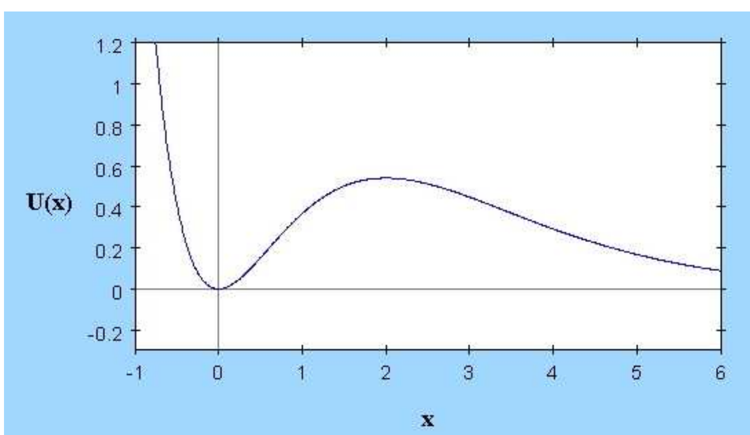


Figure 17.1: The potential $U(x)$. Distances are here measured in units of a, and the potential in units of U_0.

(b) Local minima of a potential $U(x)$ give rise to centers in the (x, v) plane, while local maxima give rise to saddles. In Fig. 17.2 we sketch the phase curves. There is a center at

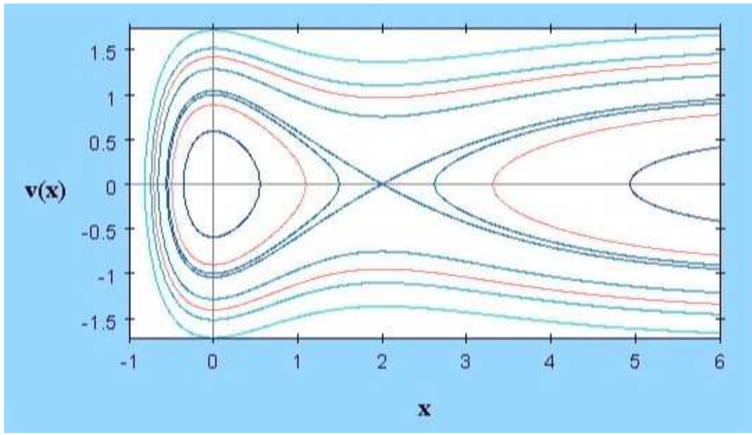


Figure 17.2: Phase curves for the potential $U(x)$. The red curves show phase curves for $E = \frac{2}{5}\,U_0$ (interior, disconnected red curves, $|v| < 1$) and $E = U_0$ (outlying red curve). The separatrix is the dark blue curve which forms a saddle at $(x, v) = (2, 0)$, and corresponds to an energy $E = 4\,e^{-2}\,U_0$.

$(0, 0)$ and a saddle at $(2a, 0)$. There is one separatrix, at energy $E = U(2a) = 4\,e^{-2}\,U_0 \approx 0.541\,U_0$.

(c) Even without a calculator, it is easy to verify that $4\,e^{-2} > \frac{2}{5}$. One simple way is to multiply both sides by $\frac{5}{2}\,e^2$ to obtain $10 > e^2$, which is true since $e^2 < (2.71828\ldots)^2 < 10$. Thus, the energy $E = \frac{2}{5}\,U_0$ lies below the local maximum value of $U(2a)$, which means that there are *two* phase curves with $E = \frac{2}{5}\,U_0$.

It is also quite obvious that the second energy value given, $E = U_0$, lies above $U(2a)$, which means that there is a single phase curve for this energy. One finds bound motions only for $x < 2$ and $0 \leq E < U(2a)$. The phase curves corresponding to total energy $E = \frac{2}{5}\,U_0$ and $E = U_0$ are shown in Fig. 17.2.

(d) Expanding $U(x)$ in a Taylor series about $x = 0$, we have

$$U(x) = \frac{U_0}{a^2}\left\{x^2 - \frac{x^3}{a} + \frac{x^4}{2a^2} + \ldots\right\} . \tag{17.3}$$

The leading order term is sufficient for $|x| \ll a$. The potential energy is then equivalent to that of a spring, with spring constant $k = 2U_0/a^2$. The period is

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{ma^2}{2U_0}}} . \tag{17.4}$$

[2] A forced, damped oscillator obeys the equation

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f_0\,\cos(\omega_0 t)\ . \tag{17.5}$$

You may assume the oscillator is underdamped.

(a) Write down the most general solution of this differential equation.

(b) Your solution should involve two constants. Derive two equations relating these constants to the initial position $x(0)$ and the initial velocity $\dot{x}(0)$. *You do not have to solve these equations.*

(c) Suppose $\omega_0 = 5.0\,\mathrm{s}^{-1}$, $\beta = 4.0\,\mathrm{s}^{-1}$, and $f_0 = 8\,\mathrm{cm\,s}^{-2}$. Suppose further you are told that $x(0) = 0$ and $x(T) = 0$, where $T = \frac{\pi}{6}\,\mathrm{s}$. Derive an expression for the initial velocity $\dot{x}(0)$.

Solution: (a) The general solution with forcing $f(t) = f_0\,\cos(\Omega t)$ is

$$x(t) = x_{\mathrm{h}}(t) + A(\Omega)\,f_0\,\cos\big(\Omega t - \delta(\Omega)\big)\ , \tag{17.6}$$

with

$$A(\Omega) = \Big[(\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\Big]^{-1/2} \quad , \quad \delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right) \tag{17.7}$$

and

$$x_{\mathrm{h}}(t) = C\,e^{-\beta t}\,\cos(\nu t) + D\,e^{-\beta t}\,\sin(\nu t)\ , \tag{17.8}$$

with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

In our case, $\Omega = \omega_0$, in which case $A = (2\beta\omega_0)^{-1}$ and $\delta = \frac{1}{2}\pi$. Thus, the most general solution is

$$\boxed{x(t) = C\,e^{-\beta t}\,\cos(\nu t) + D\,e^{-\beta t}\,\sin(\nu t) + \frac{f_0}{2\beta\omega_0}\,\sin(\omega_0 t)}\ . \tag{17.9}$$

(b) We determine the constants C and D by the boundary conditions on $x(0)$ and $\dot{x}(0)$:

$$\boxed{x(0) = C} \quad , \quad \boxed{\dot{x}(0) = -\beta C + \nu D + \frac{f_0}{2\beta}}\ . \tag{17.10}$$

Thus,

$$C = x(0) \quad , \quad D = \frac{\beta}{\nu}\,x(0) + \frac{1}{\nu}\,\dot{x}(0) - \frac{f_0}{2\beta\nu}\ . \tag{17.11}$$

(c) From $x(0) = 0$ we obtain $C = 0$. The constant D is then determined by the condition at time $t = T = \frac{1}{6}\pi$.

Note that $\nu = \sqrt{\omega_0^2 - \beta^2} = 3.0\,\mathrm{s}^{-1}$. Thus, with $T = \frac{1}{6}\pi$, we have $\nu T = \frac{1}{2}\pi$, and

$$x(T) = D\,e^{-\beta T} + \frac{f_0}{2\beta\omega_0}\,\sin(\omega_0 T)\ . \tag{17.12}$$

This determines D:

$$D = -\frac{f_0}{2\beta\omega_0}\, e^{\beta T}\, \sin(\omega_0 T) \ . \tag{17.13}$$

We now can write

$$\dot{x}(0) = \nu D + \frac{f_0}{2\beta} \tag{17.14}$$

$$= \frac{f_0}{2\beta}\left(1 - \frac{\nu}{\omega_0}\, e^{\beta T}\, \sin(\omega_0 T)\right) \tag{17.15}$$

$$= \boxed{\left(1 - \tfrac{3}{10}\, e^{2\pi/3}\right) \text{cm/s}} \ . \tag{17.16}$$

Numerically, the value is $\dot{x}(0) \approx 0.145\,\text{cm/s}$.

17.2 F05 Physics 110A Midterm #2

[**1**] Two blocks connected by a spring of spring constant k are free to slide frictionlessly along a horizontal surface, as shown in Fig. 17.3. The unstretched length of the spring is a.

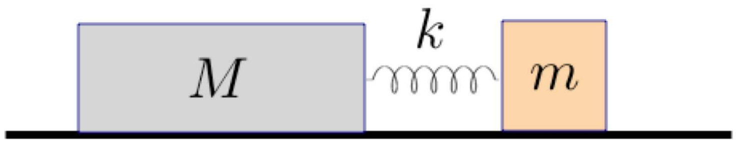


Figure 17.3: Two masses connected by a spring sliding horizontally along a frictionless surface.

(a) Identify a set of generalized coordinates and write the Lagrangian.
`[15 points]`

Solution : As generalized coordinates I choose X and u, where X is the position of the right edge of the block of mass M, and $X + u + a$ is the position of the left edge of the block of mass m, where a is the unstretched length of the spring. Thus, the extension of the spring is u. The Lagrangian is then

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{X}+\dot{u})^2 - \tfrac{1}{2}ku^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}m\dot{u}^2 + m\dot{X}\dot{u} - \tfrac{1}{2}ku^2 \ . \end{aligned} \tag{17.17}$$

(b) Find the equations of motion.
`[15 points]`

Solution : The canonical momenta are

$$p_X \equiv \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\dot{u} \quad , \quad p_u \equiv \frac{\partial L}{\partial \dot{u}} = m(\dot{X}+\dot{u}) \ . \tag{17.18}$$

The corresponding equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \qquad \Rightarrow \qquad (M+m)\ddot{X} + m\ddot{u} = 0 \tag{17.19}$$

$$\dot{p}_u = F_u = \frac{\partial L}{\partial u} \qquad \Rightarrow \qquad m(\ddot{X}+\ddot{u}) = -ku \ . \tag{17.20}$$

(c) Find all conserved quantities.
`[10 points]`

Solution : There are two conserved quantities. One is p_X itself, as is evident from the fact that L is cyclic in X. This is the conserved 'charge' Λ associated with the continuous symmetry $X \to X + \zeta$. *i.e.* $\Lambda = p_X$. The other conserved quantity is the Hamiltonian H, since L is cyclic in t. Furthermore, because the kinetic energy is homogeneous of degree two in the generalized velocities, we have that $H = E$, with

$$E = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}m\dot{u}^2 + m\dot{X}\dot{u} + \tfrac{1}{2}ku^2 \ . \tag{17.21}$$

It is possible to eliminate $\dot{X}$, using the conservation of Λ:

$$\dot{X} = \frac{\Lambda - m\dot{u}}{M+m} . \tag{17.22}$$

This allows us to write

$$E = \frac{\Lambda^2}{2(M+m)} + \frac{Mm\,\dot{u}^2}{2(M+m)} + \tfrac{1}{2}ku^2 . \tag{17.23}$$

(d) Find a complete solution to the equations of motion. As there are two degrees of freedom, your solution should involve 4 constants of integration. You need not match initial conditions, and you need not choose the quantities in part (c) to be among the constants. [10 points]

Solution : Using conservation of Λ, we may write $\ddot{X}$ in terms of $\ddot{x}$, in which case

$$\frac{Mm}{M+m}\,\ddot{u} = -ku \quad \Rightarrow \quad u(t) = A\cos(\Omega t) + B\sin(\Omega t) , \tag{17.24}$$

where

$$\Omega = \sqrt{\frac{(M+m)k}{Mm}} . \tag{17.25}$$

For the X motion, we integrate eqn. 17.22 above, obtaining

$$X(t) = X_0 + \frac{\Lambda t}{M+m} - \frac{m}{M+m}\Big(A\cos(\Omega t) - A + B\sin(\Omega t)\Big) . \tag{17.26}$$

There are thus four constants: X_0, Λ, A, and B. Note that conservation of energy says

$$E = \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}k(A^2 + B^2) . \tag{17.27}$$

Alternate solution : We could choose X as the position of the left block and x as the position of the right block. In this case,

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}k(x - X - b)^2 . \tag{17.28}$$

Here, b includes the unstretched length a of the spring, but may also include the size of the blocks if, say, X and x are measured relative to the blocks' midpoints. The canonical momenta are

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \quad , \quad p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} . \tag{17.29}$$

The equations of motion are then

$$\dot{p}_X = F_X = \frac{\partial L}{\partial X} \quad \Rightarrow \quad M\ddot{X} = k(x - X - b) \tag{17.30}$$

$$\dot{p}_x = F_x = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m\ddot{x} = -k(x - X - b) . \tag{17.31}$$

The one-parameter family which leaves L invariant is $X \to X + \zeta$ and $x \to x + \zeta$, *i.e.* simultaneous and identical displacement of both of the generalized coordinates. Then

$$\Lambda = M\dot{X} + m\dot{x} \ , \tag{17.32}$$

which is simply the x-component of the total momentum. Again, the energy is conserved:

$$E = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}k\,(x - X - b)^2 \ . \tag{17.33}$$

We can combine the equations of motion to yield

$$Mm\,\frac{d^2}{dt^2}\big(x - X - b\big) = -k\,(M+m)\big(x - X - b\big) \ , \tag{17.34}$$

which yields

$$x(t) - X(t) = b + A\cos(\Omega t) + B\sin(\Omega t) \ , \tag{17.35}$$

From the conservation of Λ, we have

$$M X(t) + m\,x(t) = \Lambda t + C \ , \tag{17.36}$$

were C is another constant. Thus, we have the motion of the system in terms of four constants: A, B, Λ, and C:

$$X(t) = -\tfrac{m}{M+m}\big(b + A\cos(\Omega t) + B\sin(\Omega t)\big) + \frac{\Lambda t + C}{M+m} \tag{17.37}$$

$$x(t) = \tfrac{M}{M+m}\big(b + A\cos(\Omega t) + B\sin(\Omega t)\big) + \frac{\Lambda t + C}{M+m} \ . \tag{17.38}$$

[2] A uniformly dense ladder of mass m and length 2ℓ leans against a block of mass M, as shown in Fig. 17.4. Choose as generalized coordinates the horizontal position X of the right end of the block, the angle θ the ladder makes with respect to the floor, and the coordinates (x, y) of the ladder's center-of-mass. These four generalized coordinates are not all independent, but instead are related by a certain set of constraints.

Recall that the kinetic energy of the ladder can be written as a sum $T_{\text{CM}} + T_{\text{rot}}$, where $T_{\text{CM}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ is the kinetic energy of the center-of-mass motion, and $T_{\text{rot}} = \frac{1}{2}I\dot{\theta}^2$, where I is the moment of inertial. For a uniformly dense ladder of length 2ℓ, $I = \frac{1}{3}m\ell^2$.

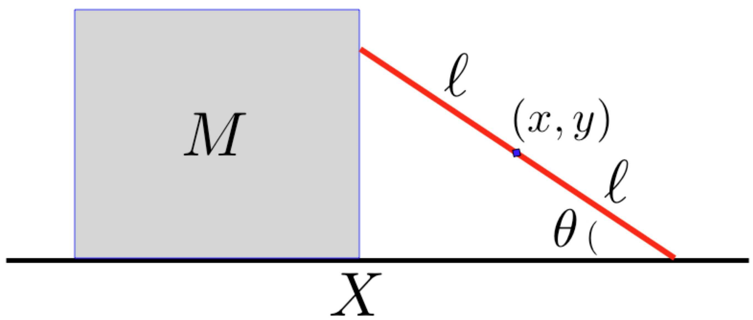


Figure 17.4: A ladder of length 2ℓ leaning against a massive block. All surfaces are frictionless..

(a) Write down the Lagrangian for this system in terms of the coordinates X, θ, x, y, and their time derivatives.
`[10 points]`

Solution : We have $L = T - U$, hence

$$L = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \ . \tag{17.39}$$

(b) Write down all the equations of constraint.
`[10 points]`

Solution : There are two constraints, corresponding to contact between the ladder and the block, and contact between the ladder and the horizontal surface:

$$G_1(X, \theta, x, y) = x - \ell\cos\theta - X = 0 \tag{17.40}$$

$$G_2(X, \theta, x, y) = y - \ell\sin\theta = 0 \ . \tag{17.41}$$

(c) Write down all the equations of motion.
`[10 points]`

Solution : Two Lagrange multipliers, λ_1 and λ_2, are introduced to effect the constraints. We have for each generalized coordinate q_σ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_{j=1}^{k} \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \equiv Q_\sigma \ , \tag{17.42}$$

where there are $k = 2$ constraints. We therefore have

$$M\ddot{X} = -\lambda_1 \tag{17.43}$$

$$m\ddot{x} = +\lambda_1 \tag{17.44}$$

$$m\ddot{y} = -mg + \lambda_2 \tag{17.45}$$

$$I\ddot{\theta} = \ell\sin\theta\,\lambda_1 - \ell\cos\theta\,\lambda_2 \ . \tag{17.46}$$

These four equations of motion are supplemented by the two constraint equations, yielding six equations in the six unknowns $\{X, \theta, x, y, \lambda_1, \lambda_2\}$.

(d) Find all conserved quantities.
`[10 points]`

Solution : The Lagrangian and all the constraints are invariant under the transformation

$$X \to X + \zeta \quad , \quad x \to x + \zeta \quad , \quad y \to y \quad , \quad \theta \to \theta \ . \tag{17.47}$$

The associated conserved 'charge' is

$$\Lambda = \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = M\dot{X} + m\dot{x} \ . \tag{17.48}$$

Using the first constraint to eliminate x in terms of X and θ, we may write this as

$$\Lambda = (M+m)\dot{X} - m\ell\sin\theta\,\dot{\theta} \ . \tag{17.49}$$

The second conserved quantity is the total energy E. This follows because the Lagrangian and all the constraints are independent of t, and because the kinetic energy is homogeneous of degree two in the generalized velocities. Thus,

$$E = \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\dot{\theta}^2 + mgy \tag{17.50}$$

$$= \frac{\Lambda^2}{2(M+m)} + \tfrac{1}{2}\Big(I + m\ell^2 - \tfrac{m}{M+m}\,m\ell^2\sin^2\theta\Big)\,\dot{\theta}^2 + mg\ell\sin\theta \ , \tag{17.51}$$

where the second line is obtained by using the constraint equations to eliminate x and y in terms of X and θ.

(e) What is the condition that the ladder detaches from the block? You do not have to solve for the angle of detachment! Express the detachment condition in terms of any quantities you find convenient.
`[10 points]`

Solution : The condition for detachment from the block is simply $\lambda_1 = 0$, *i.e.* the normal force vanishes.

Further analysis : It is instructive to work this out in detail (though this level of analysis was not required for the exam). If we eliminate x and y in terms of X and θ, we find

$$x = X + \ell\cos\theta \qquad\qquad y = \ell\sin\theta \tag{17.52}$$

$$\dot{x} = \dot{X} - \ell\sin\theta\,\dot{\theta} \qquad\qquad \dot{y} = \ell\cos\theta\,\dot{\theta} \tag{17.53}$$

$$\ddot{x} = \ddot{X} - \ell\sin\theta\,\ddot{\theta} - \ell\cos\theta\,\dot{\theta}^2 \qquad\qquad \ddot{y} = \ell\cos\theta\,\ddot{\theta} - \ell\sin\theta\,\dot{\theta}^2 \ . \tag{17.54}$$

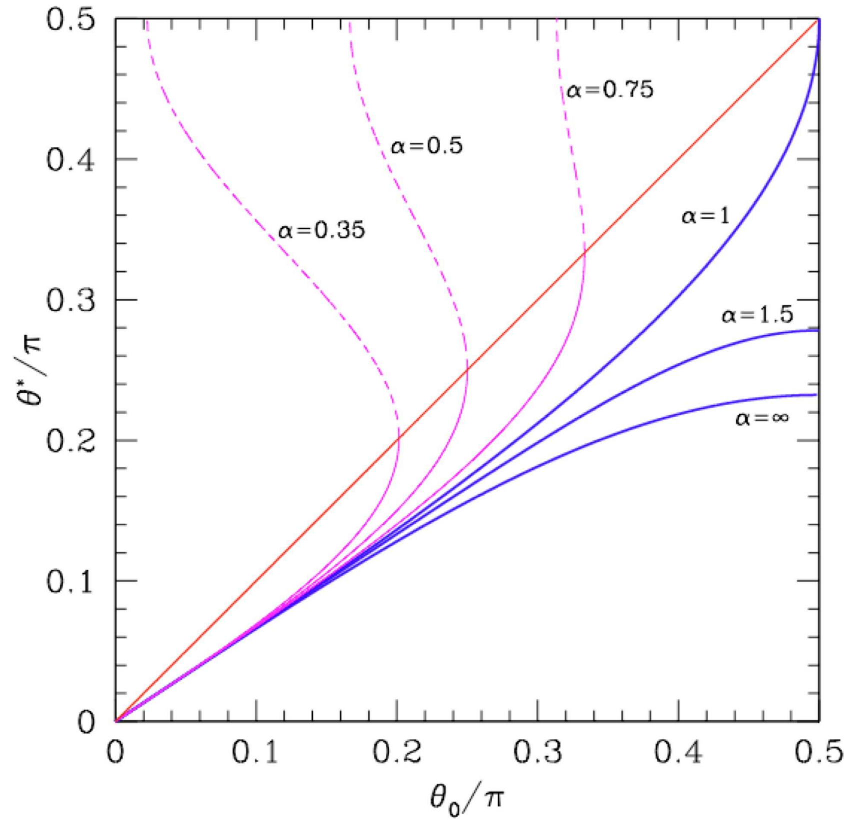


Figure 17.5: Plot of θ^* *versus* θ_0 for the ladder-block problem (eqn. 17.64). Allowed solutions, shown in blue, have $\alpha \geq 1$, and thus $\theta^* \leq \theta_0$. Unphysical solutions, with $\alpha < 1$, are shown in magenta. The line $\theta^* = \theta_0$ is shown in red.

We can now write

$$\lambda_1 = m\ddot{x} = m\ddot{X} - m\ell \sin\theta\,\ddot{\theta} - m\ell\cos\theta\,\dot{\theta}^2 = -M\ddot{X}\ , \tag{17.55}$$

which gives

$$(M+m)\ddot{X} = m\ell\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big)\ , \tag{17.56}$$

and hence

$$Q_x = \lambda_1 = -\frac{Mm}{m+m}\,\ell\,\big(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\big)\ . \tag{17.57}$$

We also have

$$\begin{aligned} Q_y = \lambda_2 &= mg + m\ddot{y} \\ &= mg + m\ell\big(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\big)\ . \end{aligned} \tag{17.58}$$

We now need an equation relating $\ddot{\theta}$ and $\dot{\theta}$. This comes from the last of the equations of motion:

$$\begin{aligned} I\ddot{\theta} &= \ell\sin\theta\,\lambda_1 - \ell\cos\theta\lambda_2 \\ &= -\tfrac{Mm}{M+m}\,\ell^2\big(\sin^2\!\theta\,\ddot{\theta} + \sin\theta\cos\theta\,\dot{\theta}^2\big) - mg\ell\cos\theta - m\ell^2\big(\cos^2\!\theta\,\ddot{\theta} - \sin\theta\cos\theta\,\dot{\theta}^2\big) \\ &= -mg\ell\cos\theta - m\ell^2\Big(1 - \tfrac{m}{M+m}\,\sin^2\!\theta\Big)\,\ddot{\theta} + \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 \ . \end{aligned} \tag{17.59}$$

Collecting terms proportional to $\ddot{\theta}$, we obtain

$$\big(I + m\ell^2 - \tfrac{m}{M+m}\,\sin^2\!\theta\big)\,\ddot{\theta} = \tfrac{m}{M+m}\,m\ell^2\,\sin\theta\cos\theta\,\dot{\theta}^2 - mg\ell\,\cos\theta \ . \tag{17.60}$$

We are now ready to demand $Q_x = \lambda_1 = 0$, which entails

$$\ddot{\theta} = -\frac{\cos\theta}{\sin\theta}\,\dot{\theta}^2 \ . \tag{17.61}$$

Substituting this into eqn. 17.60, we obtain

$$\big(I + m\ell^2\big)\,\dot{\theta}^2 = mg\ell\,\sin\theta \ . \tag{17.62}$$

Finally, we substitute this into eqn. 17.51 to obtain an equation for the detachment angle, θ^*

$$E - \frac{\Lambda^2}{2(M+m)} = \bigg(3 - \frac{m}{M+m}\cdot\frac{m\ell^2}{I+m\ell^2}\,\sin^2\!\theta^*\bigg)\cdot\tfrac{1}{2}mg\ell\,\sin\theta^* \ . \tag{17.63}$$

If our initial conditions are that the system starts from rest[1] with an angle of inclination θ_0, then the detachment condition becomes

$$\begin{aligned} \sin\theta_0 &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\Big(\tfrac{m}{M+m}\Big)\Big(\tfrac{m\ell^2}{I+m\ell^2}\Big)\sin^3\!\theta^* \\ &= \tfrac{3}{2}\sin\theta^* - \tfrac{1}{2}\,\alpha^{-1}\,\sin^3\!\theta^* \ , \end{aligned} \tag{17.64}$$

where

$$\alpha \equiv \bigg(1 + \frac{M}{m}\bigg)\bigg(1 + \frac{I}{m\ell^2}\bigg) \ . \tag{17.65}$$

Note that $\alpha \geq 1$, and that when $M/m = \infty$[2], we recover $\theta^* = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big)$. For finite α, the ladder detaches at a larger value of θ^*. A sketch of θ^* *versus* θ_0 is provided in Fig. 17.5. Note that, provided $\alpha \geq 1$, detachment always occurs for some unique value θ^* for each θ_0.

[1] 'Rest' means that the initial velocities are $\dot{X} = 0$ and $\dot{\theta} = 0$, and hence $\Lambda = 0$ as well.
[2] I must satisfy $I \leq m\ell^2$.

17.3 F05 Physics 110A Final Exam

[**1**] Two blocks and three springs are configured as in Fig. 17.6. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

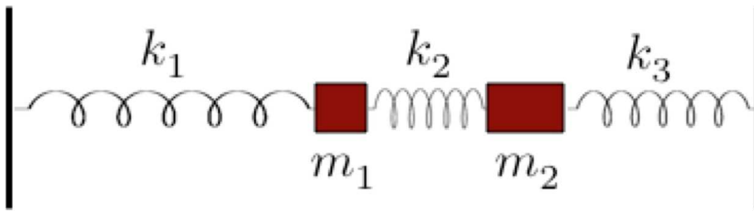


Figure 17.6: A system of masses and springs.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.
`[5 points]`

(b) Find the T and V matrices.
`[5 points]`

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \ ,$$

Find the frequencies of small oscillations.
`[5 points]`

(d) Find the normal modes of oscillation.
`[5 points]`

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.
`[5 points]`

Solution

(a) The Lagrangian is

$$\boxed{L = \tfrac{1}{2}m_1\,\dot{x}_1^2 + \tfrac{1}{2}m_2\,\dot{x}_2^2 - \tfrac{1}{2}k_1\,x_1^2 - \tfrac{1}{2}k_2\,(x_2 - x_1)^2 - \tfrac{1}{2}k_3\,x_2^2}$$

(b) The T and V matrices are

$$\boxed{\mathrm{T}_{ij} = \frac{\partial^2 T}{\partial\dot{x}_i\,\partial\dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}} \quad , \quad \boxed{\mathrm{V}_{ij} = \frac{\partial^2 U}{\partial x_i\,\partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix}}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda\,\omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathrm{T} - \mathrm{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} .$$

The determinant is

$$\begin{aligned} \det\left(\omega^2 \mathrm{T} - \mathrm{V}\right) &= (2\lambda^2 - 11\lambda + 14)\, k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\, k^2 . \end{aligned}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\boxed{\omega_- = \sqrt{\frac{2k}{m}}} \quad , \quad \boxed{\omega_+ = \sqrt{\frac{7k}{2m}}}$$

(d) The normal modes are determined from $(\omega_a^2 \mathrm{T} - \mathrm{V})\, \vec{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\vec{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\vec{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boxed{\vec{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

The standard normalization $\psi_i^{(a)}\, \mathrm{T}_{ij}\, \psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \quad , \quad \mathcal{C}_2 = \frac{1}{\sqrt{6m}} . \tag{17.66}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) .$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 .$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) . \end{aligned}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega\_t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega\_t = \omega_+ t - \pi \quad \Rightarrow \quad \boxed{t^* = \frac{2\pi}{\omega_- + \omega_+}}$$

[2] Two point particles of masses m_1 and m_2 interact via the central potential

$$U(r) = U_0 \ln\left(\frac{r^2}{r^2 + b^2}\right) ,$$

where b is a constant with dimensions of length.

(a) For what values of the relative angular momentum ℓ does a circular orbit exist? Find the radius r_0 of the circular orbit. Is it stable or unstable?
`[7 points]`

(c) For the case where a circular orbit exists, sketch the phase curves for the radial motion in the $(r, \dot{r})$ half-plane. Identify the energy ranges for bound and unbound orbits.
`[5 points]`

(c) Suppose the orbit is nearly circular, with $r = r_0 + \eta$, where $|\eta| \ll r_0$. Find the equation for the shape $\eta(\phi)$ of the perturbation.
`[8 points]`

(d) What is the angle $\Delta\phi$ through which periapsis changes each cycle? For which value(s) of ℓ does the perturbed orbit not precess?
`[5 points]`

Solution

(a) The effective potential is

$$\begin{aligned} U_{\rm eff}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^2} + U_0 \ln\left(\frac{r^2}{r^2+b^2}\right) . \end{aligned}$$

where $\mu = m_1 m_2/(m_1 + m_1)$ is the reduced mass. For a circular orbit, we must have $U'_{\rm eff}(r) = 0$, or

$$\frac{l^2}{\mu r^3} = U'(r) = \frac{2rU_0 b^2}{r^2\,(r^2+b^2)} .$$

The solution is

$$\boxed{r_0^2 = \frac{b^2\ell^2}{2\mu b^2 U_0 - \ell^2}}$$

Since $r_0^2 > 0$, the condition on ℓ is

$$\boxed{\ell < \ell_{\rm c} \equiv \sqrt{2\mu b^2 U_0}}$$

For large r, we have

$$U_{\rm eff}(r) = \left(\frac{\ell^2}{2\mu} - U_0\, b^2\right)\cdot\frac{1}{r^2} + \mathcal{O}(r^{-4}) .$$

Thus, for $\ell < \ell_{\rm c}$ the effective potential is negative for sufficiently large values of r. Thus, over the range $\ell < \ell_{\rm c}$, we must have $U_{\rm eff,min} < 0$, which must be a global minimum, since $U_{\rm eff}(0^+) = \infty$ and $U_{\rm eff}(\infty) = 0$. Therefore, the circular orbit is stable whenever it exists.

(b) Let $\ell = \epsilon\,\ell_{\rm c}$. The effective potential is then

$$U_{\rm eff}(r) = U_0\, f(r/b) ,$$

where the dimensionless effective potential is

$$f(s) = \frac{\epsilon^2}{s^2} - \ln(1+s^{-2}) .$$

The phase curves are plotted in Fig. 17.7.

(c) The energy is

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + U_{\rm eff}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\rm eff}(r) , \end{aligned}$$

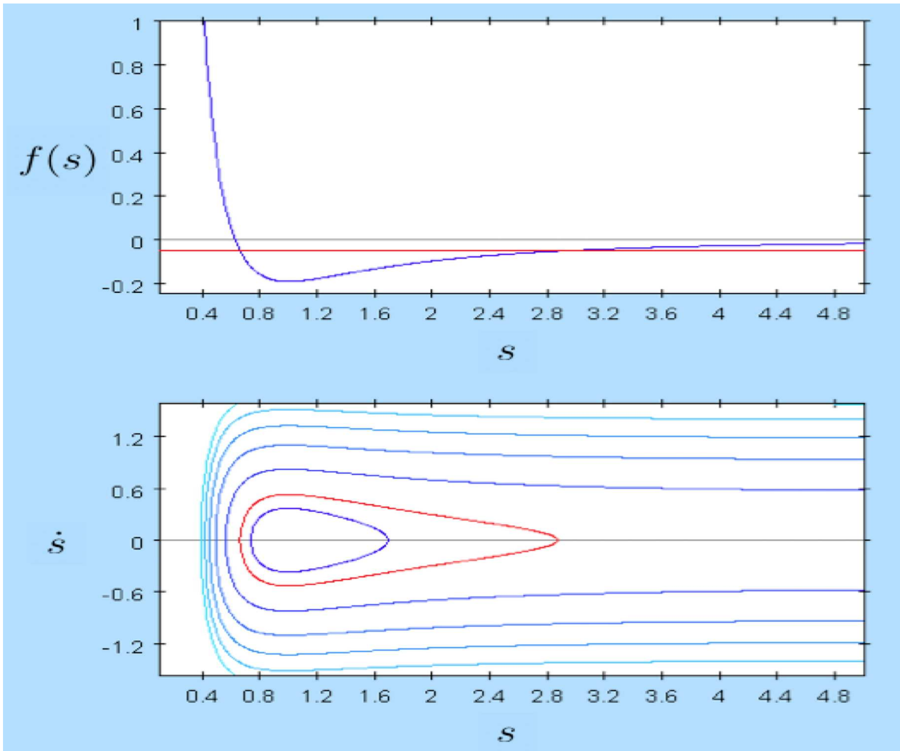


Figure 17.7: Phase curves for the scaled effective potential $f(s) = \epsilon\, s^{-2} - \ln(1 + s^{-2})$, with $\epsilon = \frac{1}{\sqrt{2}}$. Here, $\epsilon = \ell/\ell_c$. The dimensionless time variable is $\tau = t \cdot \sqrt{U_0/mb^2}$.

where we've used $\dot{r} = \dot{\phi}\, r'$ along with $\ell = \mu r^2 \dot{\phi}$. Writing $r = r_0 + \eta$ and differentiating E with respect to ϕ, we find

$$\eta'' = -\beta^2 \eta \qquad , \qquad \beta^2 = \frac{\mu r_0^4}{\ell^2}\, U''_{\text{eff}}(r_0)\ .$$

For our potential, we have

$$\boxed{\beta^2 = 2 - \frac{\ell^2}{\mu b^2 U_0} = 2\left(1 - \frac{\ell^2}{\ell_c^2}\right)}$$

The solution is

$$\boxed{\eta(\phi) = A\,\cos(\beta\phi + \delta)} \tag{17.67}$$

where A and δ are constants.

(d) The change of periapsis per cycle is

$$\boxed{\Delta\phi = 2\pi\big(\beta^{-1} - 1\big)}$$

If $\beta > 1$ then $\Delta\phi < 0$ and periapsis *advances* each cycle (*i.e.*it comes sooner with every cycle). If $\beta < 1$ then $\Delta\phi > 0$ and periapsis *recedes*. For $\beta = 1$, which means $\ell = \sqrt{\mu b^2 U_0}$, there is no precession and $\Delta\phi = 0$.

[3] A particle of charge e moves in three dimensions in the presence of a uniform magnetic field $\boldsymbol{B} = B_0\,\hat{\boldsymbol{z}}$ and a uniform electric field $\boldsymbol{E} = E_0\,\hat{\boldsymbol{x}}$. The potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = -e\,E_0\,x - \frac{e}{c}\,B_0\,x\,\dot{y}\ ,$$

where we have chosen the gauge $\boldsymbol{A} = B_0\,x\,\hat{\boldsymbol{y}}$.

(a) Find the canonical momenta p_x, p_y, and p_z.
`[7 points]`

(b) Identify all conserved quantities.
`[8 points]`

(c) Find a complete, general solution for the motion of the system $\big\{x(t), y(t), x(t)\big\}$.
`[10 points]`

Solution

(a) The Lagrangian is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{e}{c}\,B_0\,x\,\dot{y} + e\,E_0\,x\ .$$

The canonical momenta are

$$\boxed{p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}}$$

$$\boxed{p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{e}{c}\,B_0\,x}$$

$$\boxed{p_x = \frac{\partial L}{\partial \dot{z}} = m\dot{z}}$$

(b) There are three conserved quantities. First is the momentum p_y, since $F_y = \frac{\partial L}{\partial y} = 0$. Second is the momentum p_z, since $F_y = \frac{\partial L}{\partial z} = 0$. The third conserved quantity is the Hamiltonian, since $\frac{\partial L}{\partial t} = 0$. We have

$$H = p_x\,\dot{x} + p_y\,\dot{y} + p_z\,\dot{z} - L$$

$$\Rightarrow \quad \boxed{H = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\,E_0\,x}$$

(c) The equations of motion are

$$\begin{aligned}\ddot{x} - \omega_{\rm c}\,\dot{y} &= \frac{e}{m}\,E_0 \\ \ddot{y} + \omega_{\rm c}\,\dot{x} &= 0 \\ \ddot{z} &= 0\ .\end{aligned}$$

The second equation can be integrated once to yield $\dot{y} = \omega_{\rm c}(x_0 - x)$, where x_0 is a constant. Substituting this into the first equation gives

$$\ddot{x} + \omega_{\rm c}^2\,x = \omega_{\rm c}^2\,x_0 + \frac{e}{m}\,E_0\ .$$

This is the equation of a constantly forced harmonic oscillator. We can therefore write the general solution as

$$\boxed{x(t) = x_0 + \frac{eE_0}{m\omega_{\rm c}^2} + A\,\cos\big(\omega_{\rm c}t + \delta\big)}$$

$$\boxed{y(t) = y_0 - \frac{eE_0}{m\omega_{\rm c}}\,t - A\,\sin\big(\omega_{\rm c}t + \delta\big)}$$

$$\boxed{z(t) = z_0 + \dot{z}_0\,t}$$

Note that there are six constants, $\big\{A,\ \delta,\ x_0,\ y_0,\ z_0,\ \dot{z}_0\big\}$, are are required for the general solution of three coupled second order ODEs.

[4] An $N = 1$ dynamical system obeys the equation

$$\frac{du}{dt} = ru + 2bu^2 - u^3\ ,$$

where r is a control parameter, and where $b > 0$ is a constant.

(a) Find and classify all bifurcations for this system.
`[7 points]`

(b) Sketch the fixed points u^* *versus* r.
`[6 points]`

Now let $b = 3$. At time $t = 0$, the initial value of u is $u(0) = 1$. The control parameter r is then increased *very slowly* from $r = -20$ to $r = +20$, and then decreased very slowly back down to $r = -20$.

(c) What is the value of u when $r = -5$ on the *increasing* part of the cycle?
`[3 points]`

(d) What is the value of u when $r = +16$ on the *increasing* part of the cycle?
[3 points]

(e) What is the value of u when $r = +16$ on the *decreasing* part of the cycle?
[3 points]

(f) What is the value of u when $r = -5$ on the *decreasing* part of the cycle?
[3 points]

Solution

(a) Setting $\dot{u} = 0$ we obtain

$$(u^2 - 2bu - r)\, u = 0 \ .$$

The roots are

$$u = 0 \quad , \quad u = b \pm \sqrt{b^2 + r} \ .$$

The roots at $u = u_\pm = b \pm \sqrt{b^2 + r}$ are only present when $r > -b^2$. At $r = -b^2$ there is a *saddle-node bifurcation*. The fixed point $u = u_-$ crosses the fixed point at $u = 0$ at $r = 0$, at which the two fixed points exchange stability. This corresponds to a *transcritical bifurcation*. In Fig. 17.8 we plot $\dot{u}/b^3$ *versus* u/b for several representative values of r/b^2. Note that, defining $\tilde{u} = u/b$, $\tilde{r} = r/b^2$, and $\tilde{t} = b^2 t$ that our $N = 1$ system may be written

$$\frac{d\tilde{u}}{d\tilde{t}} = \left(\tilde{r} + 2\tilde{u} - \tilde{u}^2\right) \tilde{u} \ ,$$

which shows that it is only the dimensionless combination $\tilde{r} = r/b^2$ which enters into the location and classification of the bifurcations.

(b) A sketch of the fixed points u^* *versus* r is shown in Fig. 17.9. Note the two bifurcations at $r = -b^2$ (saddle-node) and $r = 0$ (transcritical).

(c) For $r = -20 < -b^2 = -9$, the initial condition $u(0) = 1$ flows directly toward the stable fixed point at $u = 0$. Since the approach to the FP is asymptotic, u remains slightly positive even after a long time. When $r = -5$, the FP at $u = 0$ is still stable. *Answer:* $\underline{u = 0}$.

(d) As soon as r becomes positive, the FP at $u^* = 0$ becomes unstable, and u flows to the upper branch u_+. When $r = 16$, we have $u = 3 + \sqrt{3^2 + 16} = 8$. *Answer:* $\underline{u = 8}$.

(e) Coming back down from larger r, the upper FP branch remains stable, thus, $u = 8$ at $r = 16$ on the way down as well. *Answer:* $\underline{u = 8}$.

(f) Now when r first becomes negative on the way down, the upper branch u_+ remains stable. Indeed it remains stable all the way down to $r = -b^2$, the location of the saddle-node bifurcation, at which point the solution $u = u_+$ simply vanishes and the flow is toward $u = 0$ again. Thus, for $r = -5$ on the way down, the system remains on the upper branch, in which case $u = 3 + \sqrt{3^2 - 5} = 5$. *Answer:* $\underline{u = 5}$.

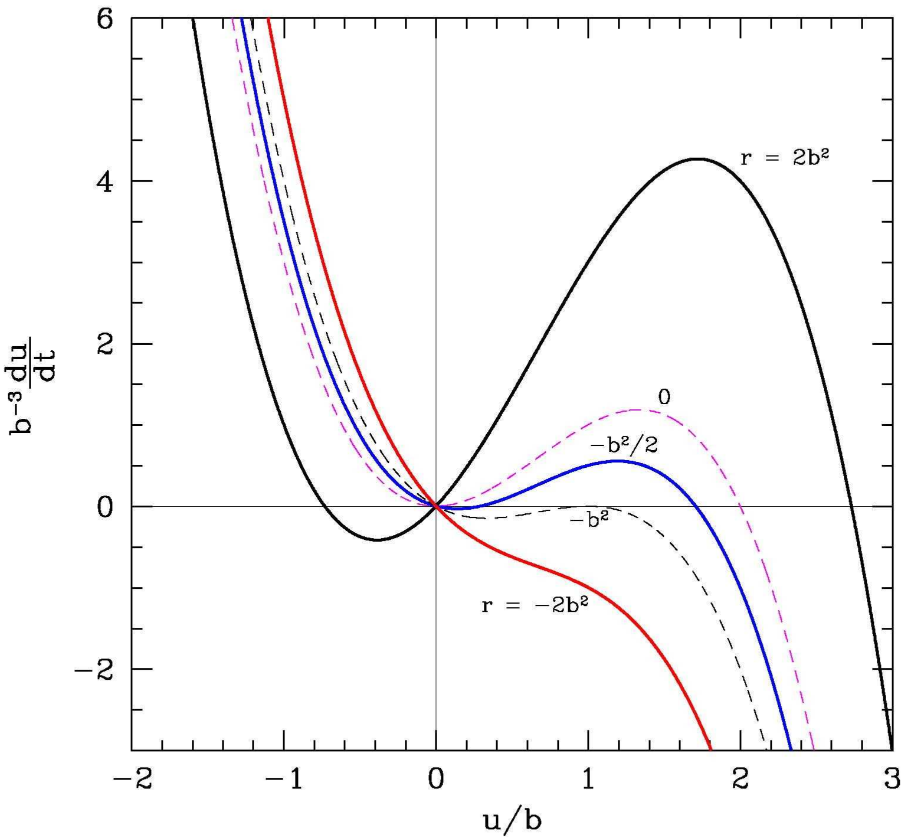


Figure 17.8: Plot of dimensionless 'velocity' $\dot{u}/b^3$ *versus* dimensionless 'coordinate' u/b for several values of the dimensionless control parameter $\tilde{r} = r/b^2$.

17.4 F07 Physics 110A Midterm #1

[**1**] A particle of mass m moves in the one-dimensional potential

$$U(x) = \frac{U_0}{a^4}\left(x^2 - a^2\right)^2 . \tag{17.68}$$

(a) Sketch $U(x)$. Identify the location(s) of any local minima and/or maxima, and be sure that your sketch shows the proper behavior as $x \to \pm\infty$.
`[15 points]`

Solution : Clearly the minima lie at $x = \pm a$ and there is a local maximum at $x = 0$.

(b) Sketch a representative set of phase curves. Be sure to sketch any separatrices which exist, and identify their energies. Also sketch all the phase curves for motions with total energy $E = \frac{1}{2}U_0$. Do the same for $E = 2\,U_0$.
`[15 points]`

Solution : See Fig. 17.10 for the phase curves. Clearly $U(\pm a) = 0$ is the minimum of the

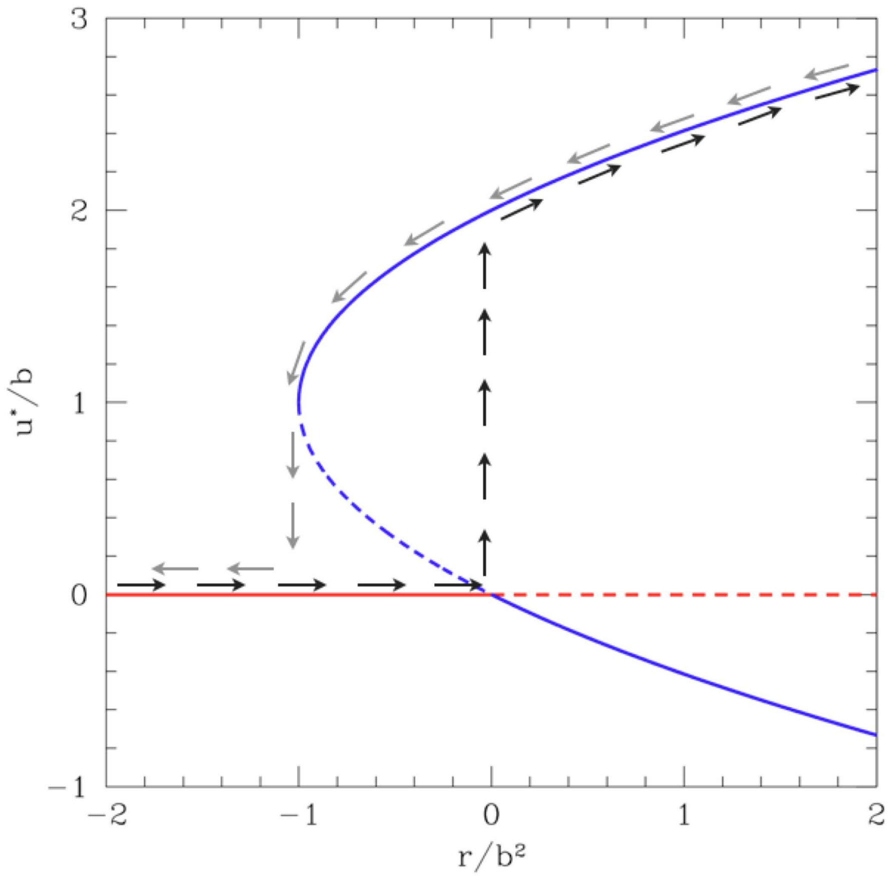


Figure 17.9: Fixed points and their stability *versus* control parameter for the $N = 1$ system $\dot{u} = ru + 2bu^2 - u^3$. Solid lines indicate stable fixed points; dashed lines indicate unstable fixed points. There is a saddle-node bifurcation at $r = -b^2$ and a transcritical bifurcation at $r = 0$. The hysteresis loop in the upper half plane $u > 0$ is shown. For $u < 0$ variations of the control parameter r are reversible and there is no hysteresis.

potential, and $U(0) = U_0$ is the local maximum and the energy of the separatrix. Thus, $E = \frac{1}{2} U_0$ cuts through the potential in both wells, and the phase curves at this energy form two disjoint sets. For $E < U_0$ there are four turning points, at

$$x_{1,<} = -a\sqrt{1 + \sqrt{\frac{E}{U_0}}} \quad , \quad x_{1,>} = -a\sqrt{1 - \sqrt{\frac{E}{U_0}}}$$

and

$$x_{2,<} = a\sqrt{1 - \sqrt{\frac{E}{U_0}}} \quad , \quad x_{2,>} = a\sqrt{1 + \sqrt{\frac{E}{U_0}}}$$

For $E = 2U_0$, the energy is above that of the separatrix, and there are only two turning points, $x_{1,<}$ and $x_{2,>}$. The phase curve is then connected.

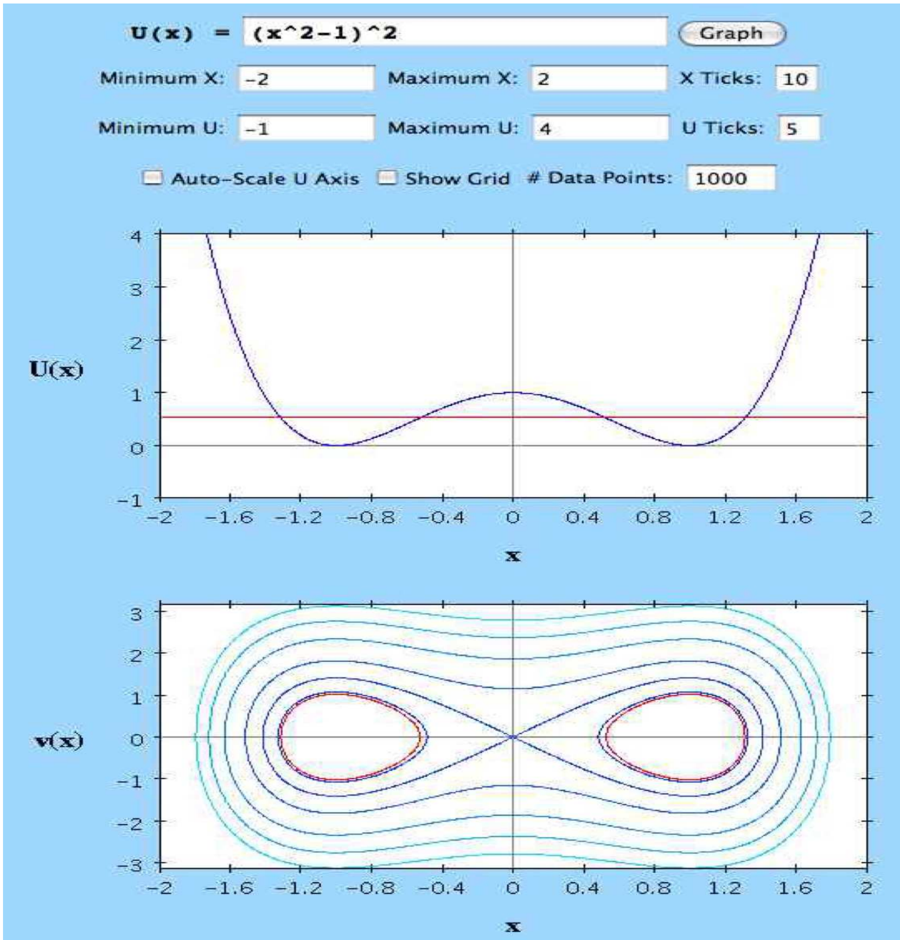


Figure 17.10: Sketch of the double well potential $U(x) = (U_0/a^4)(x^2 - a^2)^2$, here with distances in units of a, and associated phase curves. The separatrix is the phase curve which runs through the origin. Shown in red is the phase curve for $U = \frac{1}{2}\,U_0$, consisting of two deformed ellipses. For $U = 2\,U_0$, the phase curve is connected, lying outside the separatrix.

(c) The phase space dynamics are written as $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, where $\boldsymbol{\varphi} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$. Find the upper and lower components of the vector field $\boldsymbol{V}$.
`[10 points]`

Solution :

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{1}{m}\,U'(x) \end{pmatrix} = \begin{pmatrix} \dot{x} \\ -\frac{4U_0}{a^2}\,x\,(x^2 - a^2) \end{pmatrix} \ . \tag{17.69}$$

(d) Derive and expression for the period T of the motion when the system exhibits small oscillations about a potential minimum.
`[10 points]`

Solution : Set $x = \pm a + \eta$ and Taylor expand:

$$U(\pm a + \eta) = \frac{4U_0}{a^2}\,\eta^2 + \mathcal{O}(\eta^3) \ . \tag{17.70}$$

Equating this with $\frac{1}{2}k\,\eta^2$, we have the effective spring constant $k = 8U_0/a^2$, and the small oscillation frequency

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,U_0}{ma^2}} \ . \tag{17.71}$$

The period is $2\pi/\omega_0$.

[**2**] An *R-L-C* circuit is shown in fig. 17.11. The resistive element is a light bulb. The inductance is $L = 400\,\mu$H; the capacitance is $C = 1\,\mu$F; the resistance is $R = 32\,\Omega$. The voltage $V(t)$ oscillates sinusoidally, with $V(t) = V_0 \cos(\omega t)$, where $V_0 = 4\,$V. In this problem, you may neglect all transients; we are interested in the late time, steady state operation of this circuit. Recall the relevant MKS units:

$$1\,\Omega = 1\,\mathrm{V\cdot s\,/\,C} \quad , \quad 1\,\mathrm{F} = 1\,\mathrm{C\,/\,V} \quad , \quad 1\,\mathrm{H} = 1\,\mathrm{V\cdot s^2/\,C} \ .$$

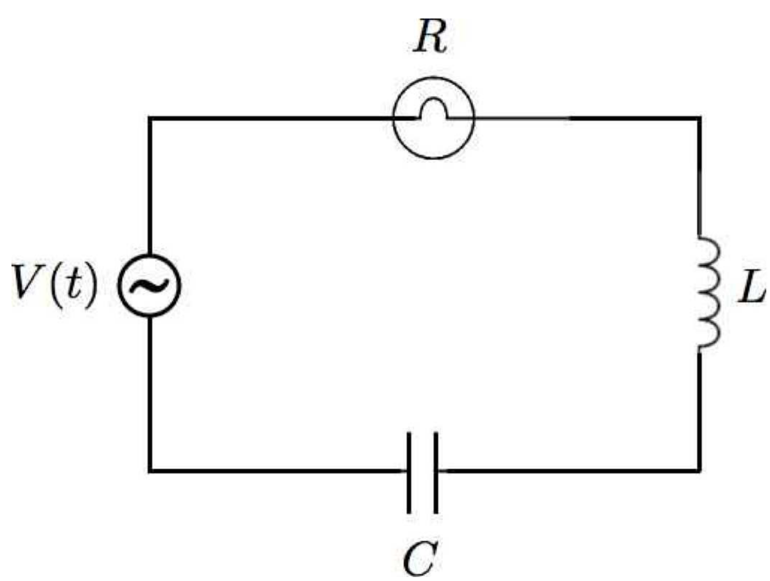


Figure 17.11: An *R-L-C* circuit in which the resistive element is a light bulb.

(a) Is this circuit underdamped or overdamped?
`[10 points]`

Solution : We have

$$\omega_0 = (LC)^{-1/2} = 5 \times 10^4\,\mathrm{s}^{-1} \quad , \quad \beta = \frac{R}{2L} = 4 \times 10^4\,\mathrm{s}^{-1} \ .$$

Thus, $\omega_0^2 > \beta^2$ and the circuit is *underdamped*.

(b) Suppose the bulb will only emit light when the average power dissipated by the bulb is greater than a threshold $P_{\text{th}} = \frac{2}{9}\,W$. For fixed $V_0 = 4\,\text{V}$, find the frequency range for ω over which the bulb emits light. Recall that the instantaneous power dissipated by a resistor is $P_R(t) = I^2(t)R$. (Average this over a cycle to get the average power dissipated.)
`[20 points]`

Solution : The charge on the capacitor plate obeys the ODE

$$L\,\ddot{Q} + R\,\dot{Q} + \frac{Q}{C} = V(t) \ .$$

The solution is

$$Q(t) = Q_{\text{hom}}(t) + A(\omega)\,\frac{V_0}{L}\,\cos\big(\omega t - \delta(\omega)\big) \ ,$$

with

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\bigg(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\bigg) \ .$$

Thus, ignoring the transients, the power dissipated by the bulb is

$$\begin{aligned} P_R(t) &= \dot{Q}^2(t)\,R \\ &= \omega^2 A^2(\omega)\,\frac{V_0^2 R}{L^2}\,\sin^2\!\big(\omega t - \delta(\omega)\big) \ . \end{aligned}$$

Averaging over a period, we have $\langle\, \sin^2(\omega t - \delta)\,\rangle = \frac{1}{2}$, so

$$\langle\, P_R \,\rangle = \omega^2 A^2(\omega)\,\frac{V_0^2 R}{2L^2} = \frac{V_0^2}{2\,R} \cdot \frac{4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \ .$$

Now $V_0^2/2R = \frac{1}{4}\,\text{W}$. So $P_{\text{th}} = \alpha V_0^2/2R$, with $\alpha = \frac{8}{9}$. We then set $\langle P_R \rangle = P_{\text{th}}$, whence

$$(1-\alpha)\cdot 4\beta^2\omega^2 = \alpha\,(\omega_0^2 - \omega^2)^2 \ .$$

The solutions are

$$\omega = \pm\sqrt{\frac{1-\alpha}{\alpha}}\,\beta + \sqrt{\bigg(\frac{1-\alpha}{\alpha}\bigg)\beta^2 + \omega_0^2} = \big(3\sqrt{3} \pm \sqrt{2}\big) \times 1000\,\text{s}^{-1} \ .$$

(c) Compare the expressions for the instantaneous power dissipated by the voltage source, $P_V(t)$, and the power dissipated by the resistor $P_R(t) = I^2(t)R$. If $P_V(t) \neq P_R(t)$, where does the power extra power go or come from? What can you say about the averages of P_V and $P_R(t)$ over a cycle? Explain your answer.
`[20 points]`

Solution : The instantaneous power dissipated by the voltage source is

$$\begin{aligned} P_V(t) = V(t)\,I(t) &= -\omega\,A\,\frac{V_0}{L}\,\sin(\omega t - \delta)\,\cos(\omega t) \\ &= \omega\,A\,\frac{V_0}{2L}\,\Big(\sin\delta - \sin(2\omega t - \delta)\Big) \ . \end{aligned}$$

As we have seen, the power dissipated by the bulb is

$$P_R(t) = \omega^2 A^2\, \frac{V_0^2 R}{L^2} \sin^2(\omega t - \delta) \ .$$

These two quantities are not identical, but they do have identical time averages over one cycle:

$$\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle = \frac{V_0^2}{2R} \cdot 4\beta^2\, \omega^2\, A^2(\omega) \ .$$

Energy conservation means

$$P_V(t) = P_R(t) + \dot{E}(t) \ ,$$

where

$$E(t) = \frac{L\dot{Q}^2}{2} + \frac{Q^2}{2C}$$

is the energy in the inductor and capacitor. Since $Q(t)$ is periodic, the average of $\dot{E}$ over a cycle must vanish, which guarantees $\langle\, P_V(t)\,\rangle = \langle\, P_R(t)\,\rangle$.

What was not asked:

(d) What is the maximum charge $Q_{\rm max}$ on the capacitor plate if $\omega = 3000\,{\rm s}^{-1}$?
`[10 points]`

Solution : Kirchoff's law gives for this circuit the equation

$$\ddot{Q} + 2\beta\,\dot{Q} + \omega_0^2\, Q = \frac{V_0}{L}\, \cos(\omega t) \ ,$$

with the solution

$$Q(t) = Q_{\rm hom}(t) + A(\omega)\, \frac{V_0}{L}\, \cos\big(\omega t - \delta(\omega)\big) \ ,$$

where $Q_{\rm hom}(t)$ is the homogeneous solution, *i.e.* the transient which we ignore, and

$$A(\omega) = \Big[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2\Big]^{-1/2} \quad , \quad \delta(\omega) = \tan^{-1}\!\bigg(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\bigg) \ .$$

Then

$$Q_{\rm max} = A(\omega)\, \frac{V_0}{L} \ .$$

Plugging in $\omega = 3000\,{\rm s}^{-1}$, we have

$$A(\omega) = \big[(5^2 - 4^2)^2 + 4 \cdot 4^2 \cdot 3^2\big]^{-1/2} \times 10^{-3}\,{\rm s}^2 = \frac{1}{8\sqrt{13}} \times 10^{-3}\,{\rm s}^2 \ .$$

Since $V_0/L = 10^4\,{\rm C/s}^2$, we have

$$Q_{\rm max} = \frac{5}{4\sqrt{13}}\, {\rm Coul} \ .$$

17.5 F07 Physics 110A Midterm #2

[1] A point mass m slides frictionlessly, under the influence of gravity, along a massive ring of radius a and mass M. The ring is affixed by horizontal springs to two fixed vertical surfaces, as depicted in fig. 17.12. All motion is within the plane of the figure.

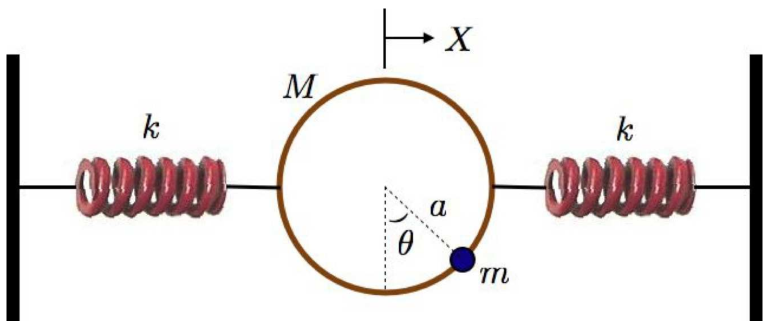


Figure 17.12: A point mass m slides frictionlessly along a massive ring of radius a and mass M, which is affixed by horizontal springs to two fixed vertical surfaces.

(a) Choose as generalized coordinates the horizontal displacement X of the center of the ring with respect to equilibrium, and the angle θ a radius to the mass m makes with respect to the vertical (see fig. 17.12). You may assume that at $X = 0$ the springs are both unstretched. Find the Lagrangian $L(X, \theta, \dot{X}, \dot{\theta}, t)$.
[15 points]

The coordinates of the mass point are

$$x = X + a\sin\theta \quad , \quad y = -a\cos\theta \ .$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{X} + a\cos\theta\,\dot{\theta}\big)^2 + \tfrac{1}{2}ma^2\sin^2\!\theta\,\dot{\theta}^2 \\ &= \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X}\,\dot{\theta} \ . \end{aligned}$$

The potential energy is

$$U = kX^2 - mga\cos\theta \ .$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma\cos\theta\,\dot{X} - kX^2 + mga\cos\theta \ .$$

(b) Find the generalized momenta p_X and p_θ, and the generalized forces F_X and F_θ
[10 points]

We have

$$p_X = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + ma\cos\theta\,\dot{\theta} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} + ma\cos\theta\,\dot{X} \ .$$

For the forces,

$$F_X = \frac{\partial L}{\partial X} = -2kX \quad , \quad F_\theta = \frac{\partial L}{\partial \theta} = -ma \sin\theta \, \dot{X} \, \dot{\theta} - mga \sin\theta \ .$$

(c) Derive the equations of motion.
`[15 points]`

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial L}{\partial q_\sigma} \ ,$$

for each generalized coordinate q_σ. For X we have

$$(M+m)\ddot{X} + ma \cos\theta \, \ddot{\theta} - ma \sin\theta \, \dot{\theta}^2 = -2kX \ .$$

For θ,

$$ma^2 \, \ddot{\theta} + ma \cos\theta \ddot{X} = -mga \sin\theta \ .$$

(d) Find expressions for all conserved quantities.
`[10 points]`

Horizontal and vertical translational symmetries are broken by the springs and by gravity, respectively. The remaining symmetry is that of time translation. From $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$, we have that $H = \sum_\sigma p_\sigma \, \dot{q}_\sigma - L$ is conserved. For this problem, the kinetic energy is a homogeneous function of degree 2 in the generalized velocities, and the potential is velocity-independent. Thus,

$$H = T + U = \tfrac{1}{2}(M+m)\dot{X}^2 + \tfrac{1}{2}ma^2\dot{\theta}^2 + ma \cos\theta \, \dot{X} \, \dot{\theta} + kX^2 - mga \cos\theta \ .$$

[2] A point particle of mass m moves in three dimensions in a helical potential

$$U(\rho,\phi,z) = U_0\,\rho\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) .$$

We call b the pitch of the helix.

(a) Write down the Lagrangian, choosing (ρ,ϕ,z) as generalized coordinates.
`[10 points]`

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) - U_0\,\rho\,\cos\left(\phi - \frac{2\pi z}{b}\right)$$

(b) Find the equations of motion.
`[20 points]`

Clearly

$$p_\rho = m\dot\rho \quad , \quad p_\phi = m\rho^2\,\dot\phi \quad , \quad p_z = m\dot z \ ,$$

and

$$F_\rho = m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_\phi = U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \quad , \quad F_z = -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) .$$

Thus, the equation of motion are

$$\begin{aligned} m\ddot\rho &= m\rho\,\dot\phi^2 - U_0\,\cos\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\rho^2\,\ddot\phi + 2m\rho\,\dot\rho\,\dot\phi &= U_0\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) \\ m\ddot z &= -\frac{2\pi U_0}{b}\,\rho\,\sin\!\left(\phi - \frac{2\pi z}{b}\right) . \end{aligned}$$

(c) Show that there exists a continuous one-parameter family of coordinate transformations which leaves L invariant. Find the associated conserved quantity, Λ. Is anything else conserved?
`[20 points]`

Due to the helical symmetry, we have that

$$\phi \to \phi + \zeta \quad , \quad z \to z + \frac{b}{2\pi}\,\zeta$$

is such a continuous one-parameter family of coordinate transformations. Since it leaves the combination $\phi - \frac{2\pi z}{b}$ unchanged, we have that $\frac{dL}{d\zeta} = 0$, and

$$\begin{aligned}
\Lambda &= p_\rho \, \frac{\partial\rho}{\partial\zeta}\bigg|_{\zeta=0} + p_\phi \, \frac{\partial\phi}{\partial\zeta}\bigg|_{\zeta=0} + p_z \, \frac{\partial z}{\partial\zeta}\bigg|_{\zeta=0} \\
&= p_\phi + \frac{b}{2\pi} \, p_z \\
&= m\rho^2 \, \dot\phi + \frac{mb}{2\pi} \, \dot z
\end{aligned}$$

is the conserved Noether 'charge'. The other conserved quantity is the Hamiltonian,

$$H = \tfrac{1}{2} m\big(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2\big) + U_0 \, \rho \, \cos\left(\phi - \frac{2\pi z}{b}\right) .$$

Note that $H = T + U$, because T is homogeneous of degree 2 and U is homogeneous of degree 0 in the generalized velocities.

17.6 F07 Physics 110A Final Exam

[**1**] Two masses and two springs are configured linearly and externally driven to rotate with angular velocity ω about a fixed point on a horizontal surface, as shown in fig. 17.13. The unstretched length of each spring is a.

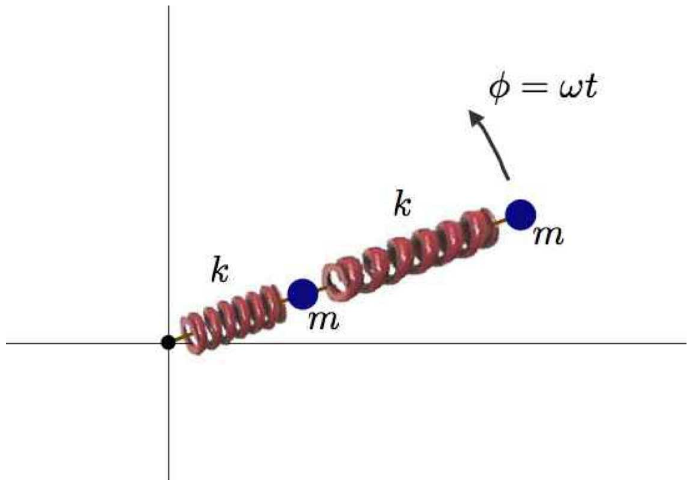


Figure 17.13: Two masses and two springs rotate with angular velocity ω.

(a) Choose as generalized coordinates the radial distances $r_{1,2}$ from the origin. Find the Lagrangian $L(r_1, r_2, \dot{r}_1, \dot{r}_2, t)$.
[5 points]

The Lagrangian is

$$L = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2 + \omega^2\, r_1^2 + \omega^2\, r_2^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.72)$$

(b) Derive expressions for all conserved quantities.
[5 points]

The Hamiltonian is conserved. Since the kinetic energy is not homogeneous of degree 2 in the generalized velocities, $H \neq T + U$. Rather,

$$H = \sum_\sigma p_\sigma\, \dot{q}_\sigma - L \qquad (17.73)$$

$$= \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2\big) - \tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.74)$$

We could define an effective potential

$$U_{\text{eff}}(r_1, r_2) = -\tfrac{1}{2}m\omega^2\big(r_1^2 + r_2^2\big) + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2 \ . \qquad (17.75)$$

Note the first term, which comes from the kinetic energy, has an interpretation of a fictitious potential which generates a *centrifugal* force.

(c) What equations determine the equilibrium radii r_1^0 and r_2^0? (You do not have to solve these equations.)
[5 points]

The equations of equilibrium are $F_\sigma = 0$. Thus,

$$0 = F_1 = \frac{\partial L}{\partial r_1} = m\omega^2 r_1 - k\,(r_1 - a) + k\,(r_2 - r_1 - a) \tag{17.76}$$

$$0 = F_2 = \frac{\partial L}{\partial r_2} = m\omega^2 r_2 - k\,(r_2 - r_1 - a)\ . \tag{17.77}$$

(d) Suppose now that the system is not externally driven, and that the angular coordinate ϕ is a dynamical variable like r_1 and r_2. Find the Lagrangian $L(r_1, r_2, \phi, \dot{r}_1, \dot{r}_2, \dot{\phi}, t)$.
[5 points]

Now we have

$$L = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2 + r_1^2\,\dot{\phi}^2 + r_2^2\,\dot{\phi}^2\big) - \tfrac{1}{2}k\,(r_1 - a)^2 - \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{17.78}$$

(e) For the system described in part (d), find expressions for all conserved quantities.
[5 points]

There are two conserved quantities. One is p_ϕ, owing to the fact the ϕ is cyclic in the Lagrangian. *I.e.* $\phi \to \phi + \zeta$ is a continuous one-parameter coordinate transformation which leaves L invariant. We have

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\big(r_1^2 + r_2^2\big)\,\dot{\phi}\ . \tag{17.79}$$

The second conserved quantity is the Hamiltonian, which is now $H = T + U$, since T is homogeneous of degree 2 in the generalized velocities. Using conservation of momentum, we can write

$$H = \tfrac{1}{2}m\big(\dot{r}_1^2 + \dot{r}_2^2\big) + \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ . \tag{17.80}$$

Once again, we can define an effective potential,

$$U_{\rm eff}(r_1, r_2) = \frac{p_\phi^2}{2m(r_1^2 + r_2^2)} + \tfrac{1}{2}k\,(r_1 - a)^2 + \tfrac{1}{2}k\,(r_2 - r_1 - a)^2\ , \tag{17.81}$$

which is different than the effective potential from part (b). However in both this case and in part (b), we have that the radial coordinates obey the equations of motion

$$m\ddot{r}_j = -\frac{\partial U_{\rm eff}}{\partial r_j}\ , \tag{17.82}$$

for $j = 1, 2$. Note that this equation of motion follows directly from $\dot{H} = 0$.

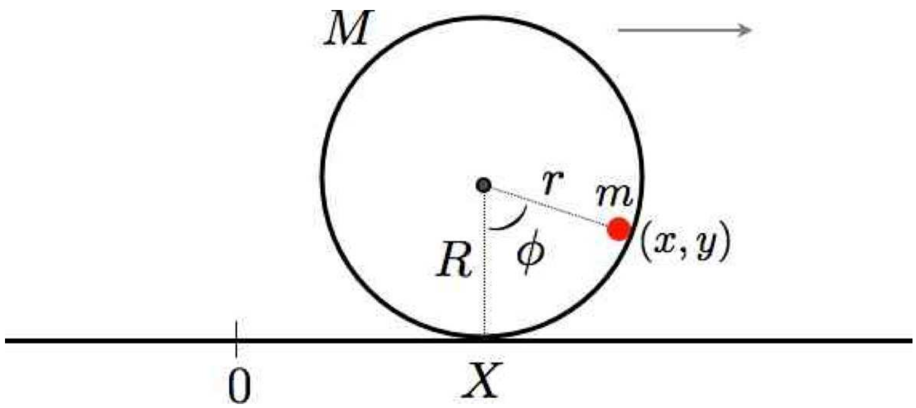


Figure 17.14: A mass point m rolls inside a hoop of mass M and radius R which rolls without slipping on a horizontal surface.

[2] A point mass m slides inside a hoop of radius R and mass M, which itself rolls without slipping on a horizontal surface, as depicted in fig. 17.14.

Choose as general coordinates (X, ϕ, r), where X is the horizontal location of the center of the hoop, ϕ is the angle the mass m makes with respect to the vertical ($\phi = 0$ at the bottom of the hoop), and r is the distance of the mass m from the center of the hoop. Since the mass m slides inside the hoop, there is a constraint:

$$G(X, \phi, r) = r - R = 0 \ .$$

Nota bene: The kinetic energy of the moving hoop, including translational and rotational components (but not including the mass m), is $T_{\rm hoop} = M\dot{X}^2$ (*i.e.* twice the translational contribution alone).

(a) Find the Lagrangian $L(X, \phi, r, \dot{X}, \dot{\phi}, \dot{r}, t)$.
[5 points]

The Cartesian coordinates and velocities of the mass m are

$$x = X + r\sin\phi \qquad \dot{x} = \dot{X} + \dot{r}\sin\phi + r\dot{\phi}\cos\phi \tag{17.83}$$

$$y = R - r\cos\phi \qquad \dot{y} = -\dot{r}\cos\phi + r\dot{\phi}\sin\phi \tag{17.84}$$

The Lagrangian is then

$$L = \overbrace{(M + \tfrac{1}{2}m)\dot{X}^2 + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + m\dot{X}(\dot{r}\sin\phi + r\dot{\phi}\cos\phi)}^{T} \; - \; \overbrace{mg(R - r\cos\phi)}^{U} \tag{17.85}$$

Note that we are not allowed to substitute $r = R$ and hence $\dot{r} = 0$ in the Lagrangian *prior* to obtaining the equations of motion. Only *after* the generalized momenta and forces are computed are we allowed to do so.

(b) Find *all* the generalized momenta p_σ, the generalized forces F_σ, and the forces of constraint Q_σ.
[10 points]

The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} + m\dot{X}\,\sin\phi \tag{17.86}$$

$$p_X = \frac{\partial L}{\partial \dot{X}} = (2M+m)\dot{X} + m\dot{r}\,\sin\phi + mr\dot{\phi}\,\cos\phi \tag{17.87}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} + mr\dot{X}\,\cos\phi \tag{17.88}$$

The generalized forces and the forces of constraint are

$$F_r = \frac{\partial L}{\partial r} = mr\dot{\phi}^2 + m\dot{X}\dot{\phi}\,\cos\phi + mg\cos\phi \qquad Q_r = \lambda\,\frac{\partial G}{\partial r} = \lambda \tag{17.89}$$

$$F_X = \frac{\partial L}{\partial X} = 0 \qquad Q_X = \lambda\,\frac{\partial G}{\partial X} = 0 \tag{17.90}$$

$$F_\phi = \frac{\partial L}{\partial \phi} = m\dot{X}\dot{r}\,\cos\phi - m\dot{X}\dot{\phi}\,\sin\phi - mgr\sin\phi \qquad Q_\phi = \lambda\,\frac{\partial G}{\partial \phi} = 0\ . \tag{17.91}$$

The equations of motion are

$$\dot{p}_\sigma = F_\sigma + Q_\sigma\ . \tag{17.92}$$

At this point, we can legitimately invoke the constraint $r = R$ and set $\dot{r} = 0$ in all the p_σ and F_σ.

`(c)` Derive expressions for all conserved quantities.
`[5 points]`

There are two conserved quantities, which each derive from continuous invariances of the Lagrangian *which respect the constraint*. The first is the total momentum p_X:

$$F_X = 0 \quad \Longrightarrow \quad P \equiv p_X = \text{constant}\ . \tag{17.93}$$

The second conserved quantity is the Hamiltonian, which in this problem turns out to be the total energy $E = T + U$. Incidentally, we can use conservation of P to write the energy in terms of the variable ϕ alone. From

$$\dot{X} = \frac{P}{2M+m} - \frac{mR\,\cos\phi}{2M+m}\,\dot{\phi}\ , \tag{17.94}$$

we obtain

$$\begin{aligned} E &= \tfrac{1}{2}(2M+m)\dot{X}^2 + \tfrac{1}{2}mR^2\dot{\phi}^2 + mR\dot{X}\dot{\phi}\,\cos\phi + mgR(1-\cos\phi) \\ &= \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2}mR^2\bigg(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\bigg)\,\dot{\phi}^2 + mgR(1-\cos\phi)\ , \end{aligned} \tag{17.95}$$

where we've defined the dimensionless ratio $\alpha \equiv m/2M$. It is convenient to define the quantity

$$\Omega^2 \equiv \bigg(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\bigg)\,\dot{\phi}^2 + 2\omega_0^2(1-\cos\phi)\ , \tag{17.96}$$

with $\omega_0 \equiv \sqrt{g/R}$. Clearly Ω^2 is conserved, as it is linearly related to the energy E:

$$E = \frac{\alpha P^2}{2m(1+\alpha)} + \tfrac{1}{2} m R^2 \Omega^2 \ . \tag{17.97}$$

(d) Derive a differential equation of motion involving the coordinate $\phi(t)$ alone. *I.e.* your equation should not involve r, X, or the Lagrange multiplier λ.
[5 points]

From conservation of energy,

$$\frac{d(\Omega^2)}{dt} = 0 \quad \Longrightarrow \quad \left(\frac{1+\alpha\,\sin^2\!\phi}{1+\alpha}\right)\ddot{\phi} + \left(\frac{\alpha\,\sin\phi\,\cos\phi}{1+\alpha}\right)\dot{\phi}^2 + \omega_0^2\,\sin\phi = 0 \ , \tag{17.98}$$

again with $\alpha = m/2M$. Incidentally, one can use these results in eqns. 17.96 and 17.98 to eliminate $\dot{\phi}$ and $\ddot{\phi}$ in the expression for the constraint force, $Q_r = \lambda = \dot{p}_r - F_r$. One finds

$$\begin{aligned} \lambda &= -mR\,\frac{\dot{\phi}^2 + \omega_0^2\,\cos\phi}{1+\alpha\sin^2\!\phi} \\ &= -\frac{mR\,\omega_0^2}{(1+\alpha\sin^2\!\phi)^2}\left\{(1+\alpha)\left(\frac{\Omega^2}{\omega_0^2} - 4\,\sin^2(\tfrac{1}{2}\phi)\right) + (1+\alpha\sin^2\!\phi)\,\cos\phi\right\} \ . \end{aligned} \tag{17.99}$$

This last equation can be used to determine the angle of detachment, where λ vanishes and the mass m falls off the inside of the hoop. This is because the hoop can only supply a repulsive normal force to the mass m. This was worked out in detail in my lecture notes on constrained systems.

[3] Two objects of masses m_1 and m_2 move under the influence of a central potential $U = k\,\big|\boldsymbol{r}_1 - \boldsymbol{r}_2\big|^{1/4}$.

(a) Sketch the effective potential $U_{\rm eff}(r)$ and the phase curves for the radial motion. Identify for which energies the motion is bounded.
[5 points]

The effective potential is

$$U_{\rm eff}(r) = \frac{\ell^2}{2\mu r^2} + k r^n \tag{17.100}$$

with $n = \frac{1}{4}$. In sketching the effective potential, I have rendered it in dimensionless form,

$$U_{\rm eff}(r) = E_0\,\mathcal{U}_{\rm eff}(r/r_0) \ , \tag{17.101}$$

where $r_0 = (\ell^2/nk\mu)^{(n+2)^{-1}}$ and $E_0 = \big(\frac{1}{2} + \frac{1}{n}\big)\ell^2/\mu r_0^2$, which are obtained from the results of part (b). One then finds

$$\mathcal{U}_{\rm eff}(x) = \frac{n\,x^{-2} + 2\,x^n}{n+2} \ . \tag{17.102}$$

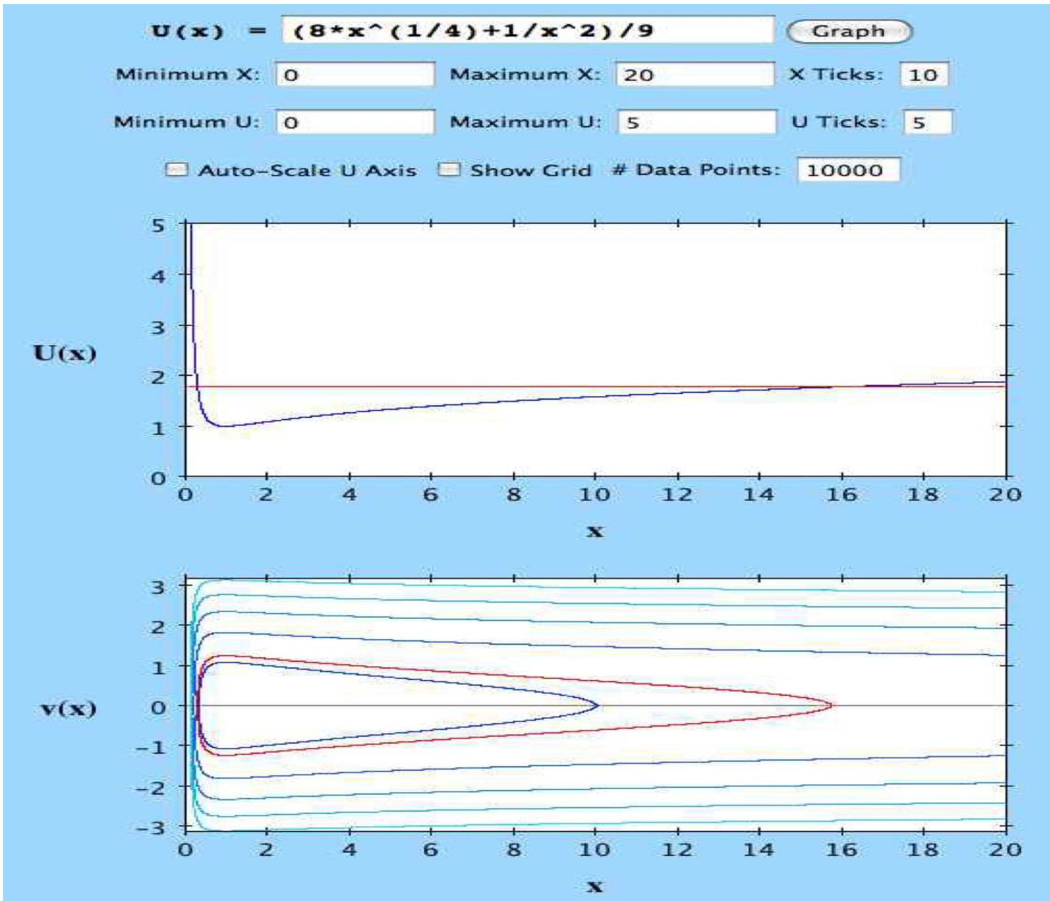


Figure 17.15: The effective $U_{\rm eff}(r) = E_0\, \mathcal{U}_{\rm eff}(r/r_0)$, where r_0 and E_0 are the radius and energy of the circular orbit.

Although it is not obvious from the detailed sketch in fig. 17.15, the effective potential does diverge, albeit slowly, for $r \to \infty$. Clearly it also diverges for $r \to 0$. Thus, the relative coordinate motion is bounded for all energies; the allowed energies are $E \geq E_0$.

(b) What is the radius r_0 of the circular orbit? Is it stable or unstable? Why?
[5 points]

For the general power law potential $U(r) = kr^n$, with $nk > 0$ (attractive force), setting $U'_{\rm eff}(r_0) = 0$ yields

$$-\frac{\ell^2}{\mu r_0^3} + nkr_0^{n-1} = 0 \ . \tag{17.103}$$

Thus,

$$r_0 = \left(\frac{\ell^2}{nk\mu}\right)^{\frac{1}{n+2}} = \left(\frac{4\ell^2}{k\mu}\right)^{\frac{4}{9}} \ . \tag{17.104}$$

The orbit $r(t) = r_0$ is stable because the effective potential has a local minimum at $r = r_0$, *i.e.* $U''_{\rm eff}(r_0) > 0$. This is obvious from inspection of the graph of $U_{\rm eff}(r)$ but can also be computed explicitly:

$$U''_{\rm eff}(r_0) = \frac{3\ell^2}{\mu r_0^4} + n(n-1)kr_0^n$$
$$= (n+2)\,\frac{\ell^2}{\mu r_0^4} \ . \tag{17.105}$$

Thus, provided $n > -2$ we have $U''_{\rm eff}(r_0) > 0$.

(c) For small perturbations about a circular orbit, the radial coordinate oscillates between two values. Suppose we compare two systems, with $\ell'/\ell = 2$, but $\mu' = \mu$ and $k' = k$. What is the ratio ω'/ω of their frequencies of small radial oscillations?
[5 points]

From the radial coordinate equation $\mu\ddot{r} = -U'_{\rm eff}(r)$, we expand $r = r_0 + \eta$ and find

$$\mu\ddot{\eta} = -U''_{\rm eff}(r_0)\,\eta + \mathcal{O}(\eta^2) \ . \tag{17.106}$$

The radial oscillation frequency is then

$$\omega = (n+2)^{1/2}\,\frac{\ell}{\mu r_0^2} = (n+2)^{1/2}\, n^{\frac{2}{n+2}}\, k^{\frac{2}{n+2}}\, \mu^{-\frac{n}{n+2}}\, \ell^{\frac{n-2}{n+2}} \ . \tag{17.107}$$

The ℓ dependence is what is key here. Clearly

$$\frac{\omega'}{\omega} = \left(\frac{\ell'}{\ell}\right)^{\frac{n-2}{n+2}} \ . \tag{17.108}$$

In our case, with $n = \frac{1}{4}$, we have $\omega \propto \ell^{-7/9}$ and thus

$$\frac{\omega'}{\omega} = 2^{-7/9} \ . \tag{17.109}$$

(d) Find the equation of the shape of the slightly perturbed circular orbit: $r(\phi) = r_0 + \eta(\phi)$. That is, find $\eta(\phi)$. Sketch the shape of the orbit.
[5 points]

We have that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$, with

$$\beta = \frac{\omega}{\dot{\phi}} = \frac{\mu r_0^2}{\ell}\cdot\omega = \sqrt{n+2} \ . \tag{17.110}$$

With $n = \frac{1}{4}$, we have $\beta = \frac{3}{2}$. Thus, the radial coordinate makes three oscillations for every two rotations. The situation is depicted in fig. 17.21.

(e) What value of n would result in a perturbed orbit shaped like that in fig. 17.22?
[5 points]

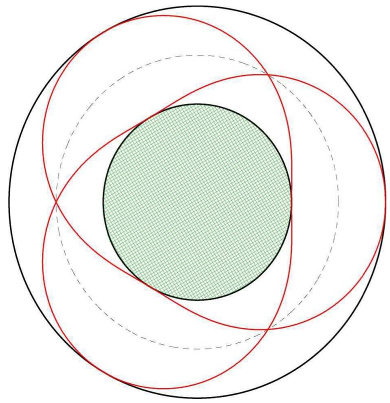


Figure 17.16: Radial oscillations with $\beta = \frac{3}{2}$.

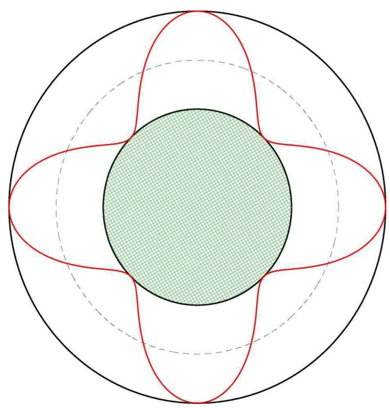


Figure 17.17: Closed precession in a central potential $U(r) = kr^n$.

Clearly $\beta = \sqrt{n+2} = 4$, in order that $\eta(\phi) = \eta_0 \cos(\beta\phi + \delta_0)$ executes four complete periods over the interval $\phi \in [0, 2\pi]$. This means $n = 14$.

[**4**] Two masses and three springs are arranged as shown in fig. 17.18. You may assume that in equilibrium the springs are all unstretched with length a. The masses and spring constants are simple multiples of fundamental values, *viz.*

$$m_1 = m \quad , \quad m_2 = 4m \quad , \quad k_1 = k \quad , \quad k_2 = 4k \quad , \quad k_3 = 28k \ . \tag{17.111}$$

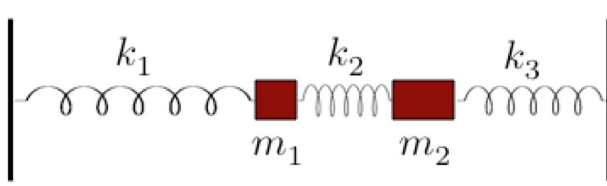


Figure 17.18: Coupled masses and springs.

`(a)` Find the Lagrangian.
`[5 points]`

Choosing displacements relative to equilibrium as our generalized coordinates, we have

$$T = \tfrac{1}{2} m\, \dot{\eta}_1^2 + 2m\, \dot{\eta}_2^2 \tag{17.112}$$

and

$$U = \tfrac{1}{2} k\, \eta_1^2 + 2k\, (\eta_2 - \eta_1)^2 + 14k\, \eta_2^2 \ . \tag{17.113}$$

Thus,

$$L = T - U = \tfrac{1}{2} m\, \dot{\eta}_1^2 + 2m\, \dot{\eta}_2^2 - \tfrac{1}{2} k\, \eta_1^2 - 2k\, (\eta_2 - \eta_1)^2 - 14k\, \eta_2^2 \ . \tag{17.114}$$

You are not required to find the equilibrium values of x_1 and x_2. However, suppose all the unstretched spring lengths are a and the total distance between the walls is L. Then, with $x_{1,2}$ being the location of the masses relative to the left wall, we have

$$U = \tfrac{1}{2} k_1\, (x_1 - a)^2 + \tfrac{1}{2} k_2\, (x_2 - x_1 - a)^2 + \tfrac{1}{2} k_3\, (L - x_2 - a)^2 \ . \tag{17.115}$$

Differentiating with respect to $x_{1,2}$ then yields

$$\frac{\partial U}{\partial x_1} = k_1\, (x_1 - a) - k_2\, (x_2 - x_1 - a) \tag{17.116}$$

$$\frac{\partial U}{\partial x_2} = k_2\, (x_2 - x_1 - a) - k_3\, (L - x_2 - a) \ . \tag{17.117}$$

Setting these both to zero, we obtain

$$(k_1 + k_2)\, x_1 - k_2\, x_2 = (k_1 - k_2)\, a \tag{17.118}$$

$$-k_2\, x_1 + (k_2 + k_3)\, x_2 = (k_2 - k_3)\, a + k_3 L \ . \tag{17.119}$$

Solving these two inhomogeneous coupled linear equations for $x_{1,2}$ then yields the equilibrium positions. However, we don't need to do this to solve the problem.

(b) Find the T and V matrices.
[5 points]

We have

$$\mathrm{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial \dot{\eta}_\sigma \partial \dot{\eta}_{\sigma'}} = \begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix} \tag{17.120}$$

and

$$\mathrm{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial \eta_\sigma \partial \eta_{\sigma'}} = \begin{pmatrix} 5k & -4k \\ -4k & 32k \end{pmatrix} \ . \tag{17.121}$$

(c) Find the eigenfrequencies ω_1 and ω_2.
[5 points]

We have

$$\begin{aligned} \mathrm{Q}(\omega) \equiv \omega^2\, \mathrm{T} - \mathrm{V} &= \begin{pmatrix} m\omega^2 - 5k & 4k \\ 4k & 4m\omega^2 - 32k \end{pmatrix} \\ &= k \begin{pmatrix} \lambda - 5 & 4 \\ 4 & 4\lambda - 32 \end{pmatrix} \ , \end{aligned} \tag{17.122}$$

where $\lambda = \omega^2/\omega_0^2$, with $\omega_0 = \sqrt{k/m}$. Setting $\det \mathrm{Q}(\omega) = 0$ then yields

$$\lambda^2 - 13\lambda + 36 = 0 \ , \tag{17.123}$$

the roots of which are $\lambda_- = 4$ and $\lambda_+ = 9$. Thus, the eigenfrequencies are

$$\omega_- = 2\,\omega_0 \qquad , \qquad \omega_+ = 3\,\omega_0 \ . \tag{17.124}$$

(d) Find the modal matrix $\mathrm{A}_{\sigma i}$.
[5 points]

To find the normal modes, we set

$$\begin{pmatrix} \lambda_\pm - 5 & 4 \\ 4 & 4\lambda_\pm - 32 \end{pmatrix} \begin{pmatrix} \psi_1^{(\pm)} \\ \psi_2^{(\pm)} \end{pmatrix} = 0 \ . \tag{17.125}$$

This yields two linearly dependent equations, from which we can determine only the ratios $\psi_2^{(\pm)}/\psi_1^{(\pm)}$. Plugging in for $\lambda_\pm$, we find

$$\begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = \mathcal{C}_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \qquad , \qquad \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ . \tag{17.126}$$

We then normalize by demanding $\psi_\sigma^{(i)}\,\mathrm{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)} = \delta_{ij}$. We can practically solve this by inspection:

$$20m\,|\mathcal{C}_-|^2 = 1 \qquad , \qquad 5m\,|\mathcal{C}_+|^2 = 1 \ . \tag{17.127}$$

We may now write the modal matrix,

$$\mathrm{A} = \frac{1}{\sqrt{5m}} \begin{pmatrix} 2 & 1 \\ \frac{1}{2} & -1 \end{pmatrix} \ . \tag{17.128}$$

(e) Write down the most general solution for the motion of the system.
[5 points]

The most general solution is

$$\begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} = B_- \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cos(2\omega_0 t + \varphi_-) + B_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(3\omega_0 t + \varphi_+) \ . \tag{17.129}$$

Note that there are four constants of integration: $B_\pm$ and $\varphi_\pm$.

17.7 W08 Physics 110B Midterm Exam

[1] Two identical semi-infinite lengths of string are joined at a point of mass m which moves vertically along a thin wire, as depicted in fig. 17.21. The mass moves with friction coefficient γ, *i.e.* its equation of motion is

$$m\ddot{z} + \gamma\dot{z} = F \; , \tag{17.130}$$

where z is the vertical displacement of the mass, and F is the force on the mass due to the string segments on either side. In this problem, gravity is to be neglected. It may be convenient to define $K \equiv 2\tau/mc^2$ and $Q \equiv \gamma/mc$.

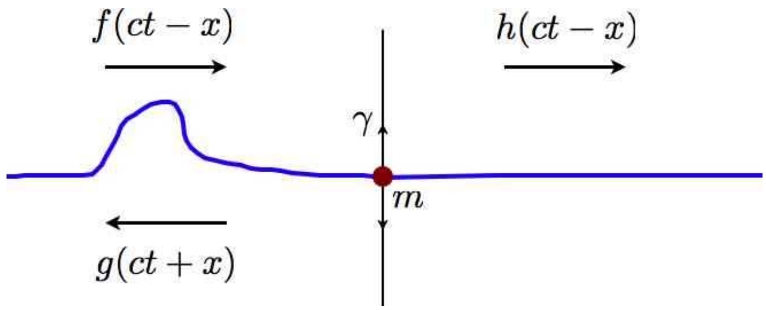


Figure 17.19: A point mass m joining two semi-infinite lengths of identical string moves vertically along a thin wire with friction coefficient γ.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct-x) + g(ct+x) & (x<0) \\ h(ct-x) & (x>0) \; . \end{cases}$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$.
[20 points]

The first equation is continuity at $x = 0$:

$$\boxed{f(\xi) = g(\xi) + h(\xi)}$$

where $\xi = ct$ ranges over the real line $[-\infty, \infty]$. The second equation comes from Newton's 2nd law $F = ma$ applied to the mass point:

$$m\,\ddot{y}(0,t) + \gamma\,\dot{y}(0,t) = \tau\,y'(0^+,t) - \tau\,y'(0^-,t) \; .$$

Expressed in terms of the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$, and dividing through by mc^2, this gives

$$f''(\xi) + g''(\xi) + Q\,f'(\xi) + Q\,g'(\xi) = -\tfrac{1}{2}\,K\,h'(\xi) + \tfrac{1}{2}\,K\,f'(\xi) - \tfrac{1}{2}\,K\,g'(\xi).$$

Integrating once, and invoking $h = f + g$, this second equation becomes

$$\boxed{f'(\xi) + Q\, f(\xi) = -g'(\xi) - (K+Q)\, g(\xi)}$$

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \ ,$$

et cetera for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\big|r(k)\big|^2 + \big|t(k)\big|^2$. Show that this sum is always less than or equal to unity, and interpret this fact.
[20 points]

Using $d/d\xi \longrightarrow ik$, we have

$$(Q + ik)\, \hat{f}(k) = -(K + Q + ik)\, \hat{g}(k) \ . \tag{17.131}$$

Therefore,

$$\boxed{r(k) = \frac{\hat{g}(k)}{\hat{f}(k)} = -\frac{Q+ik}{Q+K+ik}} \tag{17.132}$$

To find the transmission amplitude, we invoke $h(\xi) = f(\xi) + g(\xi)$, in which case

$$\boxed{t(k) = \frac{\hat{h}(k)}{\hat{f}(k)} = -\frac{K}{Q+K+ik}} \tag{17.133}$$

The sum of reflection and transmission coefficients is

$$\boxed{\big|r(k)\big|^2 + \big|t(k)\big|^2 = \frac{Q^2 + K^2 + k^2}{(Q+K)^2 + k^2}} \tag{17.134}$$

Clearly the RHS of this equation is bounded from above by unity, since both Q and K are nonnegative.

(c) Find an expression in terms of the functions f, g, and h (and/or their derivatives) for the rate $\dot{E}$ at which energy is lost by the string. Do this by evaluating the energy current on either side of the point mass. Your expression should be an overall function of time t.
[10 points]

Recall the formulae for the energy density in a string,

$$\mathcal{E}(x,t) = \tfrac{1}{2}\, \mu\, \dot{y}^2(x,t) + \tfrac{1}{2}\, \tau\, {y'}^2(x,t) \tag{17.135}$$

and

$$j_{\mathcal{E}}(x,t) = -\tau\,\dot{y}(x,t)\,y'(x,t)\ . \tag{17.136}$$

The energy continuity equation is $\partial_t \mathcal{E} + \partial_x j_{\mathcal{E}} = 0$. Assuming $j_{\mathcal{E}}(\pm\infty,t) = 0$, we have

$$\begin{aligned}\frac{dE}{dt} &= \int\limits_{-\infty}^{0^-}\!dx\,\frac{\partial\mathcal{E}}{\partial t} + \int\limits_{0^+}^{\infty}\!dx\,\frac{\partial\mathcal{E}}{\partial t} \\ &= -j_{\mathcal{E}}(\infty,t) + j_{\mathcal{E}}(0^+,t) + j_{\mathcal{E}}(-\infty,t) - j_{\mathcal{E}}(0^-,t)\ .\end{aligned} \tag{17.137}$$

Thus,

$$\boxed{\frac{dE}{dt} = c\tau\Big(\big[g'(ct)\big]^2 + \big[h'(ct)\big]^2 - \big[f'(ct)\big]^2\Big)} \tag{17.138}$$

Incidentally, if we integrate over all time, we obtain the total energy change in the string:

$$\begin{aligned}\Delta E &= \tau\int\limits_{-\infty}^{\infty}\!d\xi\,\Big(\big[g'(\xi)\big]^2 + \big[h'(\xi)\big]^2 - \big[f'(\xi)\big]^2\Big) \\ &= -\tau\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,\frac{2QK\,k^2}{(Q+K)^2+k^2}\,\big|\hat{f}(k)\big|^2\ .\end{aligned} \tag{17.139}$$

Note that the initial energy in the string, at time $t=-\infty$, is

$$E_0 = \tau\int\limits_{-\infty}^{\infty}\!\frac{dk}{2\pi}\,k^2\,\big|\hat{f}(k)\big|^2\ . \tag{17.140}$$

If the incident wave packet is very broad, say described by a Gaussian $f(\xi) = A\,\exp(-x^2/2\sigma^2)$ with $\sigma K \gg 1$ and $\sigma Q \gg 1$, then k^2 may be neglected in the denominator of eqn. 17.139, in which case

$$\Delta E \approx -\frac{2QK}{(Q+K)^2}\,E_0 \geq -\tfrac{1}{2}E_0\ . \tag{17.141}$$

[**2**] Consider a rectangular cube of density ρ and dimensions $a \times b \times c$, as depicted in fig. 17.22.

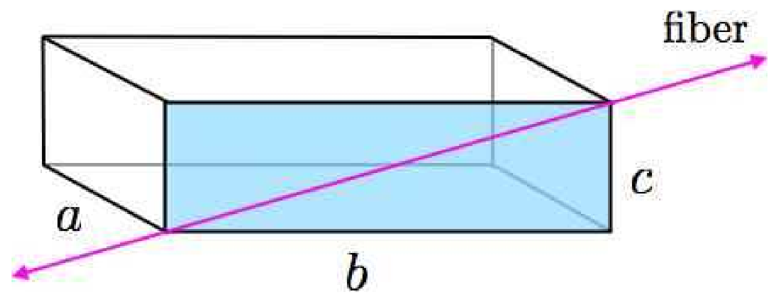


Figure 17.20: A rectangular cube of dimensions $a \times b \times c$. In part `(c)`, a massless torsional fiber is attached along the diagonal of one of the $b \times c$ faces.

`(a)` Compute the inertia tensor $I_{\alpha\beta}$ along body-fixed principle axes, with the origin at the center of mass.
`[15 points]`

We first compute I_{zz}:

$$I^{\rm CM}_{zz} = \rho \int\limits_{-a/2}^{a/2} \!dx \int\limits_{-b/2}^{b/2} \!dy \int\limits_{-c/2}^{c/2} \!dz \, \big(x^2 + y^2\big) = \tfrac{1}{12}\, M\, \big(a^2 + b^2\big) \ , \qquad (17.142)$$

where $M = \rho\, abc$. Corresponding expressions hold for the other moments of inertia. Thus,

$$I^{\rm CM} = \tfrac{1}{12} M \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} \qquad (17.143)$$

`(b)` Shifting the origin to the center of either of the $b \times c$ faces, and keeping the axes parallel, compute the new inertia tensor.
`[15 points]`

We shift the origin by a distance $\boldsymbol{d} = -\frac{1}{2} a\, \hat{\boldsymbol{x}}$ and use the parallel axis theorem,

$$I_{\alpha\beta}(\boldsymbol{d}) = I_{\alpha\beta}(0) + M \big(\boldsymbol{d}^2 \delta_{\alpha\beta} - d_\alpha d_\beta\big) \ , \qquad (17.144)$$

resulting in

$$I = \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & 4a^2 + c^2 & 0 \\ 0 & 0 & 4a^2 + b^2 \end{pmatrix} \qquad (17.145)$$

`(c)` A massless torsional fiber is (masslessly) welded along the diagonal of either $b \times c$ face. The potential energy in this fiber is given by $U(\theta) = \frac{1}{2} Y \theta^2$, where Y is a constant and θ is

the angle of rotation of the fiber. Neglecting gravity, find an expression for the oscillation frequency of the system.
[20 points]

Let θ be the twisting angle of the fiber. The kinetic energy in the fiber is

$$\begin{aligned} T &= \tfrac{1}{2}\, I_{\alpha\beta}\, \omega_\alpha\, \omega_\beta \\ &= \tfrac{1}{2}\, n_\alpha\, I_{\alpha\beta}\, n_\beta\, \dot{\theta}^2 \ , \end{aligned} \tag{17.146}$$

where

$$\hat{\boldsymbol{n}} = \frac{b\,\hat{\boldsymbol{y}}}{\sqrt{b^2+c^2}} + \frac{c\,\hat{\boldsymbol{z}}}{\sqrt{b^2+c^2}} \ . \tag{17.147}$$

We then find

$$I_{\rm axis} \equiv n_\alpha\, I_{\alpha\beta}\, n_\beta = \tfrac{1}{3} M a^2 + \tfrac{1}{6} M \frac{b^2\, c^2}{b^2+c^2} \ . \tag{17.148}$$

The frequency of oscillation is then $\Omega = \sqrt{Y/I_{\rm axis}}$, or

$$\boxed{\Omega = \sqrt{\frac{6Y}{M} \cdot \frac{b^2+c^2}{2a^2\big(b^2+c^2\big) + b^2\, c^2}}} \tag{17.149}$$

17.8 W08 Physics 110B Final Exam

[**1**] Consider a string with uniform mass density μ and tension τ. At the point $x = 0$, the string is connected to a spring of force constant K, as shown in the figure below.

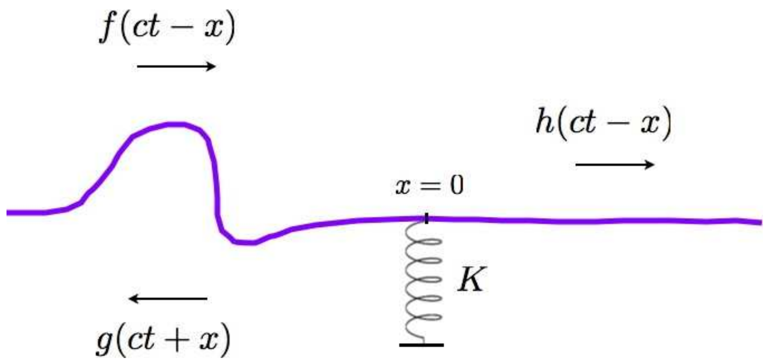


Figure 17.21: A string connected to a spring.

(a) The general solution with an incident wave from the left is written

$$y(x,t) = \begin{cases} f(ct - x) + g(ct + x) & (x < 0) \\ h(ct - x) & (x > 0)\ . \end{cases}$$

Find two equations relating the functions $f(\xi)$, $g(\xi)$, and $h(\xi)$. [10 points]

SOLUTION : The first equation is continuity at $x = 0$:

$$\boxed{f(\xi) + g(\xi) = h(\xi)}$$

where $\xi = ct$ ranges over the real line $[-\infty, \infty]$. The second equation comes from Newton's 2nd law $F = ma$ applied to the mass point:

$$\tau\, y'(0^+, t) - \tau\, y'(0^-, t) - K\, y(0, t) = 0\ ,$$

or

$$\boxed{-\,\tau\, h'(\xi) + \tau\, f'(\xi) - \tau\, g'(\xi) - K\,\big[f(\xi) + g(\xi)\big] = 0}$$

(b) Solve for the reflection amplitude $r(k) = \hat{g}(k)/\hat{f}(k)$ and the transmission amplitude $t(k) = \hat{h}(k)/\hat{f}(k)$. Recall that

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad \Longleftrightarrow \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi}\ ,$$

et cetera for the Fourier transforms. Also compute the sum of the reflection and transmission coefficients, $\big|r(k)\big|^2 + \big|t(k)\big|^2$. [10 points]

SOLUTION : Taking the Fourier transform of the two equations from part (a), we have

$$\hat{f}(k) + \hat{g}(k) = \hat{h}(k)$$

$$\hat{f}(k) + \hat{g}(k) = \frac{i\tau k}{K}\Big[\hat{f}(k) - \hat{g}(k) - \hat{h}(k)\Big] .$$

Solving for $\hat{g}(k)$ and $\hat{h}(k)$ in terms of $\hat{f}(k)$, we find

$$\boxed{\hat{g}(k) = r(k)\,\hat{f}(k) \qquad , \qquad \hat{h}(k) = t(k)\,\hat{f}(k)}$$

where the reflection coefficient $r(k)$ and the transmission coefficient $t(k)$ are given by

$$\boxed{r(k) = -\frac{K}{K + 2i\tau k} \qquad , \qquad t(k) = \frac{2i\tau k}{K + 2i\tau k}}$$

Note that

$$\boxed{\big|r(k)\big|^2 + \big|t(k)\big|^2 = 1}$$

which says that the energy flux is conserved.

(c) For the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2 - \tfrac{1}{4}\gamma\left(\frac{\partial y}{\partial x}\right)^4 ,$$

find the Euler-Lagrange equations of motion. [7 points]

SOLUTION : For a Lagrangian density $\mathcal{L}(y, \dot{y}, y')$, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) .$$

Thus, the wave equation for this system is

$$\boxed{\mu\,\ddot{y} = \tau\,y'' + 3\gamma\,\big(y'\big)^2\,y''}$$

(d) For the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\mu \left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\alpha\, y^2 - \tfrac{1}{2}\tau \left(\frac{\partial y}{\partial x}\right)^2 - \tfrac{1}{4}\beta \left(\frac{\partial^2 y}{\partial x^2}\right)^2 ,$$

find the Euler-Lagrange equations of motion. [7 points]

SOLUTION : For a Lagrangian density $\mathcal{L}(y, \dot{y}, y', y'')$, the Euler-Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial y''}\right) .$$

The last term arises upon integrating by parts twice in the integrand of the variation of the action δS. Thus, the wave equation for this system is

$$\boxed{\mu\, \ddot{y} = -\alpha\, y + \tau\, y'' - \beta\, y''''}$$

[2] Consider single species population dynamics governed by the differential equation

$$\frac{dN}{dt} = \gamma N - \frac{N^2}{K} - \frac{HN}{N+L} \ ,$$

where γ, K, L, and H are constants.

(a) Show that by rescaling N and t that the above ODE is equivalent to

$$\frac{du}{ds} = r\,u - u^2 - \frac{h\,u}{u+1} \ .$$

Give the definitions of u, s, r, and h. [5 points]

SOLUTION : From the denominator $u+1$ in the last term of the scaled equation, we see that we need to define $N = Lu$. We then write $t = \tau s$, and substituting into the original ODE yields

$$\frac{L}{\tau}\,\frac{du}{ds} = \gamma L u - \frac{L^2}{K}\,u^2 - \frac{H\,u}{u+1} \ .$$

Multiplying through by τ/L then gives

$$\frac{du}{ds} = \gamma\tau\,u - \frac{L\tau}{K}\,u^2 - \frac{\tau H}{L}\,\frac{u}{u+1} \ .$$

We set the coefficient of the second term on the RHS equal to -1 to obtain the desired form. Thus, $\tau = K/L$ and

$$\boxed{u = \frac{N}{L} \quad , \quad s = \frac{Lt}{K} \quad , \quad r = \frac{\gamma K}{L} \quad , \quad h = \frac{KH}{L^2}}$$

(b) Find and solve the equation for all fixed points $u^*(r,h)$. [10 points]

SOLUTION : In order for u to be a fixed point, we need $\dot{u} = 0$, which requires

$$\boxed{u\left(r - u - \frac{h}{u+1}\right) = 0}$$

One solution is always $\boxed{u^* = 0}$. The other roots are governed by the quadratic equation

$$(u-r)(u+1) + h = 0 \ ,$$

with roots at

$$\boxed{u^* = \tfrac{1}{2}\left(r - 1 \pm \sqrt{(r+1)^2 - 4h}\right)}$$

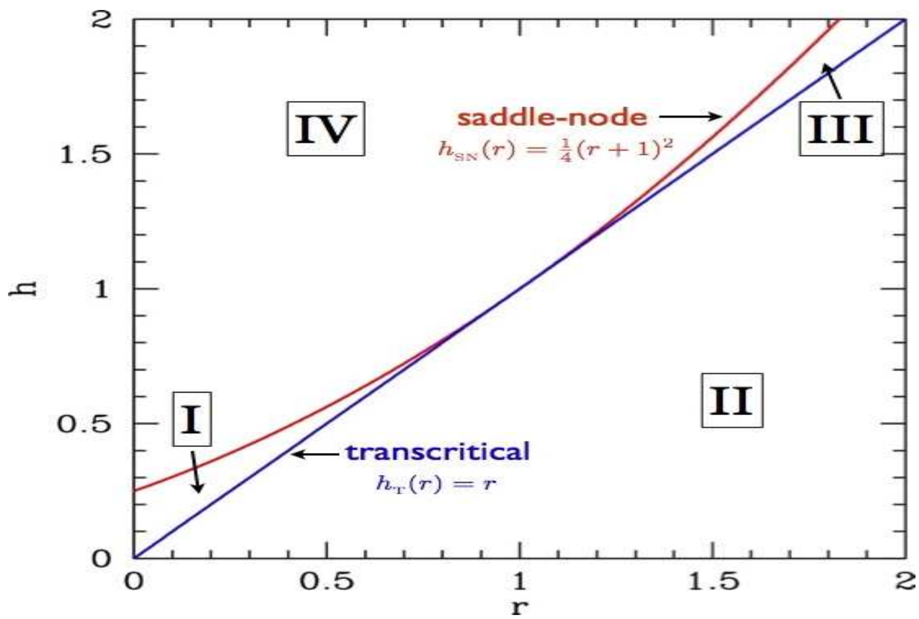


Figure 17.22: Bifurcation curves for the equation $\dot{u} = ru - u^2 - hu/(u+1)$. Red curve: $h_{SN}(r) = \frac{1}{4}(r+1)^2$, corresponding to saddle-node bifurcation. Blue curve: $h_T(r) = r$, corresponding to transcritical bifurcation.

(c) Sketch the upper right quadrant of the (r, h) plane. Show that there are four distinct regions:

Region I	:	3 real fixed points (two negative)
Region II	:	3 real fixed points (one positive, one negative)
Region III	:	3 real fixed points (two positive)
Region IV	:	1 real fixed point

Find the equations for the boundaries of these regions. These boundaries are the locations of bifurcations. Classify the bifurcations. (Note that negative values of u are unphysical in the context of population dynamics, but are legitimate from a purely mathematical standpoint.) [10 points]

SOLUTION : From the quadratic equation for the non-zero roots, we see the discriminant vanishes for $h = \frac{1}{4}(r+1)^2$. For $h > \frac{1}{4}(r+1)^2$, the discriminant is negative, and there is one real root at $u^* = 0$. Thus, the curve $\boxed{h_{SN}(r) = \frac{1}{4}(r+1)^2}$ corresponds to a curve of saddle-node bifurcations. Clearly the largest value of u^* must be a stable node, because for large u the $-u^2$ dominates on the RHS of $\dot{u} = f(u)$. In cases where there are three fixed points, the middle one must be unstable, and the smallest stable. There is another bifurcation, which occurs when the root at $u^* = 0$ is degenerate. This occurs at

$$r - 1 = \sqrt{(r+1)^2 - 4h} \qquad \Longrightarrow \qquad h = r\ .$$

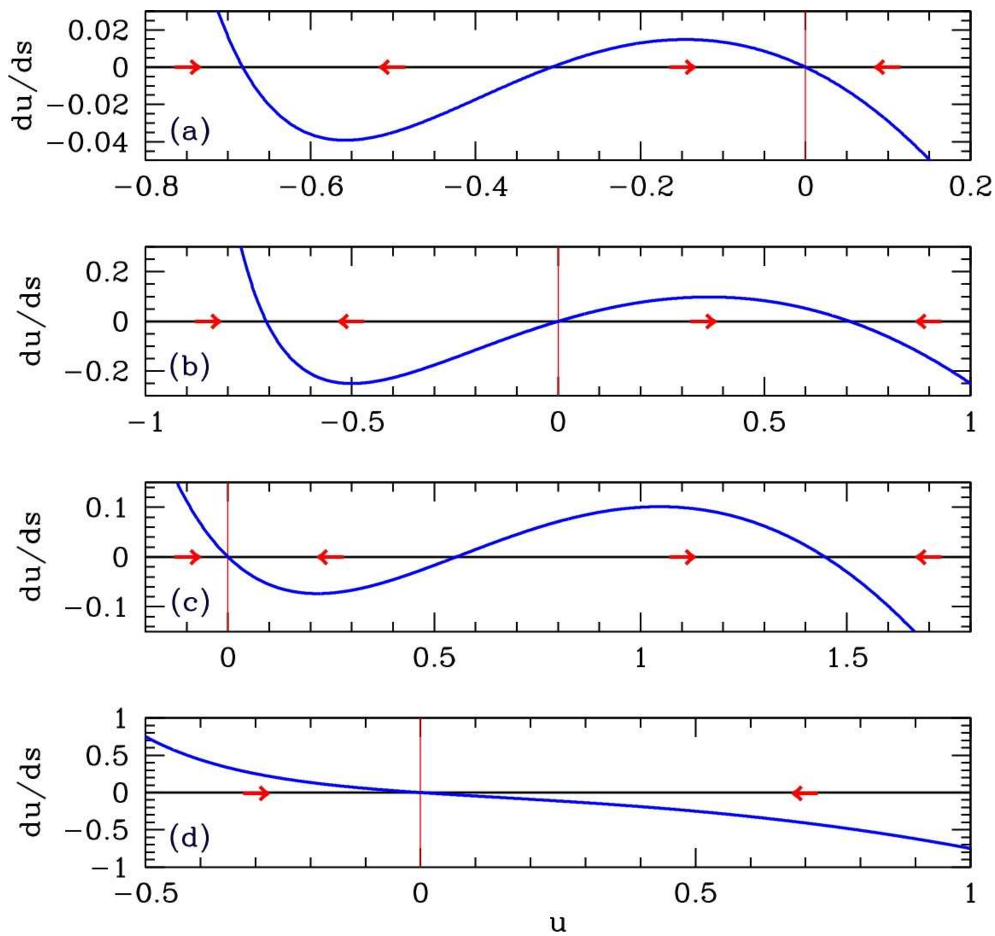


Figure 17.23: Examples of phase flows for the equation $\dot{u} = ru - u^2 - hu/(u+1)$. (a) $r = 1$, $h = 0.22$ (region I) ; (b) $r = 1$, $h = 0.5$ (region II) ; (c) $r = 3$, $h = 3.8$ (region III) ; (d) $r = 1$, $h = 1.5$ (region IV).

This defines the curve for transcritical bifurcations: $\boxed{h_{\rm T}(r) = r}$. Note that $h_{\rm T}(r) \le h_{\rm SN}(r)$, since $h_{\rm SN}(r) - h_{\rm T}(r) = \frac{1}{4}(r-1)^2 \ge 0$. For $h < r$, one root is positive and one negative, corresponding to region II.

The (r, h) control parameter space is depicted in fig. 17.22, with the regions I through IV bounded by sections of the bifurcation curves, as shown.

(d) Sketch the phase flow for each of the regions I through IV. [8 points]

SOLUTION : See fig. 17.23.

[**3**] Two brief relativity problems:

(a) A mirror lying in the (x, y) plane moves in the $\hat{\boldsymbol{z}}$ direction with speed u. A monochromatic ray of light making an angle θ with respect to the $\hat{\boldsymbol{z}}$ axis in the laboratory frame reflects off the moving mirror. Find (i) the angle of reflection, measured in the laboratory frame, and (ii) the frequency of the reflected light. [17 points]

SOLUTION : The reflection is simplest to consider in the frame of the mirror, where $\tilde{p}_z \to -\tilde{p}_z$ upon reflection. In the laboratory frame, the 4-momentum of a photon in the beam is

$$P^\mu = \big(E\,,\,0\,,\,E\sin\theta\,,\,E\cos\theta\big)\ ,$$

where, without loss of generality, we have taken the light ray to lie in the (y, z) plane, and where we are taking $c = 1$. Lorentz transforming to the frame of the mirror, we have

$$\tilde{P}^\mu = \big(\gamma E(1 - u\cos\theta)\,,\,0\,,\,E\sin\theta\,,\,\gamma E(-u + \cos\theta)\big)\ .$$

which follows from the general Lorentz boost of a 4-vector Q^μ,

$$\begin{aligned}\tilde{Q}^0 &= \gamma Q^0 - \gamma u Q_{\|}\\ \tilde{Q}_{\|} &= -\gamma u Q^0 + \gamma Q_{\|}\\ \tilde{\boldsymbol{Q}}_\perp &= \boldsymbol{Q}_\perp\ ,\end{aligned}$$

where frame $\tilde{K}$ moves with velocity $\boldsymbol{u}$ with respect to frame K.

Upon reflection, we reverse the sign of $\tilde{P}^3$ in the frame of the mirror:

$$\tilde{P}'^\mu = \big(\gamma E(1 - u\cos\theta)\,,\,0\,,\,E\sin\theta\,,\,\gamma E(u - \cos\theta)\big)\ .$$

Transforming this back to the laboratory frame yields

$$\begin{aligned}E' = P'^0 &= \gamma^2 E\,(1 - u\cos\theta) + \gamma^2 E\,u\,(u - \cos\theta)\\ &= \gamma^2 E\,\big(1 - 2u\,\cos\theta + u^2\big)\end{aligned}$$

$$P'^1 = 0$$

$$P'^2 = E\sin\theta$$

$$\begin{aligned}P'^3 &= \gamma^2 E\,u\,(1 - u\cos\theta) + \gamma^2 E\,(u - \cos\theta)\\ &= -\gamma^2 E\,\big((1 + u^2)\cos\theta - 2u\big)\end{aligned}$$

Thus, the angle of reflection is

$$\boxed{\cos\theta' = \left|\frac{P'^3}{P'^0}\right| = \frac{(1 + u^2)\cos\theta - 2u}{1 - 2u\cos\theta + u^2}}$$

and the reflected photon frequency is $\nu' = E'/h$, where

$$\boxed{E' = \left(\frac{1 - 2u\,\cos\theta + u^2}{1 - u^2}\right) E}$$

(b) Consider the reaction $\pi^+ + \text{n} \to \text{K}^+ + \Lambda^0$. What is the threshold kinetic energy of the pion to create kaon at an angle of 90° in the rest frame of the neutron? Express your answer in terms of the masses m_π, m_n, m_K, and m_Λ. [16 points]

SOLUTION : We have conservation of 4-momentum, giving

$$P_\pi^\mu + P_\text{n}^\mu = P_\text{K}^\mu + P_\Lambda^\mu \ .$$

Thus,

$$\begin{aligned} P_\Lambda^2 &= (E_\pi + E_\text{n} - E_\text{K})^2 - (\boldsymbol{P}_\pi + \boldsymbol{P}_\text{n} - \boldsymbol{P}_\text{K})^2 \\ &= (E_\pi^2 - \boldsymbol{P}_\pi^2) + (E_\text{n}^2 - \boldsymbol{P}_\text{n}^2) + (E_\text{K}^2 - \boldsymbol{P}_\text{K}^2) \\ &\qquad + 2E_\pi E_\text{n} - 2E_\pi E_\text{K} - 2E_\text{n} E_\text{K} - 2\boldsymbol{P}_\pi \cdot \boldsymbol{P}_\text{n} + 2\boldsymbol{P}_\pi \cdot \boldsymbol{P}_\text{K} + 2\boldsymbol{P}_\text{n} \cdot \boldsymbol{P}_\text{K} \\ &= E_\Lambda^2 - \boldsymbol{P}_\Lambda^2 = m_\Lambda^2 \ . \end{aligned}$$

Now in the laboratory frame the neutron is at rest, so

$$P_\text{n}^\mu = (m_\text{n} \, , \, \mathbf{0}) \ .$$

Thus, $\mathbf{P}_\pi \cdot \mathbf{P}_\text{n} = \mathbf{P}_\text{n} \cdot \mathbf{P}_\text{K} = 0$. We are also told that the pion and the kaon make an angle of 90° in the laboratory frame, hence $\mathbf{P}_\pi \cdot \mathbf{P}_\text{K} = 0$. And of course for each particle we have $E^2 - \mathbf{P}^2 = m^2$. Thus, we have

$$m_\Lambda^2 = m_\pi^2 + m_\text{n}^2 + m_\text{K}^2 - 2m_\text{n} E_\text{K} + 2(m_\text{n} - E_\text{K})\, E_\pi \ ,$$

or, solving for E_π,

$$E_\pi = \frac{m_\Lambda^2 - m_\pi^2 - m_\text{n}^2 - m_\text{K}^2 + 2m_\text{n} E_\text{K}}{2(m_\text{n} - E_\text{K})} \ .$$

The threshold pion energy is the minimum value of E_π, which must occur when E_K takes its minimum allowed value, $E_\text{K} = m_\text{K}$. Thus,

$$\boxed{T_\pi = E_\pi - m_\pi \geq \frac{m_\Lambda^2 - m_\pi^2 - m_\text{n}^2 - m_\text{K}^2 + 2m_\text{n} m_\text{K}}{2(m_\text{n} - m_\text{K})} - m_\pi}$$

[**4**] Sketch what a bletch might look like. [10,000 quatloos extra credit]
[-50 points if it looks like your professor]

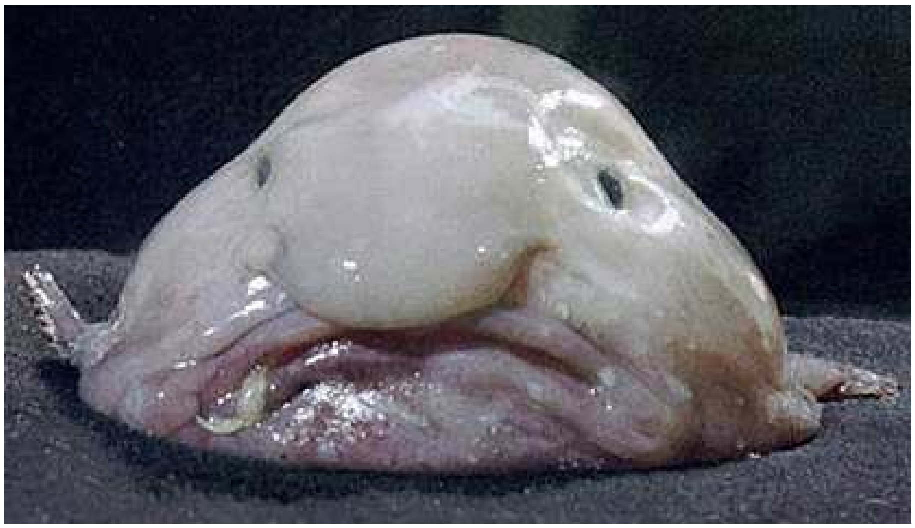


Figure 17.24: The putrid bletch, from the (underwater) Jkroo forest, on planet Barney.