PHYSICS 200A : CLASSICAL MECHANICS
PRACTICE FINAL EXAMINATION
Do #5 and any three from #1–4

[**1**] A point particle of mass m in two-dimensions moves along a one-dimensional surface under the influence of gravity $\boldsymbol{g} = -g\,\hat{\boldsymbol{y}}$. The equation of the surface is

$$y = x - \frac{x^3}{3a^2} \ .$$

The particle is released from rest at a point along the curve $(x_0\,,\,y_0)$. The particle flies off the curve at $x = a$. Determine y_0.

[**2**] Consider the two coupled strings of fig. 1. Both strings are described by identical mass density σ and tension τ. On each string, at $x = 0$, a point mass m is affixed. The two masses are connected via a spring of constant κ. When the two masses are identically displaced, *i.e.* when $u_1(0,t) = u_2(0,t)$, the spring is unstretched. There are no other forces aside from the tension in the strings and the restoring force of the spring.

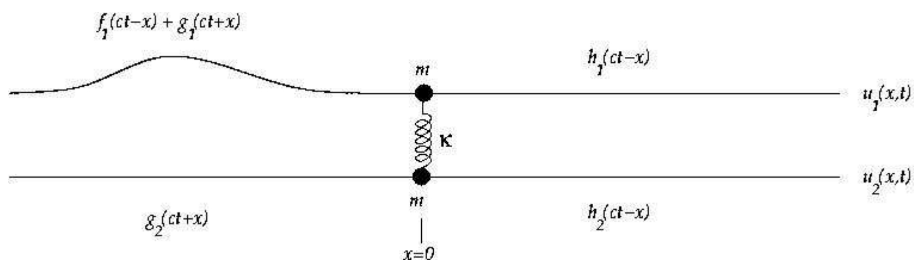

Figure 1: Two identical strings with masses m at $x = 0$, coupled via a spring of constant κ.

(a) Let $u_i(x,t)$ be the displacement field of each string ($i = 1, 2$). Write down the equations of motion for the two masses.

(b) Taking advantage of the symmetry under interchange of the two strings, define sum and difference fields,
$$u_\pm(x,t) \equiv u_1(x,t) \pm u_2(x,t) \ ,$$
and rewrite the equations from part (a) in terms of $u_\pm(0,t)$.

(c) In the distant past, a pulse of shape $f(\xi)$ is incident from the left on string #1. Given the definition of the functions $g_i(\xi)$ and $h_i(\xi)$ implicit in the figure, find the complex reflection and transmission coefficients,

$$r(k) = \frac{\hat{g}_1(k)}{\hat{f}(k)} \quad , \quad t(k) = \frac{\hat{h}_1(k)}{\hat{f}(k)} \quad , \quad \tilde{t}(k) = \frac{\hat{g}_2(k)}{\hat{f}(k)} \quad , \quad \tilde{t}'(k) = \frac{\hat{h}_2(k)}{\hat{f}(k)} \ ,$$

where $\hat{f}(k)$ is the Fourier transform of $f(\xi)$, *etc.*[1] For notational convenience, you should define $Q = \tau/mc^2$ and $P = \sqrt{2\kappa/mc^2}$.

(d) Do your answers make sense in the limits $m \to \infty$ and $\kappa \to 0$?

(e) Consider the limit $m \to 0$ with κ, τ, and σ held fixed. Find the transmission and reflection coefficients $T = |t|^2$, $R = |r|^2$, $\tilde{T} = |\tilde{t}|^2$, $\tilde{T}' = |\tilde{t}'|^2$, and show that energy flux is conserved.

(f) Write down the Lagrangian density $\mathcal{L}$ for this system.

[3] A particle of charge e moves in the (x, y) plane under the influence of a static uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The potential is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = e\,\phi(\boldsymbol{r}) - \tfrac{e}{c}\,\boldsymbol{A}(\boldsymbol{r}) \cdot \dot{\boldsymbol{r}} \ .$$

Choose the gauge

$$\boldsymbol{A} = -\tfrac{1}{2}By\,\hat{\boldsymbol{x}} + \tfrac{1}{2}Bx\,\hat{\boldsymbol{y}} \ .$$

(a) Derive the Hamiltonian $H(x, y, p_x, p_y)$.

(b) Define the cyclotron coordinates $(\zeta_x\,,\,\zeta_y)$ and the guiding center coordinates $\{R_x\,,\,R_y\}$ as follows:

$$\zeta_x = \tfrac{1}{2}\,x - \tfrac{c}{eB}\,p_y \qquad\qquad R_x = \tfrac{1}{2}\,x + \tfrac{c}{eB}\,p_y$$

$$\zeta_y = \tfrac{1}{2}\,y + \tfrac{c}{eB}\,p_x \qquad\qquad R_y = \tfrac{1}{2}\,y - \tfrac{c}{eB}\,p_x \ .$$

Compute the Poisson brackets $\{\zeta_\mu\,,\,\zeta_\nu\}$, $\{R_\mu\,,\,R_\nu\}$, and $\{\zeta_\mu\,,\,R_\nu\}$, for all possible pairings of $\mu, \nu = x$ or y.

(c) Show that π_x, the momentum conjugate to ζ_x, is a constant times ζ_y, and that κ_y, the momentum conjugate to R_y, is a constant times R_x.

(d) Write the equations of motion solely in terms of the cyclotron and guiding center coordinates. Note that

$$\phi(x, y) = \phi\big(R_x + \zeta_x, R_y + \zeta_y\big) \ .$$

(e) When the cyclotron frequency $\omega_{\rm c} = eB/mc$ is large, show that the motion of the cyclotron coordinates is approximately harmonic.

[1] Even though the g_2 wave moves to the left, we consider this transmission from one branch of string to another, rather than reflection into the same branch.

[**4**] A particle of mass m moves in the potential $U(q) = A\,|q|$. The Hamiltonian is thus

$$H_0(q,p) = \frac{p^2}{2m} + A\,|q| \ ,$$

where A is a constant.

(a) List all independent conserved quantities.

(b) Show that the action variable J is related to the energy E according to $J = \beta\,E^{3/2}/A$, where β is a constant, involving m. Find β.

(c) Find $q = q(\phi, J)$ in terms of the action-angle variables.

(d) Find $H_0(J)$ and the oscillation frequency $\nu_0(J)$.

(e) The system is now perturbed by a quadratic potential, so that

$$H(q,p) = \frac{p^2}{2m} + A\,|q| + \epsilon\,B\,q^2 \ ,$$

where ϵ is a small dimensionless parameter. Compute the shift $\Delta\nu$ to lowest nontrivial order in ϵ, in terms of ν_0 and constants.

[**5**] Provide short but accurate answers to the following questions:

(a) Write down a generating function for a canonical transformation which generates a dilation: $Q = \lambda\,q$, $P = \lambda^{-1}p$.

(b) Give an explicit example of a two-dimensional phase flow which is invertible but not volume preserving.

(c) What is Noether's theorem? Give an example and be explicit.

(d) Consider the Lagrangian,

$$L = \tfrac{1}{2}m_\perp(t)\left(\dot{x}^2 + \dot{y}^2\right) + \tfrac{1}{2}m_z\dot{z}^2 - \tfrac{1}{4}\left(x^4 + 2x^2y^2 + y^4\right) \ ,$$

where $m_\perp(t)$ is time-dependent. List and provide expressions for all conserved quantities.

(e) How are the Euler angles defined?

(f) Explain the content and physics of the 'tennis racket theorem'.