PHYSICS 200A : CLASSICAL MECHANICS
PROBLEM SET #4

[1] A tilted coin (*i.e.* a sharp-edged uniform disk) of radius a and mass M rolls without slipping on a horizontal plane in a circle of radius b. A set of orthogonal coordinate axes has its origin at the center of mass, with $\hat{\mathbf{e}}_3$ perpendicular to the face of the coin, $\hat{\mathbf{e}}_2$ in the plane of the coin and passing through the point of contact, and $\hat{\mathbf{e}}_1$ parallel to the horizontal plane and tangent to the trajectory. Introduce the angles θ, ϕ, and γ that specify the orientation of the coin as indicated in Fig. 1. The particular motion of interest (neglecting rolling friction which eventually slows down the coin) is characterized by

$$\dot{\theta} = 0 \quad , \quad \dot{\phi} = \text{const.} \quad , \quad \dot{\gamma} = \text{const.} \; . \tag{1}$$

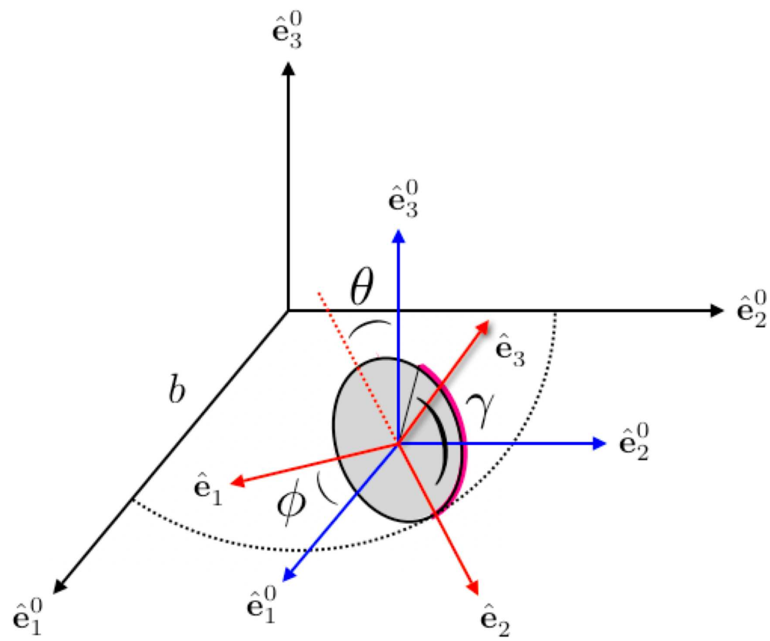

Figure 1: A rolling coin of radius a. The geometry is described in the text of problem 1.

(a) Use the equations of rigid body motion to eliminate the reaction force at the point of contact and obtain

$$\frac{d\boldsymbol{L}}{dt}\bigg|_{\text{CM}} = Ma\big[g\sin\theta - \dot{\phi}^2(b - a\sin\theta)\,\cos\theta\big]\,\hat{\mathbf{e}}_1 \; , \tag{2}$$

where the CM frame has its origin at the instantaneous center-of-mass of the coin, and whose axes are parallel to $\{\hat{\mathbf{e}}^0_1, \hat{\mathbf{e}}^0_2, \hat{\mathbf{e}}^0_3\}$ of the inertial frame.

(b) Show, in general, that

$$\boldsymbol{\omega} = \dot{\theta}\,\hat{\mathbf{e}}_1 + \dot{\phi}\,\cos\theta\,\hat{\mathbf{e}}_2 - \dot{\phi}\,\sin\theta\,\hat{\mathbf{e}}_3 \tag{3}$$

and

$$\boldsymbol{L}_{\text{CM}} = I_1\,\dot{\theta}\,\hat{\mathbf{e}}_1 + I_1\,\dot{\phi}\,\cos\theta\,\hat{\mathbf{e}}_2 + I_3\,(\dot{\gamma} - \dot{\phi}\,\sin\theta)\,\hat{\mathbf{e}}_3 \; , \tag{4}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of the body-fixed frame as seen in the CM frame.

(c) Use the relation between $\dot{\boldsymbol{L}}_{\text{CM}}$ and $\dot{\boldsymbol{L}}_{\text{body}}$ to show that the period τ for motion around the circle and the angle of inclination must satisfy the equation

$$\frac{\tau^2}{4\pi^2} = \dot{\phi}^{-2} = \frac{\cos\theta}{4g\,\sin\theta}\left(6b - 5a\sin\theta\right) . \tag{5}$$

Recall that for a disk, $I_1 = I_2 = \frac{1}{4}Ma^2$, and $I_3 = \frac{1}{2}Ma^2$.

NB: This is Fetter and Walecka problem 5.3.

[2] A symmetric top with one fixed point in a gravitational field moves with its symmetry axis nearly vertical ($\theta \ll 1$) and $p_\phi = p_\psi$.

(a) Expand the effective potential through terms of order θ^4.

(b) If $p_\psi^2 > 4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a minimum at $\theta = 0$. Sketch $U_{\text{eff}}(\theta)$ for small θ. Prove that the frequency of small oscillations about this configuration is given by

$$\Omega^2 = \frac{p_\psi^2 - 4I_1 Mg\ell}{4I_1^2} . \tag{6}$$

(c) If p_ψ^2 is slightly smaller than $4I_1 Mg\ell$, show that $U_{\text{eff}}(\theta)$ has a maximum at $\theta = 0$ and a minimum at some finite value θ^*. Find θ^*, and sketch $U_{\text{eff}}(\theta)$ for small θ, and find the frequency of small oscillations about $\theta = \theta^*$.