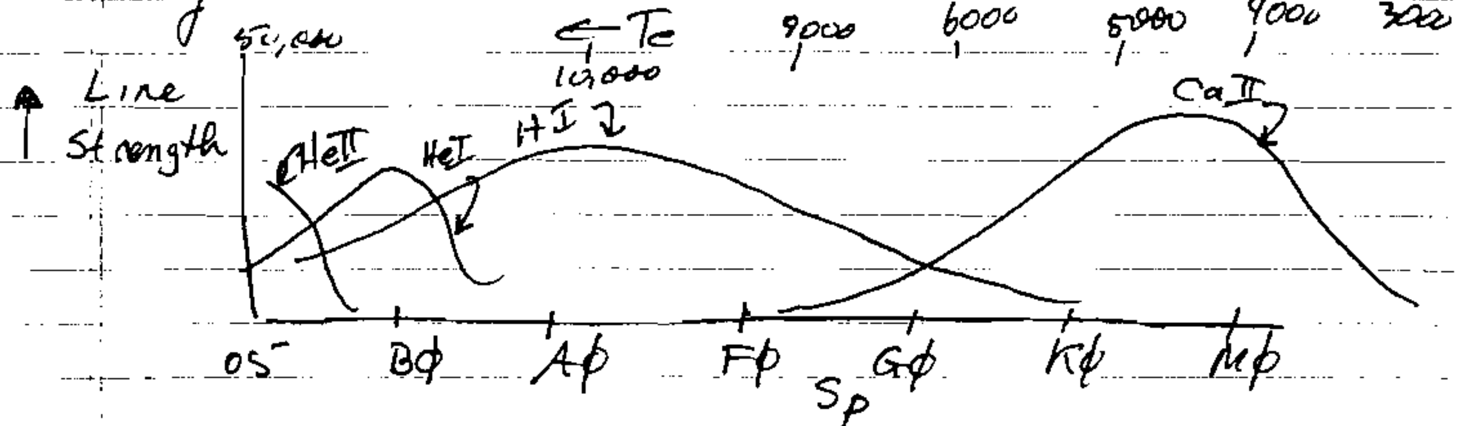
Classification of Stellar Spectra

Recap: Last lecture we discussed systematics of stellar spectra. We want to understand why absorption-line strengths depend on Sp (or equivalently on T_e) in the way they do.



Basic Physics

(1) distribution for fraction of free particles in kinetic energy interval $(\varepsilon, \varepsilon + d\varepsilon)$: translational deg. of freedom

Maxwellian:

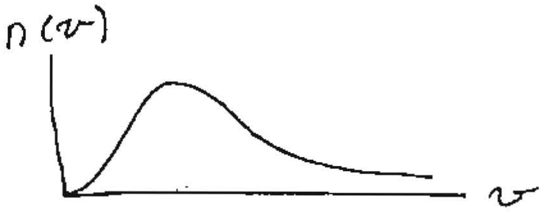
(2) Probability of finding atom in a given excited bound state

(3) Fraction of element in a given state of ionization.

Translational dof: (1)

$$n_v(v)\,dv = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left(-\frac{\frac{1}{2}mv^2}{kT}\right)4\pi v^2\,dv$$

Maxwellian



L4-2

RMS speed: $\langle v^2 \rangle = \dfrac{\int v^2 n_v(\vec{v})\,dv}{n} = \dfrac{3kT}{m}$

Examples

$$v_{rms} = \langle v^2 \rangle^{1/2} = \sqrt{\frac{3kT}{m}}$$

(a) H atoms at $T = 10^4$ K

$$v_{rms}^{(H)} = \sqrt{\frac{3 \times 1.38\times10^{-16}\times10^4}{1.67\times10^{-24}}} = 1.57\times10^6 \text{ cm/s}$$

$$= 15.7 \text{ km/s}$$

(b) e^- at $T = 10^4$:

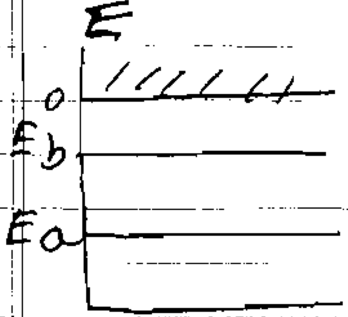
$$v_{rms}(e) = \sqrt{\frac{m_H}{m_e}}\, v_{rms}(H) = 43\, v_{rms}(H)$$

$$v_{rms}(e) = 677 \text{ km/s}$$

Bound States (2)

- Bound states excited and de-excited by
  - (a) transferring energ to and from free particle translational d.o.f.
  - (b) In T.E. principle of detailed balance says rate of excitation = rate of de-excitation.
  - (c) Let S_a stand for a specific set of quantum nos. (In H they are).
    - n : principle quant no.
    - l : magnitude of orbital angular momentum
    - m_l : projection of ang. momentum along z axis.
    - m_s : z component of e^- spin ($\frac{1}{2}$, $-\frac{1}{2}$)

In TE, ratio of probabilities of e^- occupying state b relative to state a:



$$\boxed{\frac{P(S_b)}{P(S_a)} = \exp\left[-\left(\frac{E_b - E_a}{kT}\right)\right]}$$

(a) For a given T, higher energy states are less likely to be occupied than lower energy states, since $E_b > E_a$.

(b) Also as $T \to 0$ $P(S_b)/P(S_a) \to 0$.
Cold gas contains too few e^- to transfer energy and excite upper energy levels.

(c) T is same temperature characterizing Maxwellian.

Degeneracy: In H as in most elements, many quantum states have the same energy. Those states are degenerate.

Let g_b, g_a be no. of states with energies E_b, E_a. In that case

$$\frac{P(S_b)}{P(S_a)} = \frac{g_b}{g_a} \exp\left[-\left(\frac{E_b - E_a}{kT}\right)\right]$$

Same as ratio of number of atoms in state b relative to a.

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} \exp\left[-\left(\frac{E_b - E_a}{kT}\right)\right]$$

Since $N_b = n\,P(S_b)$, $N_a = n\,P(S_a)$: n = total no. of atoms in vol.

Example: Hydrogen:

(a) No. of states with same principal quantum no. n: $g_n = 2n^2$

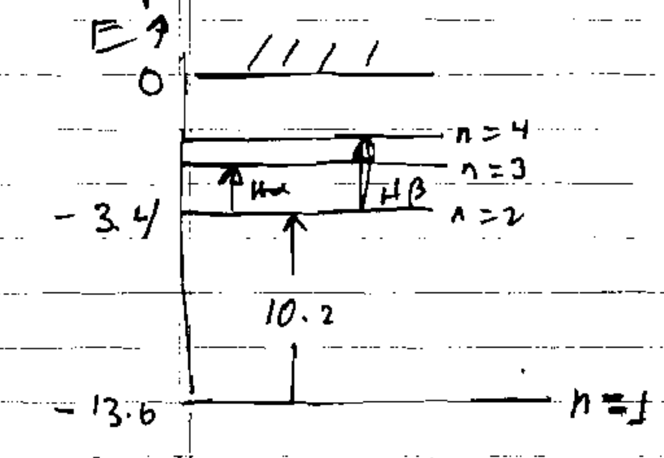
(b) But $E_n = -\frac{\chi_H}{n^2}$. Energy only depends on n

Here $\chi_H = 13.6\,eV$ (IP of H).

E does not depend on m_s, m_ℓ, ℓ.

n	ℓ	m_ℓ	m_s	E (eV)	g_n
1	0	0	1/2	−13.6	2
1	0	0	−1/2	−13.6	
2	0	0	1/2	−3.4	8
2	0	0	−1/2	"	
2	1	1	+1/2	"	
2	1	1	−1/2		
2	1	0	+1/2		
2	1	0	−1/2		
2	1	−1	1/2		
2	1	−1	−1/2		

Example Excitation of $n = 2$ level in neutral hydrogen H^0



$E_2 - E_1 = 10.2\,eV$

• First: work out kT in units of eV

$$kT = \frac{\left(1.38\times10^{-16}\times10^{4}\left(\frac{T}{10^4}\right)\right)}{1.6\times10^{-12}}$$

$$kT = 0.86\,(T/10^4)\ eV$$

• Second: Compute n_2/n_1 ratio

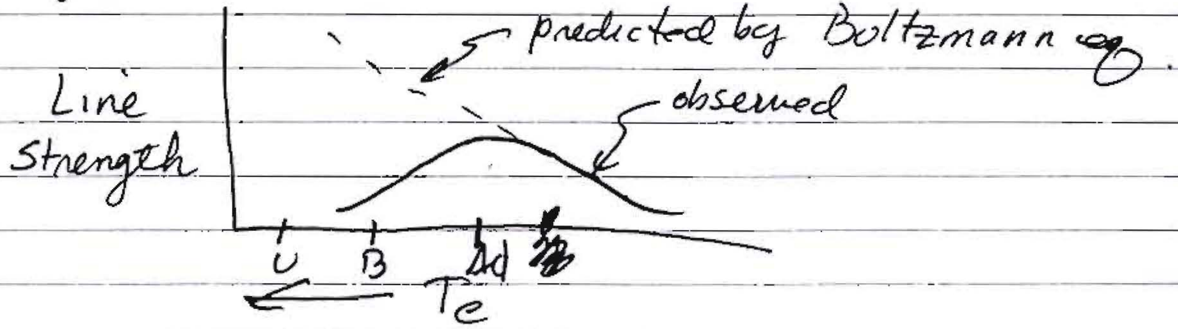
$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\left(\frac{E_2-E_1}{kT}\right)\right] = \frac{2(2)^2}{2(1)^2}\exp\left[\frac{-10.2}{.86(T/10^4K)}\right]$$

$$\frac{n_2}{n_1} = 4\exp\left[-\frac{11.9}{(T/10^4 K)}\right]$$

• **Puzzle**: We found that $H\alpha$, $H\beta$ reached maximum strength in A0 stars where $T \simeq 9500\,K$.

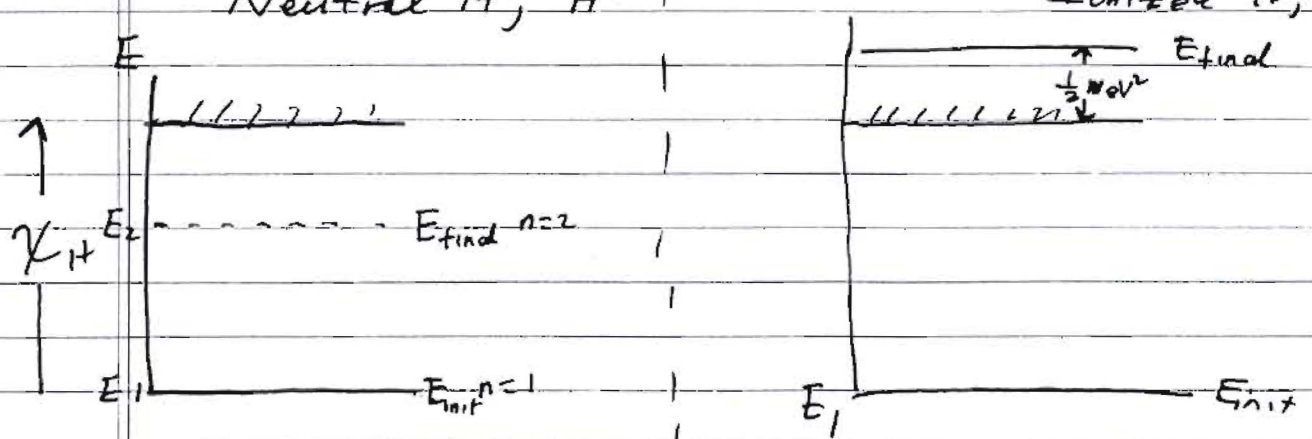
But $\frac{n_2}{n_1}(T = 9500K) = 4\exp\left[-\frac{11.9}{.95}\right] \simeq 1.5\times 10^{-5}$!

Question: why aren't $H\alpha$; $H\beta$ absorption lines stronger in stars with $T \gg 9500\,K$? Say in O or B stars with higher T?



To answer this question we generalize Boltzmann equation for population ratios to include unbound states, i.e., we

Neutral H; H^0 | Ionized H, H^+



$$E_2 - E_1 = -\frac{\chi_H}{(2)^2} - \left(-\frac{\chi_H}{(1)^2}\right)$$

$$E_f - E_{init} = E_2 - E_1 = \frac{3}{4}\chi_H$$

$$E_{final} - E_{init} = \frac{1}{2}m_e v^2 - (-\chi_H)$$

$$E_{final} - E_{imt} = \frac{1}{2}m_e v^2 + \chi_H$$

$$\frac{n_2}{n_1} = 4 \exp\left[-\frac{11.9}{(T/10^4 K)}\right]$$

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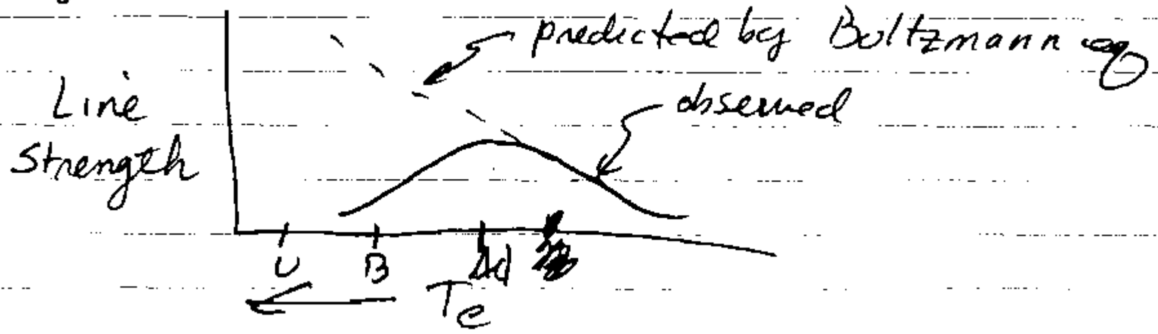

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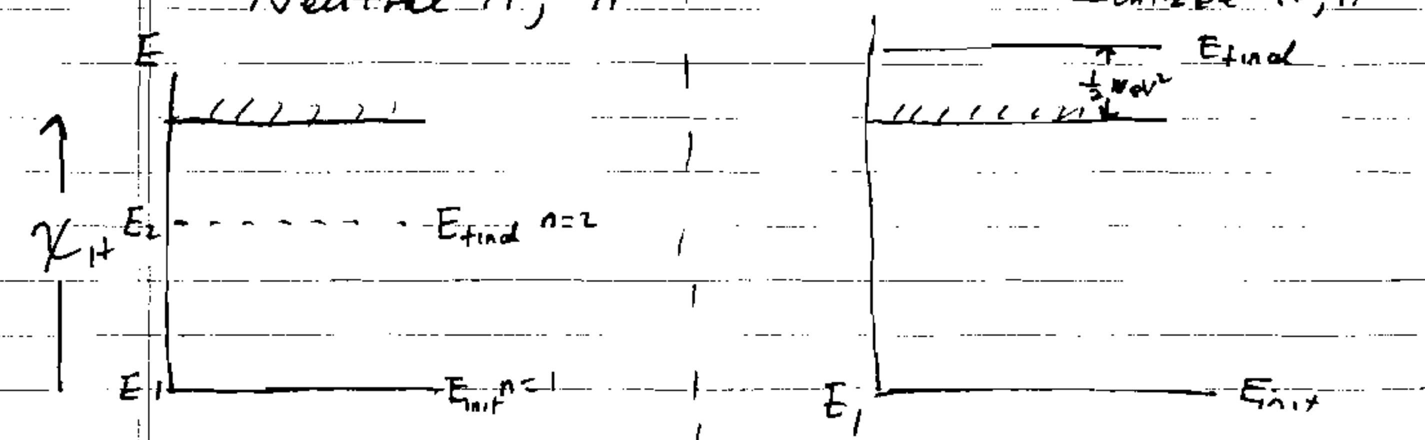

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Compare population of excited e^- state to ground state

(1) Neutral H: Ratio of bound state populations

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left(-\frac{3}{4}\frac{\chi_H}{kT}\right) = \frac{2(2)^2}{2}\exp\left(-\frac{3}{4}\frac{\chi_H}{kT}\right) =$$
$$= 4.\exp\left[-\frac{3}{4}\frac{\chi_H}{kT}\right]$$

(2) Ionized H^+: Ratio of unbound state population to bound state of H^0.

$$\frac{dn_1^+}{n_1} = \frac{g}{g_1}\exp\left[-\left(\frac{\frac{1}{2}m_e v^2 + \chi_{1+}}{kT}\right)\right]$$

dn_1^+ = no. of (a) H^+ ions in ground state
(b) free e^- with $(v, v+dv)$

g_1 = statistical weight (degeneracy) of neutral ground state $g_1 = 2(1)^2$

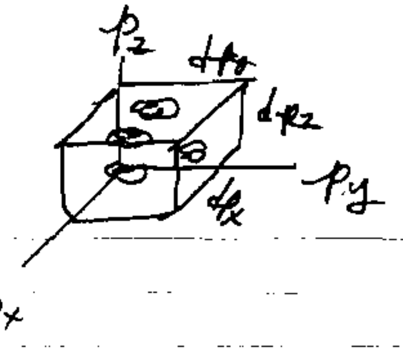
$g = g_1^+ \times dg_e$

g_1^+ = statistical weight of ion ground state
dg_e = differential stat. wt. of unbound state of free e^-

It turns out that for spin 1/2 electrons:

$$dg_e = \frac{2}{n_e}\cdot\frac{4\pi p^2 dp}{h^3} \quad \text{where } p = m_e v$$

where does this come from: How many states available per particle?



Due to Heisenberg uncertainty principle, we cannot lodge an arbitrary no. of particles in given momentum interval $d^3p = dp_x\, dp_y\, dz$

Intrinsic fuzziness in δp_x if particle confined to distance a

$$\delta p_x = h/a \qquad (\Delta x \Delta p_x = h)$$

$$\therefore\ dg_e = 2 \cdot \frac{dp_x\, dp_y\, dp_z}{\left(\frac{h}{a}\right)\left(\frac{h}{a}\right)\left(\frac{h}{a}\right)} = a^3 \cdot \frac{2 \cdot d^3p}{h^3}$$

internal d.o.f. ← { 2 spin states for Fermion e^-

minimum δp

But $a^3 = 1/n_e$ = volume per particle.
In spherical momentum coordinates $d^3p = 4\pi p^2 dp$

$$\therefore\ dg_e = 2(4\pi p^2 dp)/h^3 \cdot n_e$$

Because $p = m_e v$, $p^2 = m_e^2 v^2$, $dp = m_e dv$,

$$\boxed{dg_e = \left(\frac{8\pi}{n_e h^3}\right) \cdot m_e^3 v^2 dv} \qquad (1)$$

Therefore

$$\frac{dn_1^+}{n_1} = \frac{g_1^+}{g_1} \times \left[\frac{8\pi}{n_e h^3} \cdot m_e^3\right] \cdot \exp\left[-\frac{(\chi_H + \frac{1}{2} m_e v^2)}{kT}\right] v^2 dv$$

Now integrate over all v!

Result:

$$\frac{n_1^+ n_e}{n_1} = \frac{g_1^+}{g_1}\left(\frac{8\pi m_e^3}{h^3}\right)\cdot \exp\left(-\frac{\chi_H}{kT}\right)\int_0^\infty e^{-\frac{\frac{1}{2} m_e v^2}{kT}}\cdot v^2\,dv$$

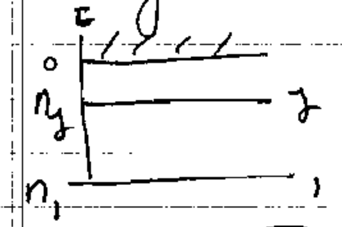
$$x = \sqrt{\frac{m_e}{2kT}}\cdot v$$

$$\frac{n_1^+ n_e}{n_1} = \frac{g_1^+}{g_1}\left(\frac{8\pi m_e^3}{h^3}\right)\cdot \exp\left(-\frac{\chi_H}{kT}\right)\left(\frac{2kT}{m_e}\right)^{3/2}\underbrace{\int_0^\infty e^{-x^2}x^2\,dx}_{\pi^{1/2}/4}$$

Therefore:

$$\boxed{\frac{n_1^+ n_e}{n_1} = \left(\frac{2g_1^+}{g_1}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right)} \quad ②$$

Atom Not finished yet. We want to express result as a function of total density of H^0 atoms, not just H^0 atoms in ground state.



Now $\frac{n_j}{n_1} = \frac{g_j}{g_1}\exp\left[-\left(\frac{E_j - E_1}{kT}\right)\right]$

Total density of atoms: $n = \sum_{j=1}^{\infty} n_j$

or $$n = \frac{n_1}{g_1}\underbrace{\sum_{j=1}^{\infty} g_j \exp\left[-\left(\frac{E_j - E_1}{kT}\right)\right]}$$

$$n = \frac{n_1}{g_1} Z(T)$$ {Z is partition function}

Ion $$n^+ = \frac{n_1^+}{g_1^+} Z_+(T)$$ (by analogy)

Substitute eqs. into eq. (2)

get rid of n_1/g_1; n_1^+/g_1^+

$$\frac{\left(\frac{\cancel{g_1^+} n^+}{Z_+}\right)\cdot n_e}{\left(\frac{g_1 n}{Z}\right)} = \frac{2\cancel{g_1^+}}{\cancel{g_1}}\cdot\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right)$$

$$\therefore \boxed{\frac{n^+}{n} = \frac{2 Z_+}{n_e Z}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right)} \quad (3)$$

More general result for ionization states $i, i+1$

$$\boxed{\frac{n_{i+1}}{n_i} = \frac{2 Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_i}{kT}\right) \quad (4)}$$

Example 1: Hydrogen

$$\frac{n(H^+)}{n(H^0)} = \frac{2\cdot Z_{H^+}(T)}{n_e Z_{H^0}(T)}\cdot\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_H}{kT}\right)$$

Problem: Given n_e, T: find $H\alpha$ line strength

(1) Compute fraction of H that is neutral, H^0

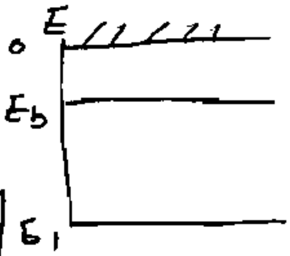
(2) Fraction of H^0 in $n=2$ state

(3) Answer: fraction of H in $n=2$ state (2×3) is proportional to line $H\alpha$ strength

Recap:

Last lecture I derived two results that will help us understand the properties of spectral classifications:

(1) Excitation:



Boltzmann equation:
$$\frac{n_b}{n_1} = \frac{g_b}{g_1}\exp\left[-\left(\frac{E_b - E_1}{kT}\right)\right]$$

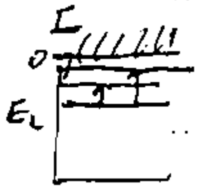
So, as T increases n_b/n_1 increases

(2) Ionization:
Applied to hydrogen

Saha Equation:
$$\frac{n(H^+)}{n(H^0)} = \frac{2 \cdot Z_{H^+}(T)}{n_e Z_{H^0}(T)}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_{I}}{kT}\right)$$

as T increases H^+/H^0 increases so H^0/H^+ decreases

(3) Problem: Compute fraction of H (i.e, $H^0 + H^+$) that is H^0 with bound e^- in excited $n=2$ state. This will give us strength of Balmer lines as a function of T.



We want:
$$\frac{n_{H^0}(n=2)}{n_H} = \frac{n_{H^0}(n=2)}{n_{H^0}} \cdot \frac{n_{H^0}}{n_H}$$

$\frac{n_{H^0}(n=2)}{n_{H^0}}$: excitation process

$\frac{n_{H^0}}{n_H}$: ionization process

First Compute partition function

$$Z = \sum_{j=1}^{\infty} g_j \cdot \exp\left[-\left(\frac{E_j - E_1}{kT}\right)\right]$$

Ion H^+: $Z_{H^+} = 1$ bound gd. state only.. no degeneracies since $\exists$ are no bound electrons.

Atomic H^0: $Z_{H^0} = g_1 + g_2 e^{-\left(\frac{E_2 - E_1}{kT}\right)} + \dots$

For our T range ($T \simeq 3000 \rightarrow 25{,}000\,K$) we can neglect all excited states because of Boltzmann guillotine factor

$$Z_{H^0} \simeq 2 + 2(2)^2 \cdot \exp\left[-\frac{10.2}{.86(T/10^4)}\right] \simeq 2$$

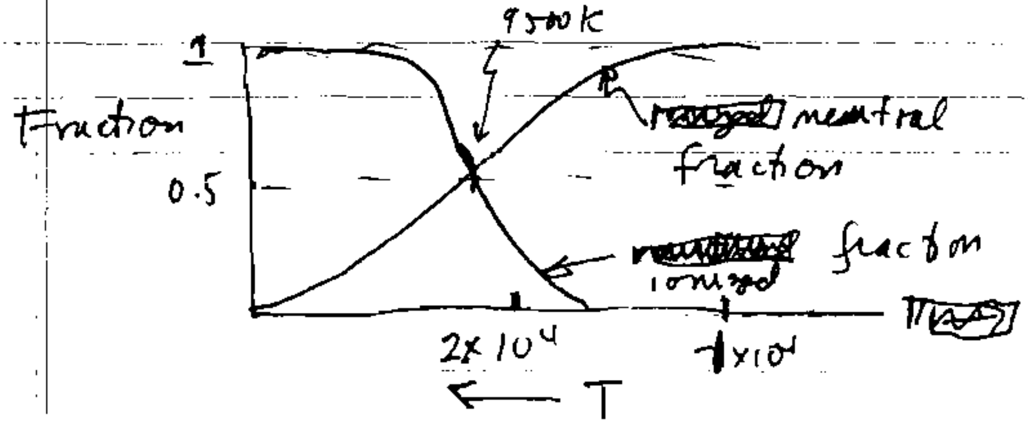
Expression:

$$\frac{n(H^+)}{n(H^0)} = \frac{2 \cdot (1)}{n_e \cdot 2} \cdot \left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} \exp\left(\frac{-13.6}{.86(T/10^4)}\right)$$

$$\frac{n(H^+)}{n(H^0)} \simeq \frac{1}{n_e} \cdot \left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} \exp\left(-\frac{1.58 \times 10^5}{T}\right)$$

Schematic result:

Neutral fraction $x_{H^0} = \dfrac{n(H^0)}{n(H^0) + n(H^+)}$; Ionized fraction $x_{H^+} = \dfrac{n(H^+)}{n(H^0) + n(H^+)}$



$n_e \simeq 10^{14}\,cm^{-3}$

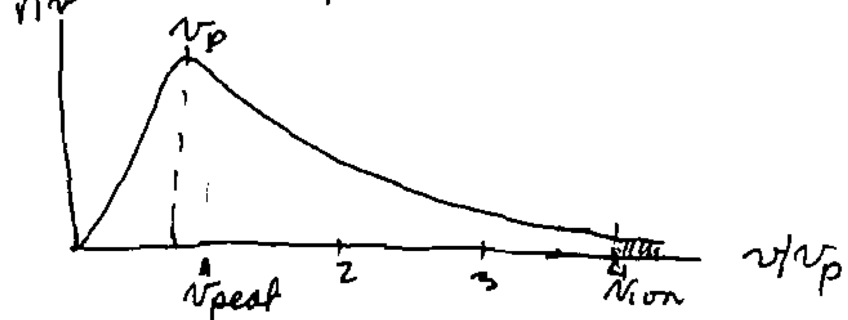
$$x_{H^+} = \frac{n(H^+)/n(H^0)}{1 + n(H^+)/n(H^0)}$$

As T increases the ionization fraction increases. At the same time the neutral fraction will decrease.

Note: H becomes significantly ionized when T rises to $\simeq 9500\,K$; i.e., $\frac{n(H^+)}{n(H^+)+n(H^0)} \simeq 1/2$ at $T \simeq 9500$

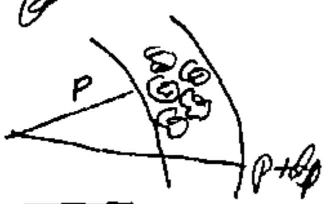
Puzzle: Peak of Maxwellian distribution $\frac{1}{2} m_e v_p^2 = kT$

$\frac{1}{2} m_e v_p^2 = 0.86(.95) = 0.8\,eV$
But ionization energy $= 13.6\,eV$ $\Big\}$ $\frac{1}{2} m_e v_{ion}^2 = \chi_H$
Therefore: $\frac{v_{ion}}{v_p} = \sqrt{\frac{13.6}{.8}} = 4$



Therefore, only a small fraction (5×10^{-7}) of e^- at $T = 9500\,K$ have kinetic energies large enough to ionize H^0. The reason the ionized fraction is large, however, is due to the large statistical weight of free electron states available to liberated e^-.

crude: $\frac{n(H^+)}{n(H^0)} \simeq \frac{g_E}{g_1} \exp\left(-\frac{\chi_H}{kT}\right)$

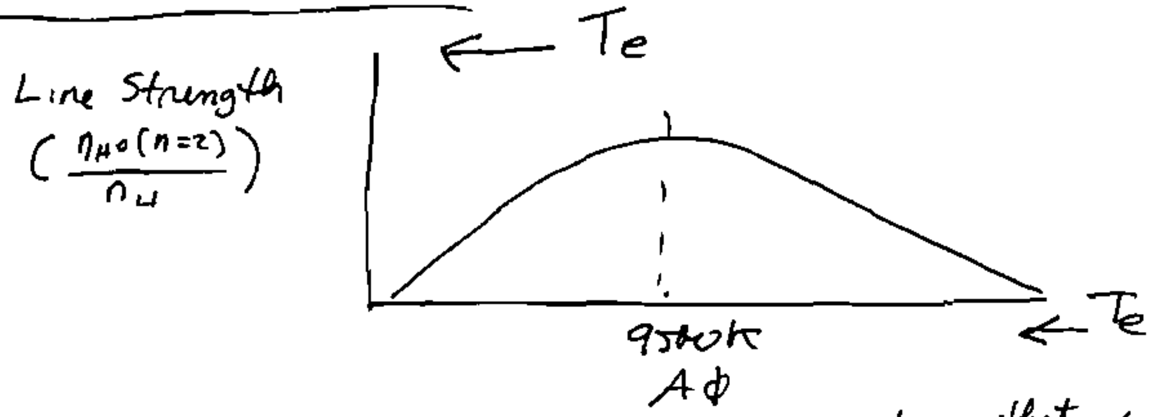


$$g_E \simeq \frac{\langle p\rangle^3 \cdot V}{h^3} \simeq \frac{\left(\sqrt{2m_e kT}\right)^3}{h^3}\frac{1}{n_e} \quad : \; p = \sqrt{2 m E}$$

$T \simeq 9500\,K$

Exact $\frac{n(H^+)}{n(H^0)} = \frac{1}{n_e}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\frac{\chi_H}{kT}} \simeq \frac{2.2\times10^{21}}{n_e} e^{-16.63} \simeq 1$ for $n_e \simeq 10^{14}\,cm^{-3}$

Back to Balmer lines:

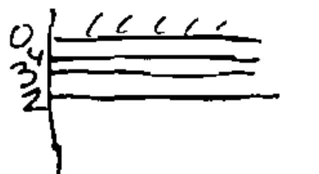


(1) So as Te increases, we just saw that neutral fraction decreases, which helps to explain decrease of Balmer lines with increasing T at T > 9500 K.

(2) But why do Balmer line strengths decrease with decreasing Te at T < 9500 K?

Recall $$\frac{n_2}{n_H} = \frac{n_2}{n_{H^0}} \times \frac{n_{H^0}}{n_H}$$ where $n_2 \equiv n_{H^0}(n=2)$

Let's compute $\frac{n_2}{n_{H^0}}$ (Excitation), since we already computed $\frac{n_{H^0}}{n_H}$ (Ionization)



$$\frac{n_2}{n_{H^0}} = \frac{n_2}{n_1+n_2+n_3+\cdots} = \frac{n_2/n_1}{1+\frac{n_2}{n_1}+\frac{n_3}{n_1}+\cdots}$$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\left(\frac{E_2-E_1}{kT}\right)\right]$$

$$\frac{n_2}{n_1} = \frac{2(2)^2}{2} * \exp\left(-\frac{1.2\times10^5}{T}\right)$$

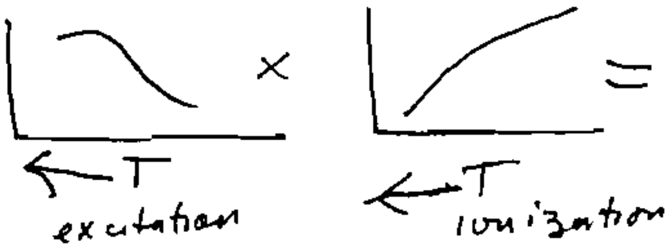
Neglecting states above n_2, we have

$$\frac{n_2}{n_{H^0}} = \frac{4\exp\left[-\frac{1.2\times10^5}{T}\right]}{1+4\exp\left[-\frac{1.2\times10^5}{T}\right]+\cdots}$$

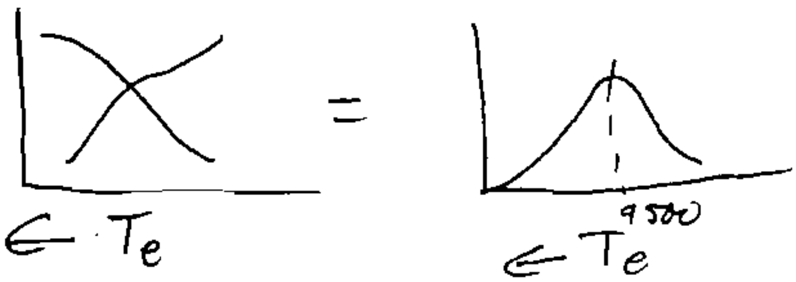
Therefore, as T increases n_2 increases.
fractional population of $n=2$ H^0 state increases.

But we saw before that as T increases $\frac{n_{H^0}}{n_H}$ decreases : neutral fraction decreas

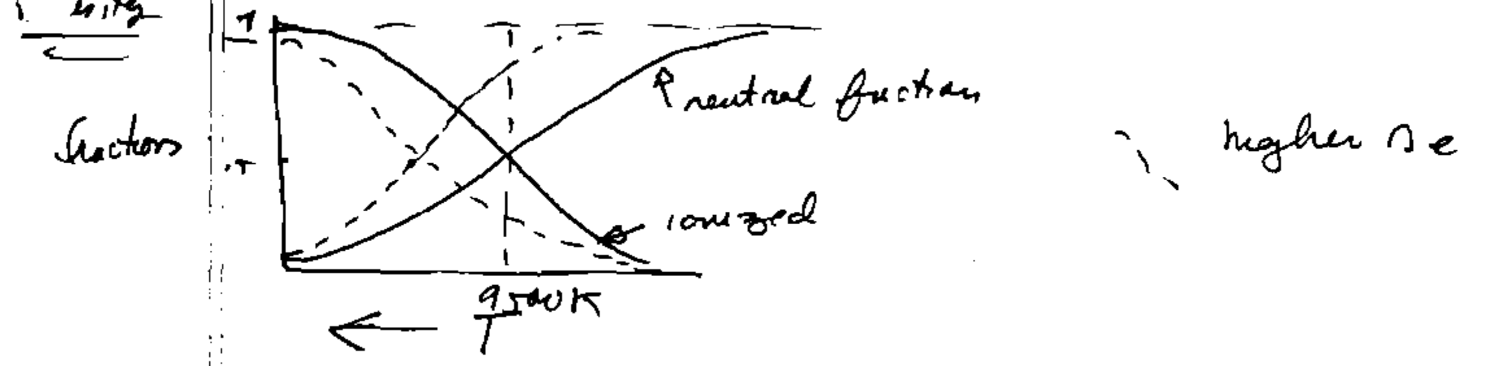
Again:
$$\frac{n_2}{n_H} = \frac{n_2}{n_{H^0}} \cdot \frac{n_{H^0}}{n_H}$$



Product of these two function leads to a maximum



Other Consequences of Saha equation:



Note if we increase n_e, $n(H^+)/n(H^0)$ is smaller at a given temperature. As a result you have to go to even higher T's before gas becomes 50% ionized.

Implication: 2 stars with same T. Star with denser stellar atmosphere is more neutral than star with lower densities

Alternative form of Saha: We shall see that electron pressures are a fundamental quantities, so we can get rid of n_e in terms of pressure.

Ideal gas law $P_e = n_e k T$

General Saha: i = no. of bound e^- removed

$$\frac{n_{i+1}}{n_i} = \frac{2 \cdot Z_{i+1}}{n_e Z_i} \cdot \left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} \exp\left(\frac{-\chi_i}{kT}\right)$$

or

$$\frac{n_{i+1}}{n_i} = \frac{2 k T \cdot Z_{i+1}}{P_e \cdot Z_i} \left(\frac{2\pi m_e k T}{h^2}\right)^{3/2} \exp\left(\frac{-\chi_i}{kT}\right)$$

Comparison between HI (Balmer) lines and Ca II lines in the sun

Solar photosphere: $T = 5800\,K$; $\underline{P}_e = 15\ dynes/cm^2$

Abundance of Ca: $\left(\frac{n(Ca)}{n(H)}\right)_{\odot} = 2\times10^{-6}$ (low)

Physics of H

(1) $\frac{n(H^+)}{n(H^0)} = 7.7\times10^{-5}$: most of H in solar photosphere is neutral

(2) $\frac{n_2}{n_H} = \frac{n_2}{n(H^0)} \times \frac{n(H^0)}{n(H)} \simeq \frac{n_2}{n(H^0)}$ since $\frac{n(H^0)}{n(H)} \simeq 1$

$$\frac{n_2}{n_H} = \frac{n_2}{n_{H^0}} = \frac{n_2/n_1}{1 + \frac{n_2}{n_1} + \cdots} \simeq 4\cdot \exp\left(-\frac{1.2\times10^5}{T}\right) \simeq 5.1\times10^{-9}$$

Explains why $H_\alpha, H_\beta, \ldots$ so weak in the sun.

Physics of Ca

(1) Parameters: $Z_i = Z_{CaI} = 1.32$; $Z_{i+1} = Z_{CaII} = 2.3$

neutral Ca — singly ionized Ca

$\chi_i = \chi_{CaI} = 6.11\,eV \Rightarrow \frac{\chi_{CaI}}{k} = 7.08\times10^4\,K$

Ionization

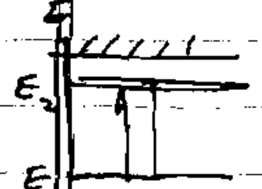
Saha: $$\frac{n(CaII)}{n(CaI)} = \frac{2\cdot kT\cdot Z_{CaII}}{P_e\cdot Z_{CaI}}\cdot\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\exp\left(-\frac{\chi_{CaI}}{kT}\right)$$

$$\frac{n(CaII)}{n(CaI)} = \frac{2\times1.38\times10^{-16}\times5.8\times10^3\cdot2.3}{15\cdot1.32}\times\left(\frac{2\pi\times9.11\times10^{-28}\times1.38\times10^{-16}\cdot5.8\times10^3}{(6.626\times10^{-27})^2}\right)^{3/2} e^{-\frac{7.08\times10^4}{5800}}$$

$$\frac{n(\mathrm{CaII})}{n(\mathrm{CaI})} = 920$$

Thus most Ca is CaII or $n_{CaII}/n_{Ca} \simeq 1$

Lines: CaII H&K $\lambda\lambda 3934, 3969$ arise from the ground state. So different conditions from those producing the Balmer lines, which arise from n = excited state in H



In this case $E_2 - E_1 = 3.12\,\mathrm{eV}$

$g_2 = 4,\ g_1 = 2$

$$\frac{n_2}{n_1} = \frac{g_2}{g_1}\exp\left[-\left(\frac{E_2-E_1}{kT}\right)\right]$$

$$\frac{E_2-E_1}{k} = \frac{3.12\times1.6\times10^{-12}}{1.38\times10^{-16}} = 3.62\times10^{4}\,K$$

therefore: $$\frac{n_2}{n_1} = 2\cdot\exp\left[-\left(\frac{3.62\times10^{4}}{5.8\times10^{3}}\right)\right] = 3.8\times10^{-3}$$

Thus most CaII ions in ground state $\left(\frac{n_1}{n_{CaII}}\right)_{CaII} \simeq 1$

Compare Hα to CaII line strengths

$$\frac{H\alpha}{CaII} = \frac{n_2(H^0)}{n_1(CaII)} = \frac{\left(\frac{n_2}{n_{H^0}}\right)\frac{n_{H^0}}{n_H}\cdot n_H}{\left(\frac{n_1}{n_{CaII}}\right)\frac{n_{CaII}}{n_{Ca}}\cdot n_{Ca}}$$

Since $n_{H^0}/n_H \simeq 1$, $n_{CaII}/n_{Ca} \simeq 1$, $n_1/n_{CaII} \simeq 1$

$$\frac{H\alpha}{CaII} \simeq \frac{(n_2/n_{H^0})(1)}{(1)\cdot(1)\,(n_{Ca}/n_H)} \simeq \frac{n_2/n_{H^0}}{(Ca/H)} \leftarrow \text{abundance}$$

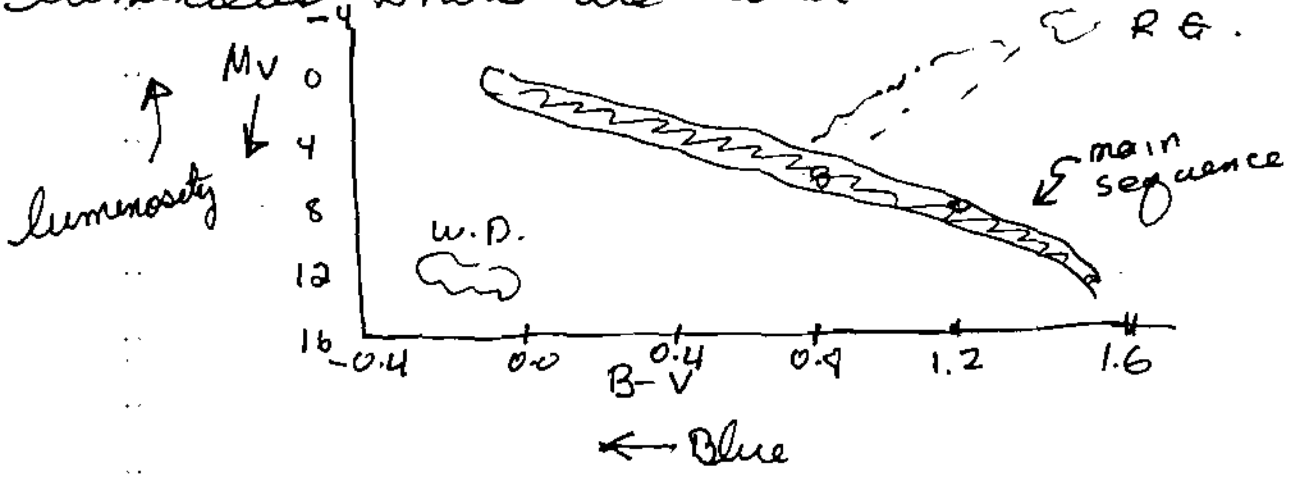
$$\therefore \frac{H\alpha}{CaII} \simeq \frac{5.1\times10^{-9}}{2\times10^{-6}} = 2.5\times10^{-3}$$ since $(Ca/H) \simeq 2\times10^{-6}$ (low abundance)

Strength of ~~CaII~~ Balmer relative to ~~Balmer~~ CaII lines does not mean H is underabundant relative to Ca. Rather it reflects low excitation level of $n=2$ state of H^0 in the sun

Hertzsprung-Russell Diagram

In the early 20th century Hertzsprung & Russell noticed some interesting regularities in Plots of absolute magnitude or equivalently luminosity versus effective temperature.

They expected to find a scatter plot. Instead, they found that the bulk of the stars were located on a strip along which M_V decreases as B-V decreased; i.e., more luminous stars are hotter



Main Sequence: Contains between 80%-90% of all stars in the HR diagram

The remaining ~15% of the stars

White Dwarfs: Bluish stars that are 8 mag. or more fainter than blue stars on the MS.

Red Giants: Red stars that are more luminous than red stars on MS.

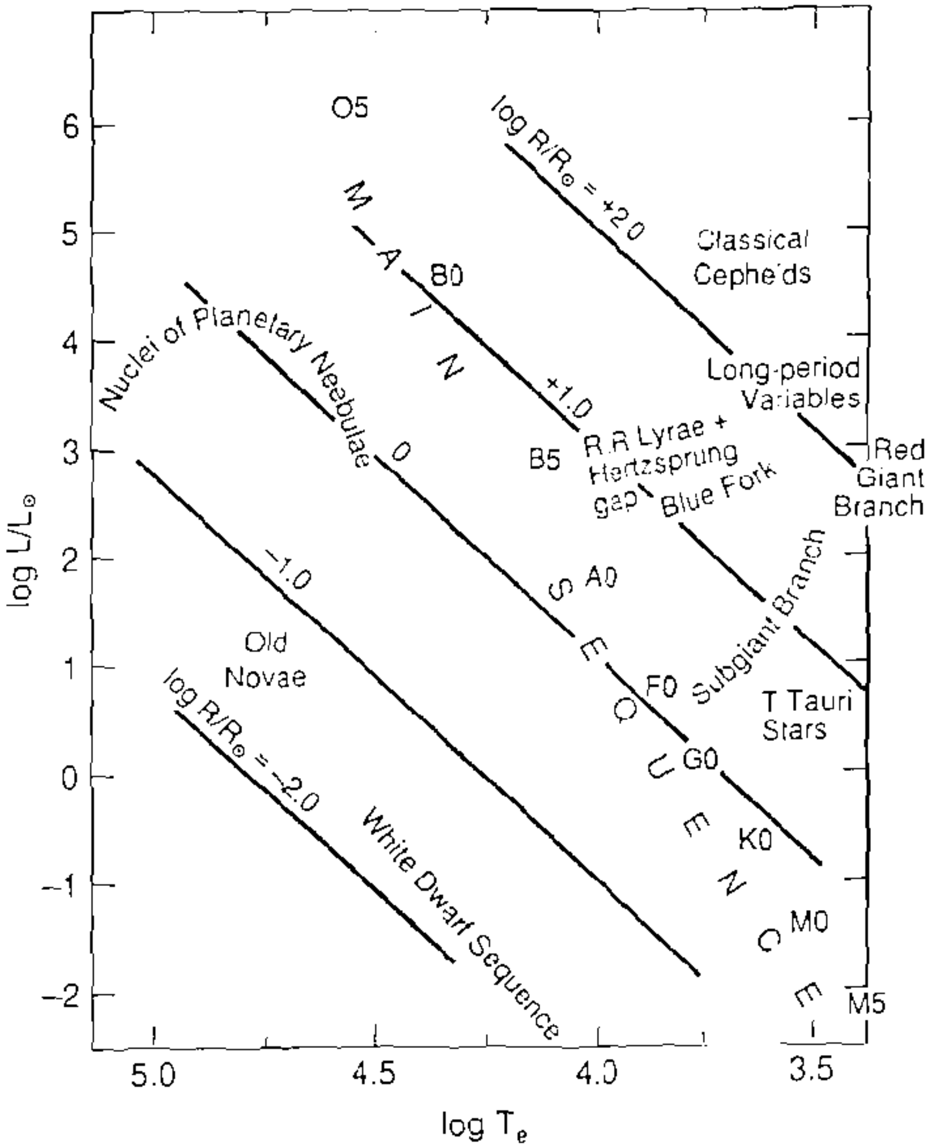


FIGURE 1.4. Schematic Hertzsprung–Russell diagram. The lines of constant slope repre stars having identical radii (see Section 4:13).

(1) Main Sequence:

(a) As we shall see this is location of stars in which H is fused into He in stellar cores. The reason most stars lie on MS is that this is the longest evolutionary phase through which all stars pass.

(b) Shape of MS indicates stars with hotter photospheres are also more luminous.

(2) Theoretical HR diagram: Let's look at physical HR diagram, which plots bolometric luminosity L versus T_e

(a) Transformation

$$M_{bol} = M_v + BC$$

BC is function only of T_e (B-V)

(b) Examples for (MS) stars

Sp	M_v	BC	M_{bol}	$L/L_\odot$
O5	-5.1	-4.4	-9.51	5×10^5
G0	+4.7	-0.18	+4.50	1.25
M0	+8.9	-1.38	+7.52	0.077

where $L/L_\odot = 10^{-.4(M_{bol} - (M_\odot)_{bol})}$: $M_\odot = +4.74$

(3) Lines of constant Radii

$$L = 4\pi R^2 \cdot \sigma T_e^4$$

$$\log L = 2\log R + 4\log T_e + \text{const.}$$

therefore for $R = \text{const}$, $\log L = 4 \log T_e + \text{const}$

Figure shows that

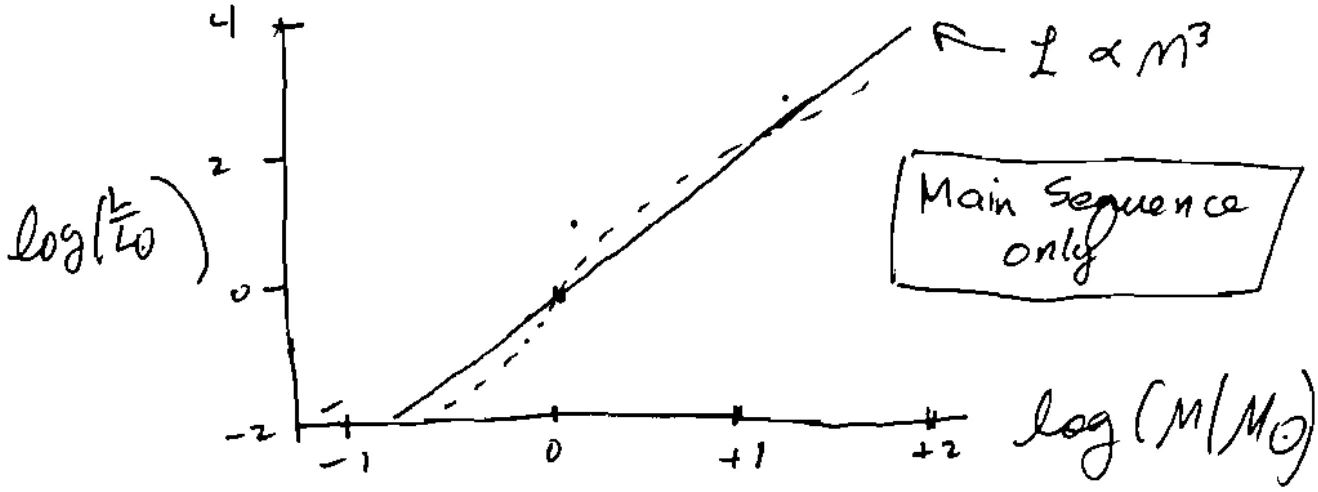
ⓐ MS rises more steeply than T_e^4.

ⓑ Implication: Upper MS stars (UMS) are larger than lower MS stars.

Example: B0 radius is about 10 × G2 radius

But throughout most of the MS, we find about a factor of 30-40 increase in R

Mass $\quad$ L versus M relationship on the MS.



As we already discussed, on the MS luminosity increases with mass roughly like M^3

Factor of a few 100 range in mass along the M.S. ~ 10^7 range in L (or more

Off the MS

Red Giants: Stars are more luminous than MS stars not because they are hotter: In fact T_e is lower than in Sun. Rather, they are more luminous because their radii are larger.

$$R_{RG} \simeq 10^2 R_\odot$$

- What is relationship of RGs to MS stars
- How do masses of RGs compare to masses of MS stars?

White Dwarfs: Stars have higher T_e but lower L than MS stars. These stars are underluminous not because T_e is low, but rather because R is low: $R \simeq 10^{-2} R_\odot$

- What is relationship of WDs to sun?
- How do mass of WDs compare to sun or to RGs?

Star Clusters: