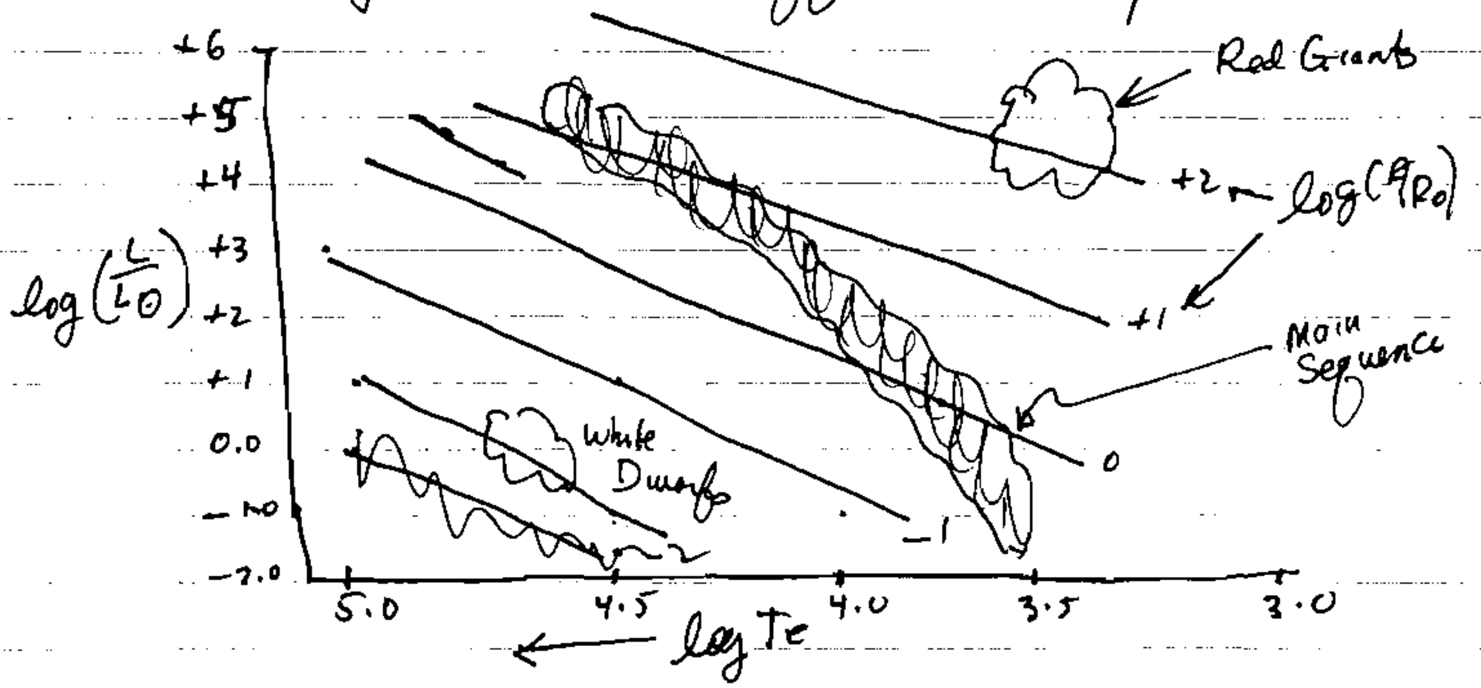
Recap (chp. 8)

Hertzsprung-Russell Diagram:

Last lecture I introduced Hertzsprung-Russell diagram. Physically, it is a plot of luminosity, L, versus effective temperature, Te.



What do we learn from HR diagram

(1) Main Sequence
- ~90% of stars in solar neighborhood on MS
- L rises faster than T_e^4. Implication is UMS stars larger than lower MS

~~(2) Red Giants~~ Since $L = 4\pi R_s^2 \cdot \sigma T_e^4$

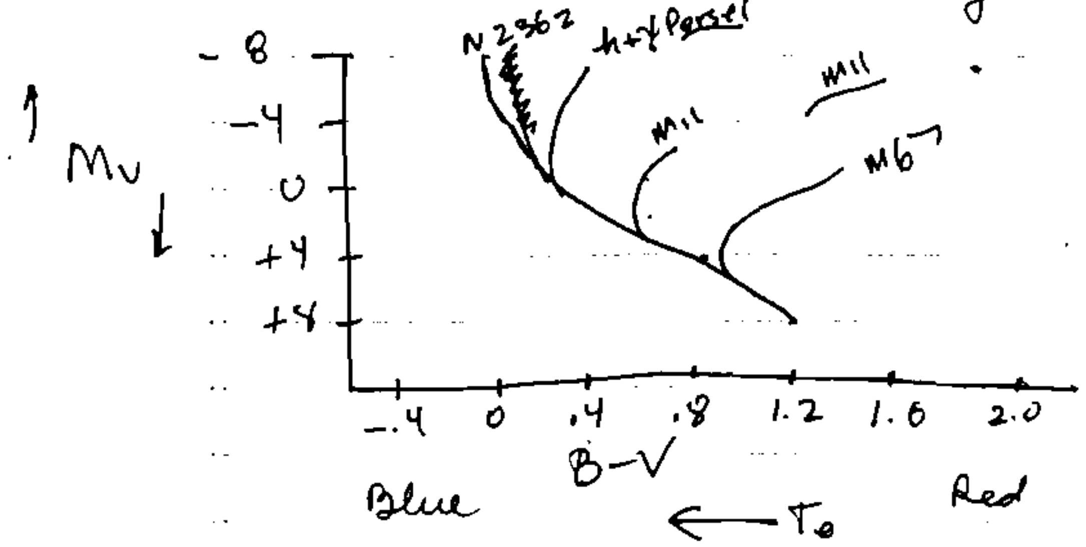
(2) Red Giants

$\frac{L_2}{L_1} = \left(\frac{R_2}{R_1}\right)^2 \left(\frac{T_2}{T_1}\right)^4$
- Cool stars about 10^4 x more luminous than MS star with same Te. Implication $R_{RG} \simeq 10^2 \cdot R_{MS}$

(3) White Dwarfs
Hot stars, but $\sim 10^{-4}$ × luminosity of MS star with similar T_e. These must have smaller radii $\sim 10^{-2} R_{MS}$.

Star Clusters: While there is fair amount of scatter (i.e. noise) in HR diagram of solar neighborhood, the HR diagrams for star clusters are much cleaner.

Star Cluster: Groupings of stars, i.e., gravitationally bound units containing between 10^3 & 10^6 stars.



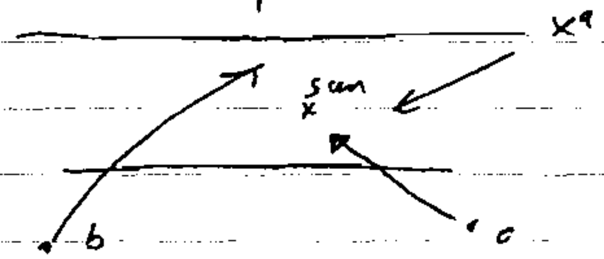
Empirical HR diagram

MS turnoffs:
- Cluster like NGC 2362 has MS stars all the way up to $M_V \simeq -7$, $B-V = -0.4$ (luminous + blue)

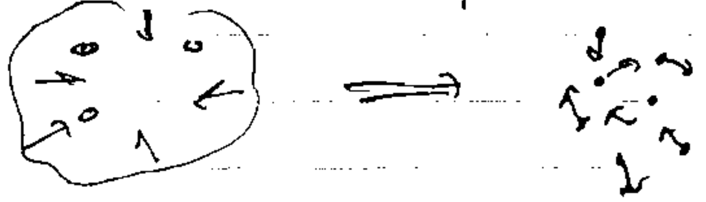
- By contrast, M67 has no MS stars more luminous than $M_V = +4$

Explanation: why are cluster HR diagrams cleaner than solar neighborhood HR diagrams?

- Solar Neighborhood: like an airport with stars passing by the sun that originate from completely different regions of the Milky Way they undoubtedly have different ages, and different chemical compositions



- Cluster: By contrast cluster stars formed out of the same gaseous cloud. think about those images of star clusters I showed. Pleiades To a good approximation, they have same age & same chemical composition



- Age Sequence: Hint at explanation for different MS turnoffs
  - NGC 2362, h+χ Persei, clusters with luminous MS turnoffs are associated with interstellar gas and dust (i.e., recently formed, young)
  - M67 cluster with underluminous turnoff has no gas or dust (older object)

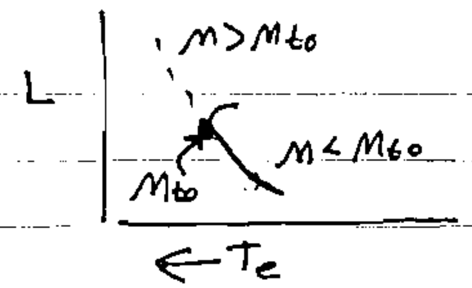
Implication: We are looking at an age sequence

Crude Explanation:

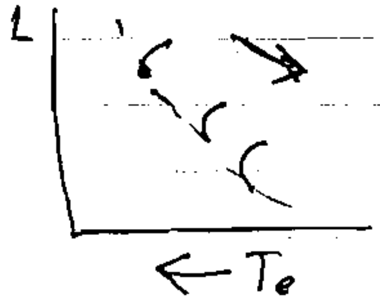
- *Energy Loss*: Rate of change of total energy E of Star is its luminosity $\frac{dE}{dt} = -L$
  - Assume $L \simeq$ const. during MS lifetime Δt
  - Integrating –
    $$\Delta E = -L \cdot \Delta t$$
    implying: $\Delta t = \frac{|\Delta E|}{L}$
  - I shall show that $|\Delta E|$ is proportional to total mass of the star: we shall see that $|\Delta E| \propto$ total nuclear binding energy of He nuclei
- *Mass luminosity Relationship*
  We have seen there is an emperical relationship
  $$L \propto M^{\alpha} \quad (\alpha \simeq 3\text{–}3.5)$$
- *MS lifetime*
  Define $\tau_{ms} \equiv \Delta t$ {we shall quantify ΔE later on ($\sim 10\%$ of M)}
  Therefore $\boxed{\tau_{ms} \propto \frac{M}{M^{\alpha}} = \frac{1}{M^{\alpha-1}}}$
  or $\boxed{\tau_{ms} \propto \frac{1}{M^{2.5}}}$

Thus higher-mass stars, UMS stars, evolve off the MS more rapidly than lower mass stars on the lower MS. As a result the location of the cluster MS turn off can be used to compute the age of a star cluster

HR Diagram



MS stars with masses $M > M_{to}$ do not sit on MS since their MS lifetimes $\tau_{ms}(M) < \tau_{ms}(M_{to})$, where $\tau_{ms}(M_{to}) = t_{age}$ of cluster.
Stars with $M < M_{to}$ have $\tau_{ms}(M) > \tau_{ms}(M_{to})$
As cluster ages $\tau_{ms}(M_{to})$ moves to lower masses and lower L. $\frac{1}{M^{\alpha-1}} \sim t_{age}$



as cluster ages location of turnoff moves to lower L(M and lower Te

Implication is that there is a universal initial main sequence -

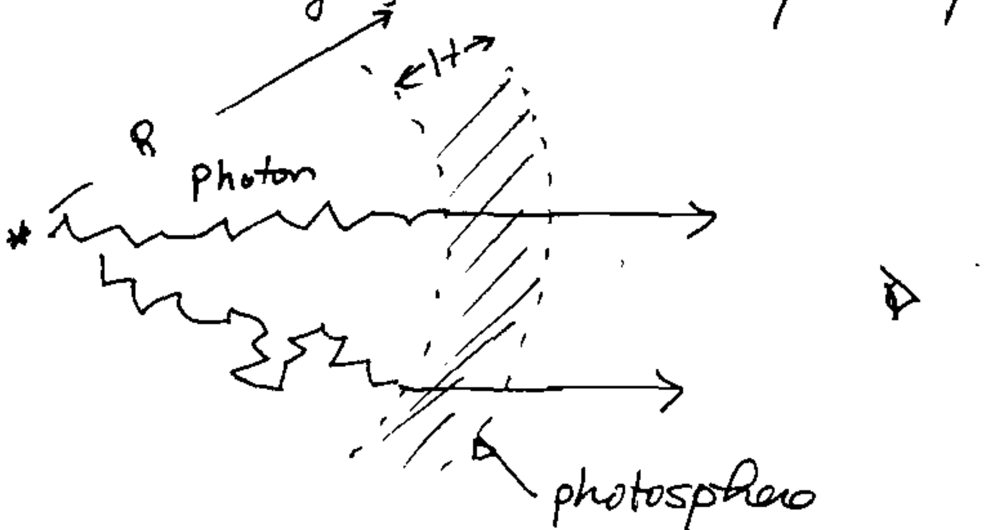
Radiative Transfer :

We are almost ready to discuss important topic of stellar interiors, but before I do that let me say a few ~~more~~ more words about stellar atmospheres and radiative transfer

~~Basic Quantities:~~
~~Intensity Recall definition..~~

Photospheres

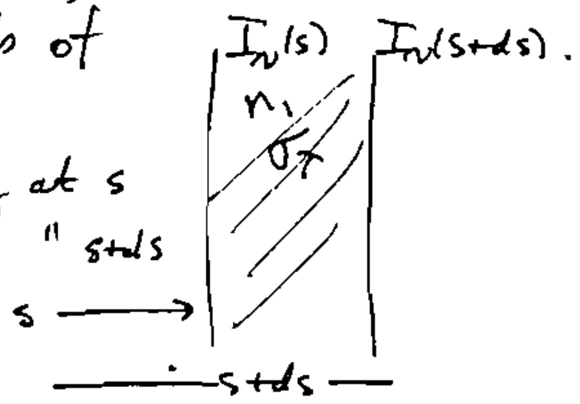
All the radiation emitted by stars propagates from a thin outer layer, the stellar photosphere -



To understand physics, let's consider the transfer of radiation.

Assume plane ∥ atmosphere, which is a good approximation, since $H \ll R$.

Consider a slab of thickness ds



- $I_\nu(s)$ is intensity at s and $I_\nu(s+ds)$ is " " $s+ds$
- gas in slab has density n
- at wavelength λ, absorption cross-section is σ_λ

Experiment: $I_\nu(s+ds) - I_\nu(s) \propto -I_\nu(s)\,ds$

Taylor Expansion:

$$\cancel{I_\nu(s)} + \left(\frac{dI_\nu}{ds}\right)_s ds - \cancel{I_\nu(s)} \propto -I_\nu(s)\,ds$$

Therefore $dI_\nu/ds \propto -I_\nu(s)$

Proportionality constant: $k_\lambda = n\sigma_K = \rho \kappa_{TR}$

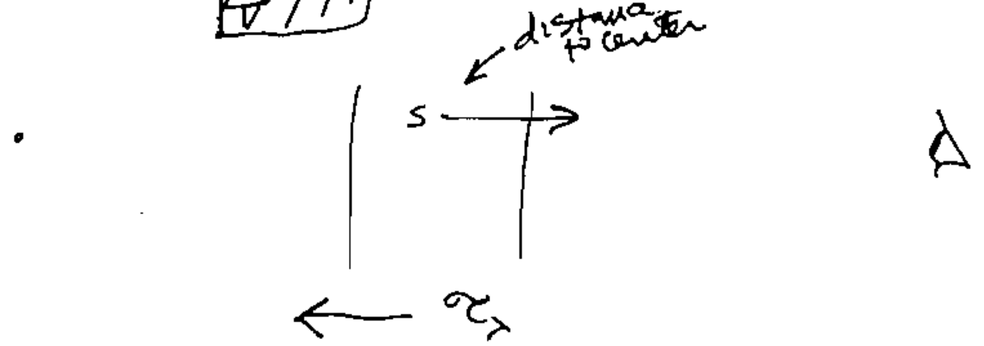
number density; cross-section; mass density; opacity

$$\boxed{\frac{dI_\nu}{ds} = -(n\sigma_\lambda) I_\nu}$$

Note, this equation assume there are <u>no sources</u> of radiation in the photosphere. Good approx. in photosphere, but not in the interior.

<u>Solution</u>: $\cancel{\int \frac{dI_\nu}{I_\nu}}$

Define optical depth: $d\tau_\lambda = -n\sigma_\lambda ds$



Rewrite above equation: $\dfrac{dI_\nu}{-n\sigma_\lambda ds} = I_\nu$

implies $\dfrac{dI_\nu}{d\tau_\nu} = I_\nu$ $(\tau_\nu = \tau_\lambda)$

<u>Integrate inward</u>: $\displaystyle\int_{I_\nu(0)}^{I_\nu(\tau_\nu)} \frac{dI_\nu}{I_\nu} = \int_0^{\tau_\nu} d\tau_\nu'$

$$\ln\left[\frac{I_\nu(\tau_\nu)}{I_\nu(0)}\right] = \tau_\nu$$

$$\frac{I_\nu(\tau_\nu)}{I_\nu(0)} = e^{\tau_\nu}$$

or $$\boxed{\underset{\substack{\uparrow \\ \text{observed}}}{I_\nu(0)} = I_\nu(\tau_\nu)e^{-\tau_\nu}}$$

$e^{-\tau_\nu}$ is a "guillotine" factor, which because of exponential cutoff, prevents from seeing beyond $\tau_\nu \sim 1$ ($\tau_\nu = 2/3$ is more accurate)

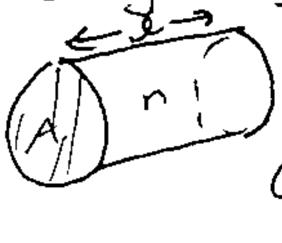
<u>Mean-free path</u>: That is $\tau_\nu = n\sigma_\lambda s = 1$

$$\Rightarrow \quad s = l_\lambda = \frac{1}{n\sigma_\lambda} \quad (\text{m.f.p.})$$

distance photon ~~travels~~ travels between scatterings.

Different look at..

Let photon travel through cylinder of matter with length l and surface area A. ~~to~~



Total number of atoms:

$$N = n \cdot A \cdot l$$

Column density $\frac{N}{A} = nl$ (~~atoms~~ atoms per unit area.

Projected area occupied by each atom: $\mathcal{A} = \frac{A}{N} = \frac{1}{nl}$

photons guaranteed to to be absorbed when $\sigma_\lambda \simeq \mathcal{A}$

~~[illegible]~~ Sky is opaque when

when n $\boxed{\cancel{[illegible]} \simeq l_\lambda}$

when $\mathcal{A} \simeq \sigma_\lambda$

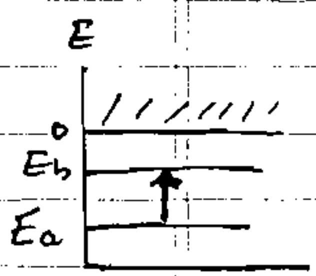
then $l_\lambda = \frac{1}{n\sigma_\lambda} \simeq l$ (size > m.f.p.)

~~[illegible] $\frac{1}{n\sigma_\lambda}$~~

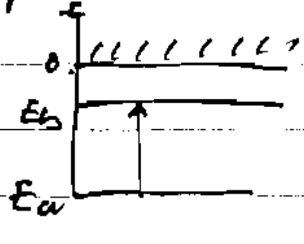
Evaluating I_λ:

Size of I_λ depends on physical process:

Bound-Bound Transitions: These are important in low-density photosphere where cooler temperatures result in low ionization. Most nuclei keep their bound electrons.

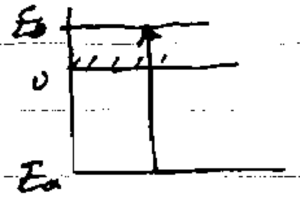


or

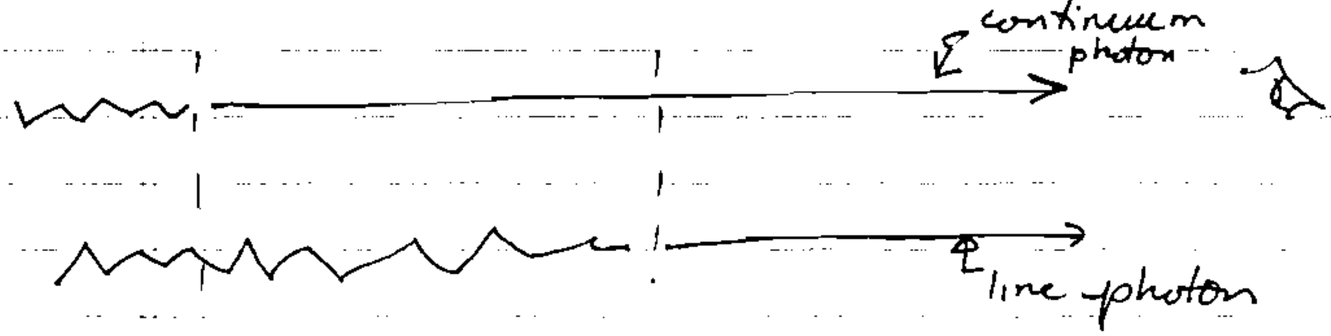


So in photosphere: H atoms normally in ground state or CaII atoms start in gd. state

Bound-Free: Transition



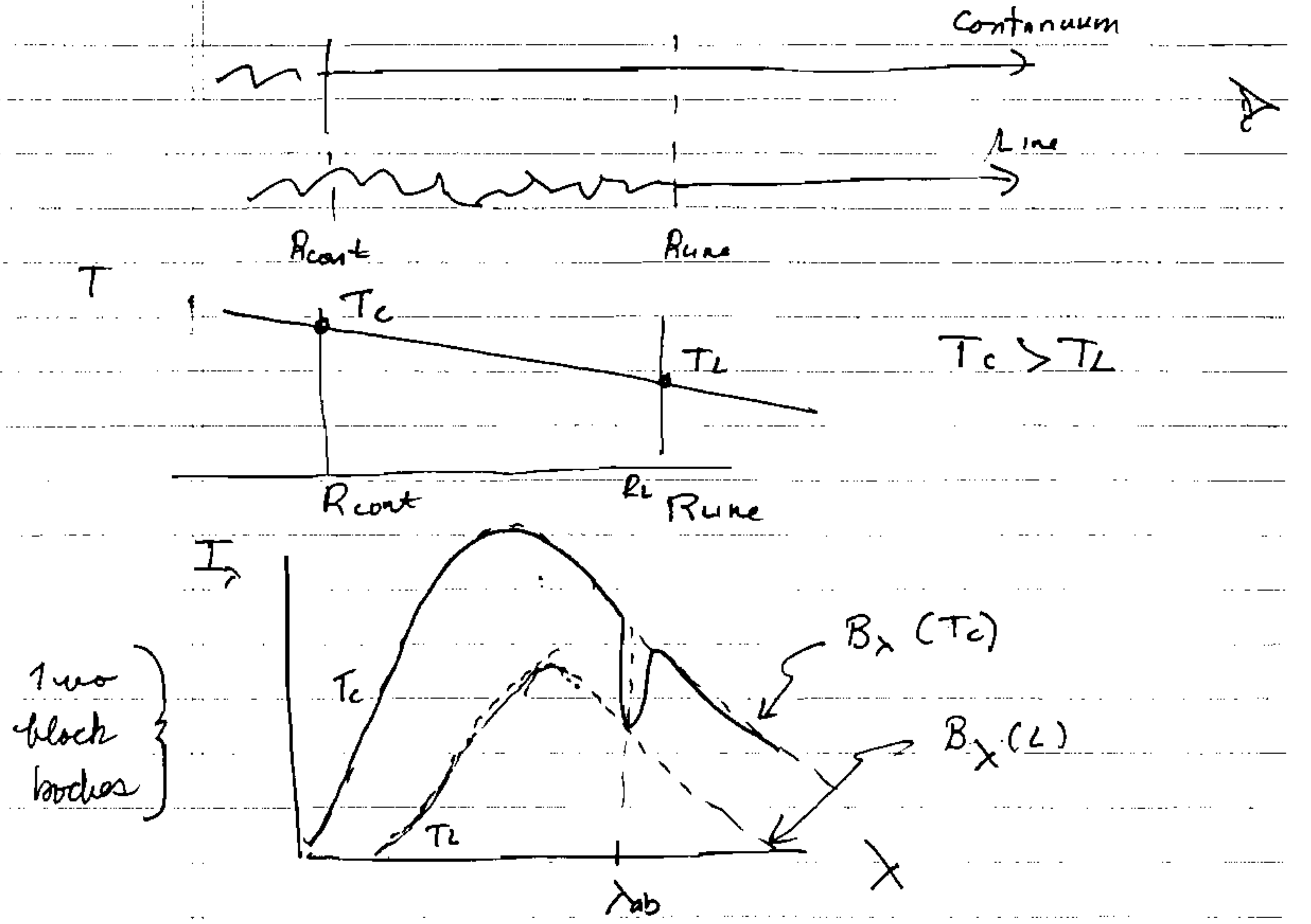
Important point is that $\sigma_{bb} \gg \sigma_{bf}$, therefore $(\tau_\nu)_{bb} \gg (\tau_\nu)_{bf}$



So we can see deeper into the star at wavelengths away from those corresponding to discrete transitions; i.e.,

$$\nu \neq \nu_{ab} = \frac{E_b - E_a}{h}$$

But we will see shortly that temperature of the star increases with decreasing radius; i.e., $dT/dR < 0$



~~[illegible]~~

at most values of λ we see back to $R = R_{cont}$ where $T = T_c$; "hot" black body. But at $\lambda = \lambda_{ab}$ ~~[illegible]~~ increased opacity allows us only to see back to R_{Line} where temperature is cooler.

This is why absorption lines form in stellar photosphere.

Stellar Interiors (Chp. 10)

So far everything I said follows directly from observed properties of the stars; i.e., from information obtained through observations of light escaping from photospheres.

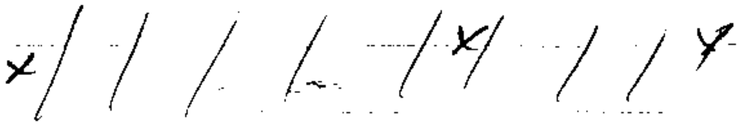
Let's see what we can learn about stellar interior.

Density. what is mean density of the sun

$$\langle\rho\rangle_\odot = \left(\frac{M}{V}\right)_\odot = \frac{M_\odot}{4\pi R_\odot^3/3}$$

$$\langle\rho\rangle_\odot = \frac{2\times10^{33}\,g}{(4\pi/3)(7\times10^{10})^3} \simeq 1.4\ g\ cm^{-3}$$

A lot is made of the fact that this density is similar to the mean density of H_2O. But that comparison is misleading. The sun is not a liquid. Rather it is an ideal gas. More meaningful comparison is with another ideal gas that pervades the ~~medium be~~ space between the stars; i.e., the interstellar medium.



$$\langle\rho\rangle_{ISM} \simeq 2\times10^{-24}\,g/cm^3$$

Let's look more closely at stellar interior

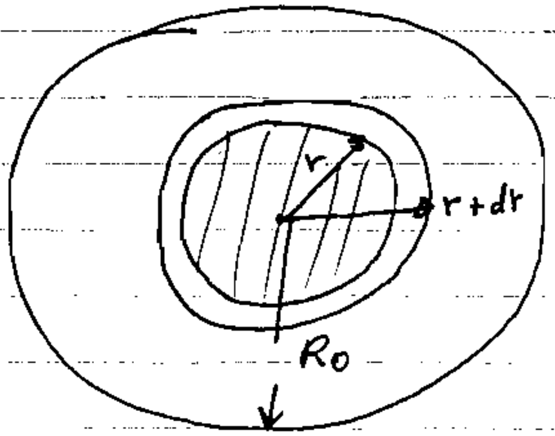
Pressure and temperature

Sun is in state of hydrostatic equilibrium. It neither contracts nor expands. Rather it is in a state of balance between two forces:

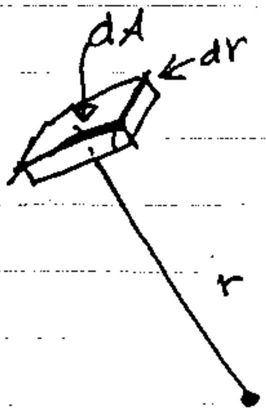
(1) short-range pressure force

(2) long-range gravitational force

Assuming spherical symmetry, let's break sun up into concentric thin spherical shells



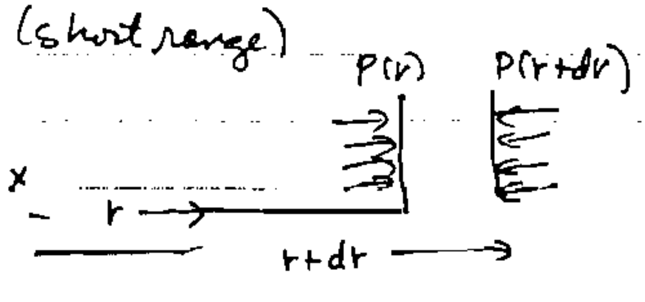
Consider these forces to act on a volume element of that shell



thickness: dr

Area : dA

(1) Pressure (short range)



Pressure on each ~~[illegible]~~ face of volume element due momentum transfer through collisions by particles moving to the right (at r) and to the left (at r + dr)

Net outward pressure force

$$dF_p = dA\left[P(r) - P(r+dr)\right]$$

where $P(r)$ is scalar pressure.

Taylor Series: Since $dr \ll r$, let

ignore

$$P(r+dr) = P(r) + \left(\frac{dP}{dr}\right)dr + \cdots O(dr^2) + \cdots$$

As a result: $dF_p = dA\left[P(r) - P(r) - \left(\frac{dP}{dr}\right)dr\right]$

or $\boxed{dF_p = +dA(-dP) = -dA\,dP}$

Crucial point: positive outward dF_p only if $dP/dr < 0 \Rightarrow$ negative pressure gradient.

(2) Gravity (long Range)

For spherical body, only mass interior to r exerts net gravitational force on our volume element or mass element dm

where $dM = \rho\, dV$ where $dV = dA \cdot dr$

$$dF_G = -\frac{GM(r)}{r^2}\,dm$$

$$\boxed{\therefore\ dF_G = -\frac{GM(r)\rho\,dA\cdot dr}{r^2}}$$

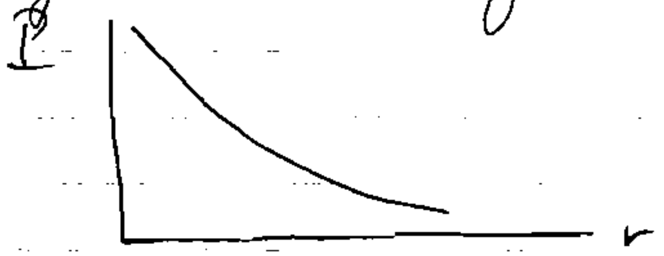
(3) <u>Force balance</u>

$$dF_p + dF_G = 0$$

$$-\cancel{d}A\,dP - \frac{GM\rho\,\cancel{d}A\,dr}{r^2} = 0$$

Result: $\boxed{\dfrac{dP}{dr} = -\dfrac{GM(r)\rho(r)}{r^2}}$

Basic eq. of stellar structure. negative pressure gradient required to balance inward gravitational force.

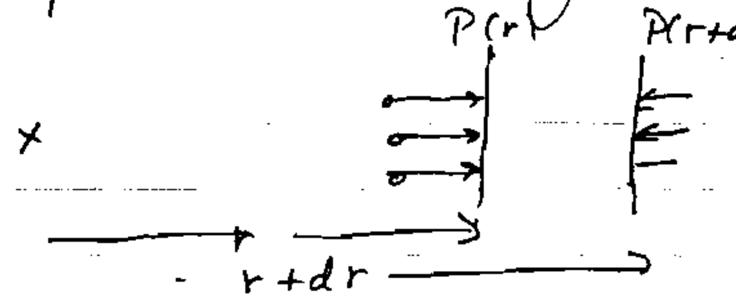


Recap:

Hydrostatic Equilibrium

Last lecture we starting investigating the hydrostatic equilibrium that is maintained in the sun and in other stars. This is a balance between outward pressure forces and inward gravitational force:

(1) Pressure force: Depends on difference between outward pressure on inner face minus inward pressure on outer face:

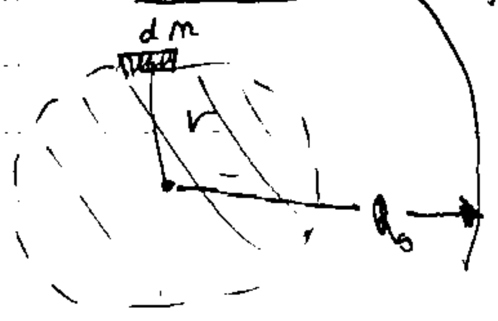


outward pressure force on A

$$dF_P = dA \cdot (P(r) - P(r+dr))$$

$$dF_p = dA\,(-dP/dr)\,dr$$

(2) Gravitational force: Only mass within r exerts force on dM in case of spherical symmetry



$$dF_G = -\frac{GM(r)}{r^2}\,dM$$

since $dM = \rho\, dV = \rho\, dA \cdot dr$

we found: $dF_G = -\frac{GM(r)}{r^2}\,\rho\, dA\, dr$

(3) Hydrostatic equilibrium condition:

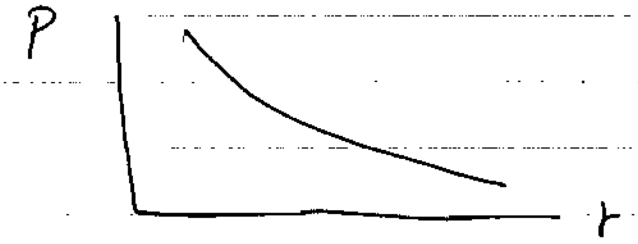
Net force on dM vanishes. So . . .

$$dF_P + dF_G = 0$$

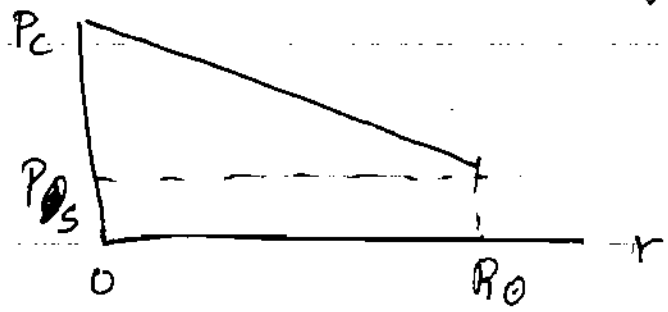
$$\cancel{dA\,dr}(-dP/dr) - \frac{GM(r)}{r^2}\,\rho\,\cancel{dA\,dr} = 0$$

$$\therefore \quad \boxed{\frac{dP}{dr} = -\frac{GM(r)\,\rho(r)}{r^2}} \quad (1)$$

Since G, M, ρ, r^2 are all positive, we see that hydrostatic equilibrium requires negative pressure gradient; i.e., $dP/dr < 0$.



Central Pressure: We can use eq. (1) to make a crude estimate of central pressure (at $r=0$). Assume $P(r)$ decreases linearly with radius



In that case $\frac{dP}{dr} = \frac{P_\odot - P_c}{R_\odot} \simeq -\frac{P_c}{R_\odot}$

Since we assume $P_s \ll P_c$.

From eq. (1) $-\frac{P_c}{R_\odot} \simeq -\frac{G M_\odot}{R_\odot^2} \langle\rho\rangle_\odot$

$$\boxed{P_c \simeq \frac{G M_\odot}{R_\odot} \langle\rho\rangle_\odot} \quad (1a)$$

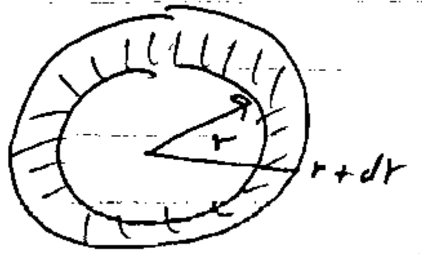
Recall $M_\odot = 2\times10^{33}$ g, $\langle\rho\rangle_\odot = 1.4$ g cm^{-3}, $R_\odot = 7\times10^{10}$ cm.

$\Rightarrow P_c = 2.68\times10^{15}$ dynes/cm² $\left(2.68\times10^{14} \frac{N}{m^2}\right)$

More rigorous estimate show $\rho(r)$ increases more steeply as r decreases. So integration of eq. (1) gives
$P_c \simeq 2.3\times10^{17}$ dynes/cm²
On earth atmospheric pressure is
$1\,atm \simeq 1\times10^6$ dynes/cm²

$\therefore P_c \simeq 10^{11}$ atm!

Second eq. of stellar structure: Mass conservation

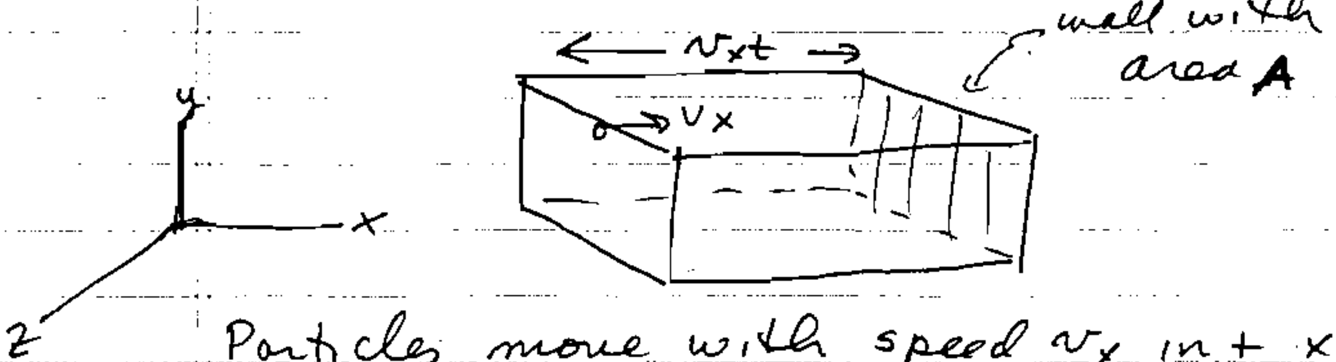


Mass in shell
$dM = \rho\, dV$
$dM = \rho \cdot 4\pi r^2 dr$

$$\boxed{\frac{dM}{dr} = 4\pi r^2 \rho(r)} \quad (2)$$

Temperature and ideal gas law

Consider the pressure on an imaginary wall



Particles move with speed v_x in $+x$ direction transfer momentum across area A in time t.

Only particles in box make it to the wall in time t.

Number of particles in the box: $N = \overset{\text{density}}{n}(\underbrace{A \cdot v_x t}_{V})$

Each equal-mass particle has momentum p_x.

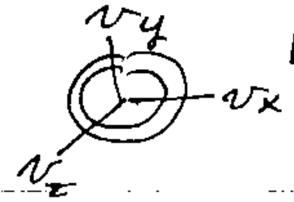
Total Momentum delivered: $P_x = N \cdot p_x$

$$P_x = n A \cdot v_x t \cdot p_x$$

$$\frac{\text{Force}}{\text{Area}} = \frac{P_x}{t \cdot A} = n v_x p_x = \text{Pressure} \rightarrow \mathcal{P}$$

In realistic case not all particles have same velocity. So take average over some distribution

$$\mathcal{P} = n\langle v_x p_x \rangle$$



For isotropic distribution:

$$\langle v_x p_x \rangle = \langle v_y p_y \rangle = \langle v_z p_z \rangle$$

or $\underset{\sim}{v} \cdot \underset{\sim}{p} = v_x p_x + v_y p_y + v_z p_z = 3 v_x p_x$

$$\therefore \langle v_x p_x \rangle = \frac{1}{3} \langle v p \rangle$$

$$\therefore \underline{P} = \frac{1}{3} n \langle v p \rangle$$

For non-relativistic gas $p = mv$

$$\therefore \boxed{P = \frac{1}{3} n \langle m v^2 \rangle = \frac{2}{3} n \langle \tfrac{1}{2} m v^2 \rangle} \quad (2b)$$

Integrate over Maxwellian:

$$\underline{P} = \frac{2}{3} \int \frac{m v^2}{2} n_v(v) dv \quad ; \quad n = \int n_v(v) dv$$

Recall: $n_v(v) dv = n \left(\frac{m}{2\pi k T} \right)^{3/2} e^{-\frac{m v^2}{2kT}} 4\pi v^2 dv$

$$\langle \tfrac{1}{2} m v^2 \rangle = \frac{3}{2} kT \quad , \quad \langle \tfrac{1}{2} m v^2 \rangle = \frac{\int \frac{1}{2} m v^2 n_v dv}{n}$$

$$\Rightarrow \boxed{\underline{P} = n k T} \quad (3)$$

<u>Mean molecular weight</u>:

Let's put eq. of state in terms of mass density ρ.

$n = \frac{\rho}{\bar{m}}$: $\bar{m}$ is average mass of free particle.

$$\therefore \underline{P} = \frac{\rho \cdot kT}{\bar{m}}$$

Mean molecular weight $\mu = \frac{\bar{m}}{m_H}$

i.e., mass of average free particle in units of hydrogen masses.

$$\therefore \boxed{P = \frac{\rho k T}{\mu m_H}} \quad (4)$$

~~[illegible]~~
We'll work this out in more detail later, but let's take sample case of pure H and assume gas is ionized. Therefore we have 2 free particles per H mass.

$$\therefore \mu = \frac{\frac{1}{2}(m_e + m_H)}{m_H} \simeq 1/2$$

$$\therefore \text{In this case } P = \frac{2 \rho k T}{m_H}$$

Central temperature:

Assume sun is ~~[illegible]~~ composed only of ionized H.

(1) Hydrostatic Equilibrium ~~[illegible]~~ $P_c \simeq \frac{G M_\odot}{R_\odot} \langle \rho \rangle_\odot$ (1a)

(2) Eq. of state $\Rightarrow P_c = \frac{2 \langle \rho \rangle_\odot k T_c}{m_H}$

Equating (1) & (2) we get

$$\therefore T_c \simeq \frac{G M_\odot}{R_\odot} \times \frac{m_H}{k} \quad \text{(forget factor of 2)}$$

Numerical estimate

$$T_c \doteq \frac{6.7\times10^{-8}\times 2\times10^{33}}{7\times10^{10}} \times \frac{1.67\times10^{-24}}{1.38\times10^{-16}}$$

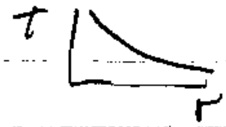
$$\boxed{T_c \simeq 2\times10^{7}\,K}$$

or $T_c \simeq 2\times10^{7}(M/M_\odot)(R_\odot/R)\,K$

Recall ~~Te~~ photosphere temperature at surface $T_e \simeq 5800\,K$. As a result

$$T_c/T_e \simeq \frac{2\times10^{7}}{.6\times10^{4}} \simeq 3500$$

So temperature also has negative radial gradient ($dT/dr < 0$).



Recall $P_c \simeq 2.3\times10^{17}$ dynes/cm^2

$$P_{surface} \simeq \frac{\rho k T}{m_H} \simeq 10^{8}\times 5\times10^{-7}\times 5800K \simeq 3\times10^{5}\,\frac{dynes}{cm^2}$$

$$\therefore \frac{P_c}{P_{surface}} \simeq \frac{2.3\times10^{17}}{3\times10^{5}} \simeq 10^{12}$$

much ~~stronger~~ steeper gradient in P than in T

Precision of Hydrostatic Equilibrium

Suppose star deviates from hydrostatic equilibrium. What are the consequences?
Let's return to net force on mass element dM.

$$dF_P + dF_G = dM\,\frac{d^2r}{dt^2} \quad \Big\{ \text{Previously set to zero}$$

$$dM\,\frac{d^2r}{dt^2} = -dP\,dA - \frac{GM(r)\,dM}{r^2}$$

$$\frac{d^2r}{dt^2} = -dP\left(\frac{dA}{dM}\right) - \frac{GM(r)}{r^2}$$

But $dM = \rho \cdot dA \cdot dr$

therefore $$\frac{d^2r}{dt^2} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2}$$

Rewrite: $$\frac{d^2r}{dt^2} = \frac{-GM(r)}{r^2}\left[1 + \frac{\frac{1}{\rho}\frac{dP}{dr}}{GM/r^2}\right]$$

Since $\frac{dP}{dr} < 0$, let $\frac{dP}{dr} = -\left|\frac{dP}{dr}\right|$

As a result: $$\frac{d^2r}{dt^2} = -\frac{GM(r)}{r^2}\left[1 - \frac{\left|\frac{1}{\rho}\frac{dP}{dr}\right|}{GM/r^2}\right]$$

Let $\alpha \equiv \left|\frac{1}{\rho}\frac{dP}{dr}\right| \Big/ GM(r)/r^2$

$$\frac{d^2r}{dt^2} = \frac{-GM(r)}{r^2}(1-\alpha)$$

In hydrostatic equilibrium, $\alpha = 1$ since $\frac{d^2r}{dt^2} = 0$

Suppose $\alpha = 0.9$; ~~[illegible]~~. What happens?

Order of magnitude Estimate

$$\left[\frac{d^2r}{dt^2}\right] \sim \frac{R_\odot}{T_{collapse}^2} \;;\; \left[\frac{GM}{r^2}\right] \sim \frac{GM_\odot}{R_\odot^2}$$

In that case eq. of motion becomes –

$$\frac{R_\odot}{T_{collapse}^2} \sim \frac{GM_\odot}{R_\odot^2}(1-\alpha)$$

or $$T_{collapse} \sim \sqrt{\frac{R_\odot^3}{GM_\odot \times (1-\alpha)}}$$

Rewrite: $\langle\rho\rangle \simeq \frac{M_\odot}{(4\pi/3)R_\odot^3}$ or $\frac{R_\odot^3}{M_\odot} = \frac{3}{4\pi\langle\rho\rangle}$

As a result:

$$T_{collapse} \sim \sqrt{\frac{3}{4\pi G(1-\alpha)\langle\rho\rangle}}$$

More accurate expression:

$$T_{collapse} = \sqrt{\frac{3\pi}{32G(1-\alpha)\langle\rho\rangle}}$$

Numerical Estimate: $\langle\rho\rangle = 1.4\,g\,cm^{-3}$

$$T_{collapse} = \frac{0.58\,hr}{\sqrt{1-\alpha}}$$

So for $\alpha = 0.9$ or $1-\alpha = 0.1$, $T_{collapse} \simeq 1.8\,hr$.

Sun would collapse in an exceedingly short time. Results would be catastropic given its stability for $t_{age} \simeq 4.6 \times 10^9 y$

Thus $T_{collapse} \gg 4.6 \times 10^9 y$

Solve for $1-\alpha$:

$$1 - \alpha = \frac{3\pi}{32 G \langle \rho \rangle T_{collapse}^2}$$

So $T_{collapse} \gg 4.6 \times 10^9 \times 3 \times 10^7 = 1.4 \times 10^{17} s$

$$1 - \alpha \ll \frac{3 \cdot \pi}{32 \cdot 6.7 \times 10^{-8} \times 1.4 \times (1.4 \times 10^{17})^2}$$

$$1 - \alpha \ll 1.6 \times 10^{-28} \; ! \, !$$

or $0.99999 \ldots < \alpha < 1$

28 decimal places

Conclusion: Hydrostatic equilibrium in the sun must be obeyed to incredible precision. Interestingly the sun is very stable. Any deviation from equilibrium last for only 1 free-fall time of $\cancel{T}_{f.f} = \sqrt{\frac{3\pi}{32 G \rho}} \sim 0.5 \text{ hr}$.

Reason: contraction raises pressure & hydrostatic equilibrium is restored.

$$\left\{ \begin{array}{l} t_{sound} \approx \frac{R_\odot}{c_s} = \frac{R_\odot}{\sqrt{\frac{kT_c}{m_H}}} \\ \approx t_{FF} \approx \sqrt{\frac{R_\odot^3}{GM}} \end{array} \right.$$

Virial Theorem:

Let's go back to equation of hydrostatic equilibrium.

$$\frac{dP}{dr} = -\frac{G M(r) \rho(r)}{r^2}$$

mult. by $4\pi r^3$ and integrate from $r = 0 \rightarrow R_s$.

$$\int_0^{R_s} dr \cdot 4\pi r^3 \frac{dP}{dr} = -G \int_0^{R_s} dr \cdot 4\pi r^3 \frac{M(r)\rho(r)}{r^2}$$

Integrate LHS by parts

$$\int_0^{R_s} 4\pi r^3 dP = \left[4\pi r^3 \cdot P\right]_0^{R_s} - \int_0^{R_s} P \cdot 3\times 4\pi r^2 dr$$

the boundary term vanishes: $P(R_s)=0$

$dV = 4\pi r^2 dr$

As a result: $-\int P \cdot 3dV = -G\int dr \cdot \frac{4\pi r^3 M(r)\rho(r)}{r^2}$ divide by r

$$= -G\int \frac{M(r)\cdot\rho(r)4\pi r^2 dr}{r}$$

But Potential Energy $\Omega_G = -G\int \frac{M(r)\,dm}{r}$

or $\boxed{-\int 3P\,dV = \Omega_G}$

Recall $\underline{P} = \frac{2}{3} n \langle \frac{1}{2} m v^2 \rangle$ (from. eq. 26)

Let $u = \langle \frac{1}{2} m v^2 \rangle$ = average thermal kinetic energy <u>per particle</u>.

$U = nu$ = thermal kinetic <u>energy density</u>

From last equation:

$$-\int 3P\,dV = -\int 3\left[\frac{2}{3} nu\right] dV = -\int 2U\,dV$$

$$\Rightarrow -\int 3P\,dV = -2E_{KE}$$ total kinetic energy

As a result:

$$-2E_{KE} = \Omega_G \quad \text{or}$$

Virial theorem $\boxed{2E_{KE} + \Omega_G = 0} \quad \boxed{E_{KE} = -\frac{1}{2}\Omega_G}$

Explicit Expressions:

(1) $u = \frac{3}{2} kT$

(2) $\mathcal{U} = nu = \frac{3}{2} nkT$

(3) $E_{KE} = \int \mathcal{U} \, dV = \frac{3}{2} NkT$ ← total no. particles

Numerical Estimates

(a) Kinetic Energy: $E_{KE} = \frac{3}{2} NkT$

$T \sim 10^7 K$; $N \simeq \frac{M}{m_H} = \frac{2\times10^{33}}{1.67\times10^{-24}} \sim 10^{57}$ H nuclei

$E_{KE} \sim 1.5 \times 10^{57} \times 1.38\times10^{-16} \times 10^7 \sim 2.5\times10^{48}$ ergs

(b) Potential Energy: $\Omega_G \sim -\frac{GM_\odot^2}{R_\odot}$

$$\Omega_G = \frac{-6.7\times10^{-8}(2\times10^{33})^2}{7\times10^{10}} = -3.8\times10^{48} \text{ ergs}$$

Consequences of the Virial Theorem

$$E = E_{KE} + \Omega_G$$

Suppose star radiates away energy $|\Delta E|$ in time interval Δt. Net energy change should $\Delta E < 0$.

But virial theorem implies

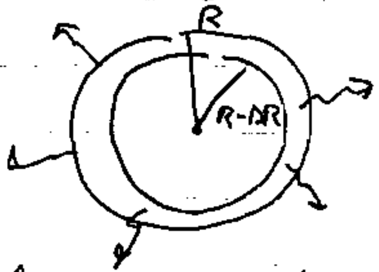
$$E = -\frac{1}{2}\Omega_G + \Omega_G = \frac{1}{2}\Omega_G$$

$$\therefore \quad \Delta E = \frac{1}{2}\Delta\Omega_G$$

Since $\Delta E < 0$, then $\Delta\Omega_G$ must also be less than zero. But for any model

$$\Omega_G = -\frac{\xi G M^2}{R} \quad (\xi \simeq 1)$$

Since M is constant, R must decrease!



Ω_G becomes more negative

What happens to ~~[illegible]~~ change in potential energy?

(A) $\Delta E = \frac{1}{2}\Delta\Omega_G$ $\therefore$ 50% of decrease in Ω_G is lost in form of radiation.

(B) $E_{KE} = -\frac{1}{2}\Omega_G$

Therefore: $\Delta E_{KE} = -\frac{1}{2}\Delta\Omega_G$

Other 50% of released gravitational potential energy goes into increasing thermal kinetic energy of the star; i.e., $\Delta E_{KE} > 0$.

~~[illegible]~~ Too much Potential energy release to be radiated. Rest heats up star.

<u>Result</u>: Star heats up. T rises! as star loses energy. Star effectively has negative heat capacity.

$$\Delta Q = C_v \Delta T$$

$\Delta Q < 0$ as energy is lost.

But $\Delta T = \dfrac{\Delta Q}{C_v}$

Normally if $C_v > 0$, $\Delta T < 0$ if $\Delta Q < 0$

But in star $C_v < 0$, thus $\Delta T > 0$, if $\Delta Q < 0$