Mass

So far we have discussed the luminosity, L, the surface (or effective) temperature, T_e, and the radius, R_s, of a star.

The next crucial parameter is mass, M. The technique for determining M relies on Newtonian gravitational physics. That is mass generates gravity and gravity generates acceleration motion in test bodies. By measuring this motion we infer the mass of the body.

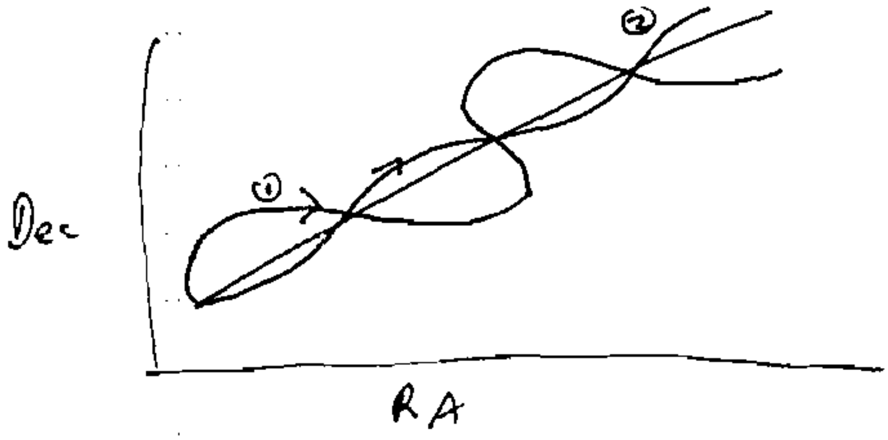
Binary Stars: the ideal testing ground for this is the sample of binary stars: these comprise about half of all known stars, so we have a good statistical sample.

- Definition

(i) Double stars in bound orbits about each other.

(A) - Visual Binaries

Small fraction of binaries are "visual binaries". These stars are far enough apart that they can be identified as separate stars



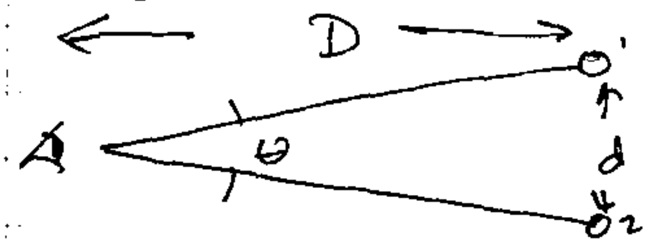
(a) st. line motion of center-of-mass

(b) Oscillator motion of each star about (C.M)

(B) - Astrometric Binary

Visual binary in which one star is too faint to be detected. One star seems to oscillate about st. line.

These visual/astrometric binaries comprise a small fraction of all binary stars. They must be far enough apart to be detected separately.



$d = \theta \cdot D$

$d = \frac{\theta(\text{arcsec})}{2\times10^5} \cdot D$

or $d = \theta(\text{arcsec})\, D(\text{pc})$ [AU] (inverse of parallax)

Since $D > 1\,\text{pc}$ (typically $D \approx 30\text{–}50\,\text{pc}$ for bright stars) and $\theta \gtrsim$ few arcsec req. for spatial resolution: say $\theta \simeq 2$ arcsec $\Rightarrow$

$$d > 2 \cdot 40 = 80\ \text{A.U.}$$

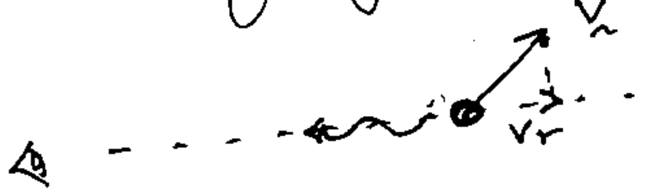
Kepler's law (below) implies $P \propto d^{3/2}$
Then for stars with $M \sim 1 M_\odot$

$$\frac{P}{P_\oplus} = (80)^{3/2} \Rightarrow P > 500\text{–}700 \text{ years}$$

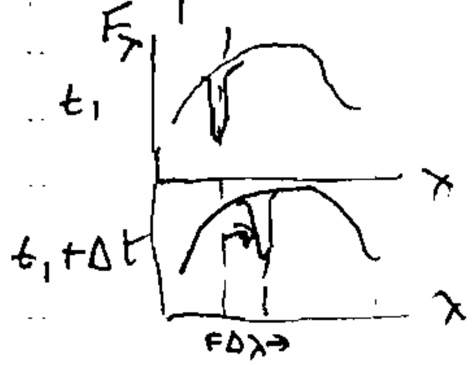
(C) Spectroscopic Binaries

Most common type. Typically $\theta < 1$ arcsec, so 2 stars appear as one star. They are noticed by periodic shift in wavelengths of stellar absorption lines.

From Doppler shifts of stellar absorption lines, we deduce velocity of star projected along the line-of-sight; i.e., radial velocity v_r.



In the limit $v_r \ll c$, which holds for all stars in Milky Way, we get spectrals shifts of absorption lines given by

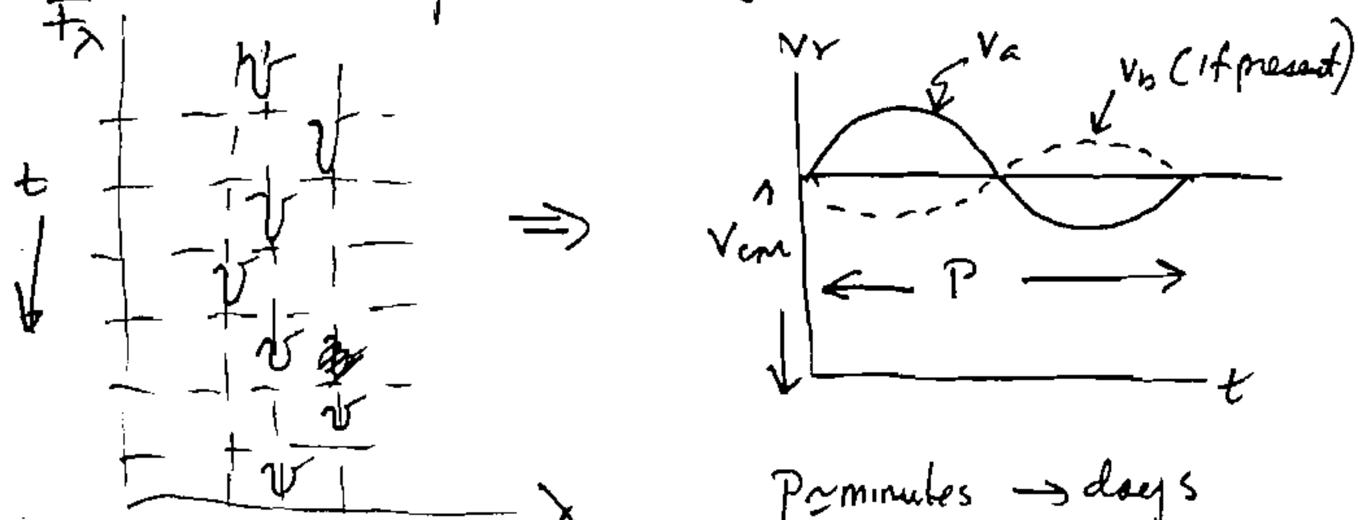


$$\frac{\Delta\lambda}{\lambda_{rest}} = \frac{\lambda_{obs} - \lambda_{rest}}{\lambda_{rest}} = \frac{v_r}{c}$$

(A) Redshift: If $v_r > 0$, then $\Delta\lambda > 0$ and line is redshifted

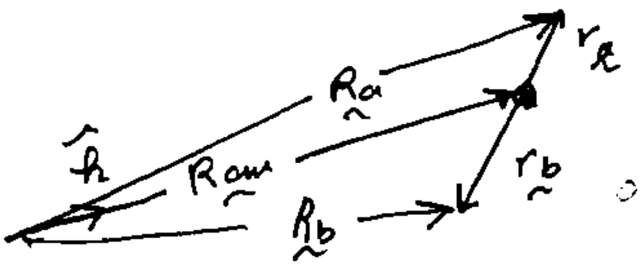
(B) Blueshift: If $v_r < 0$, then $\Delta\lambda < 0$ and line is blue-shifted.

In spectroscopic binaries lines are blueshifted and redshifted periodically - back and forth



$P \simeq$ minutes $\rightarrow$ days

Interpretation



Since $\underset{\sim}{R}_a = \underset{\sim}{R}_{cm} + \underset{\sim}{r}_a$, $\frac{d\underset{\sim}{R}_a}{dt} = \frac{d\underset{\sim}{R}_{cm}}{dt} + \frac{d\underset{\sim}{r}_a}{dt}$

As a result: $\underset{\sim}{V}_a = \underset{\sim}{V}_{cm} + \underset{\sim}{v}_a$

Radial velocity: $(V_a)_r = \hat{k} \cdot \underset{\sim}{V}_a$ $\;(|\hat{k}| = 1)$

Therefore $(V_a)_r = (V_{cm})_r + (v_a)_r$ (components $\parallel$ to $\hat{k}$)

Integrate observed velocity curve in time:

$$\int_0^P dt\, [V_a(t)]_r = (V_{cm})_r P + \int_0^P dt\, (v_a)_r$$

(const, since externall forces weaker than internal forces.)

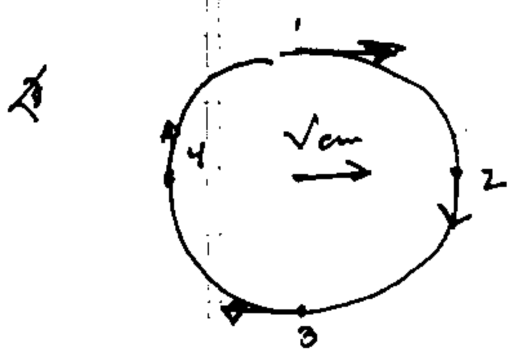
But $\int_0^P dt\, (v_a)_r = \int_0^P (dr_a)_r$

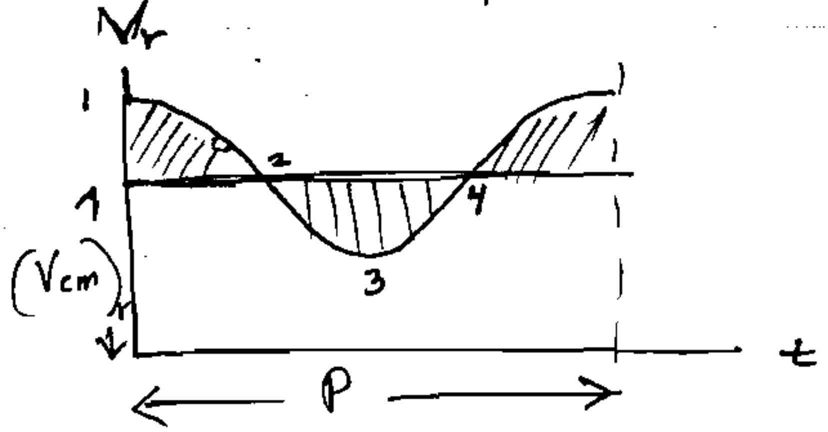
$= [(r_a)]_0^P = 0$ (No net desplacement in 1 period)

r_a

Therefore: $\boxed{(V_{cm})_r = \frac{1}{P}\int_0^P dt\, (V_a)_r}$

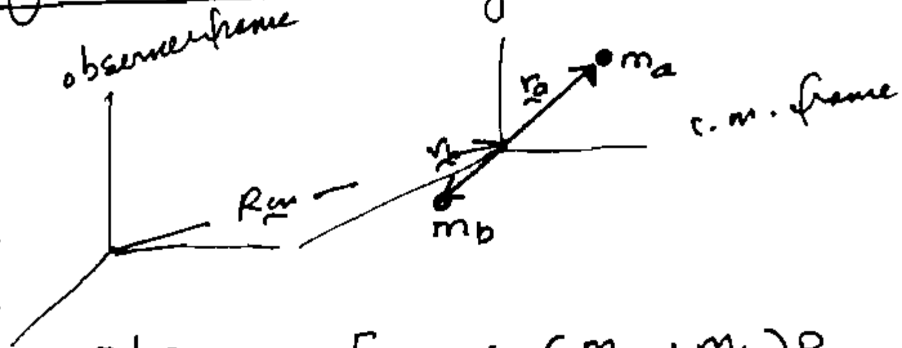
Let's consider simple case of circular orbits





So $(V_{cm})_r$ determined by letting /// area = \\\ area under periodic curves.

Center-of-mass Frame Dynamics



Definition of c.o.m.

Observer Frame: $(m_a + m_b)\tilde{R}_{cm} = m_a R_a + m_b R_b$

But, since $\tilde{R}_a = \tilde{R}_{cm} + \tilde{r}_a$; $\tilde{R}_b = \tilde{R}_{cm} + \tilde{r}_b$

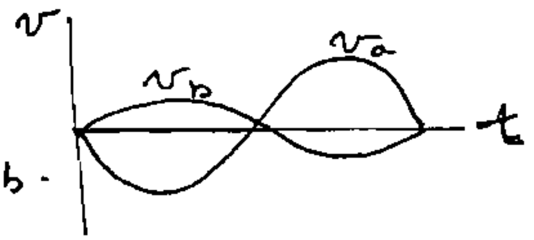
c.m. Frame: $\Rightarrow$ $m_a \tilde{r}_a + m_b \tilde{r}_b = 0$

Take time derivative: $m_a \tilde{v}_a + m_b \tilde{v}_b = 0$

$$\therefore \quad \boxed{v_b = -\frac{m_a}{m_b} \tilde{v}_a} \; ; \quad \tilde{r}_b = -\frac{m_a}{m_b} \tilde{r}_a$$

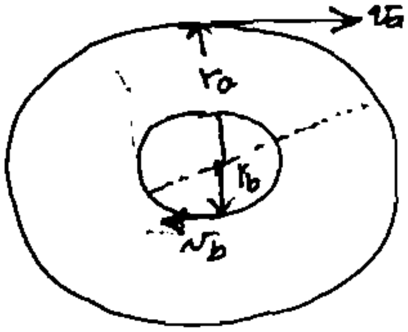
(1) Explains mirror symmetry in orbital velocities:

$m_b > m_a$ to explain lower vel amplitude of b.



(2)

1.

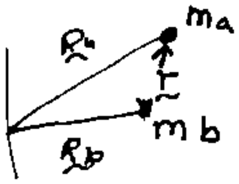


(ii) $|\tilde{r}_a| > |\tilde{r}_b|$
when $m_b > m_a$

(iii) Two stars always located on straight line going through c.m.

Observer Frame Dynamics

Define: $\tilde{r} = \tilde{R}_a - \tilde{R}_b = \tilde{r}_a - \tilde{r}_b$

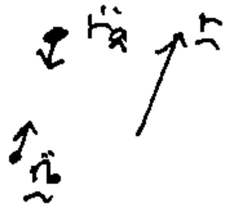


$\therefore \quad \ddot{\tilde{r}} = \ddot{\tilde{r}}_a - \ddot{\tilde{r}}_b$

Newton's law of gravitation:

$$\ddot{\tilde{r}}_a = -\frac{G m_b \tilde{r}}{|\tilde{r}|^3}$$

$$\ddot{\tilde{r}}_b = +\frac{G m_a}{|r|^3}\tilde{r}$$

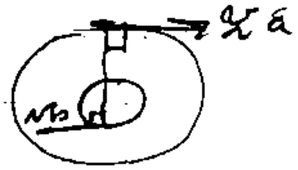


As a result: $$\ddot{\tilde{r}} = -\frac{G m_b}{|\tilde{r}|^3}\tilde{r} - \frac{G m_a \tilde{r}}{|\tilde{r}|^3}$$

$$\ddot{\tilde{r}} = -\frac{G(m_b + m_a)\tilde{r}}{|\tilde{r}|^3}$$

Magnitude: $$\boxed{\ddot{r} = \frac{G(m_b + m_a)}{r^2}}$$

This is general solution. But in case of circular motion:



$$\ddot{\underset{\sim}{r}} = \frac{v_{rel}^2}{r}$$

Since $\dot{\underset{\sim}{r}} = \underset{\sim}{v_a} - \underset{\sim}{v_b}$; $\underset{\sim}{v_a}$ in opposite direction from $\underset{\sim}{v_b}$,

$$|\dot{\underset{\sim}{r}}| = v_a + v_b \quad \text{or} \quad v_{rel} = v_a + v_b$$

From eq. we have: $\dfrac{v_{rel}^2}{r} = \dfrac{G(m_a + m_b)}{r^2}$

For circular orbits. $v_{rel} = 2\pi r / \underline{P}$

Solve for mass: $m_a + m_b = \dfrac{r}{G} v_{rel}^2$

$$m_a + m_b = \frac{r}{G}\left(\frac{2\pi r}{\underline{P}}\right)^2 = \frac{(2\pi)^2}{G} \cdot \frac{r^3}{\underline{P}^2}$$

Kepler's law:

$$\left.\begin{array}{l} \text{If } [r] = AU \\ [M] = M_\odot \\ [P] = \text{years} \end{array}\right\} \quad m_a + m_b = r^3 / P^2$$

For the sun $\underset{\text{Sun}}{m_b} \gg \underset{\text{earth}}{m_a}$

$$m_\odot = (1)^3/(1)^2 = 1 M_\odot$$

Physical Units: $m_a + m_b = \dfrac{(2\pi)^2}{6.7\times10^{-8}} \dfrac{(1.5\times10^{13})^3}{(3\times10^7)^2}$

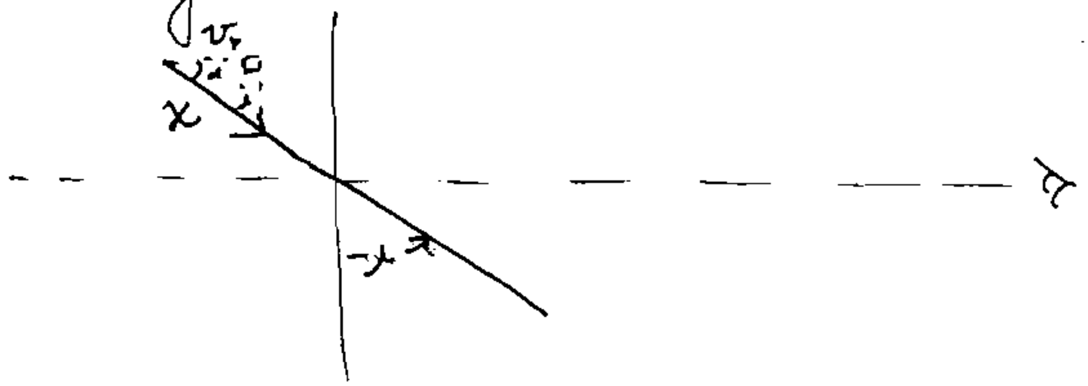
$$= 2\times10^{33} g$$

Complications

Equation on previous page is not useful for 2 reasons:

(1) We don't know r for spectroscopic binaries

(2) Orbital plane inclined to line of sight



$$v_r = v\cos\left(\tfrac{\pi}{2}-i\right) = v\cdot \sin i$$

Now in convenient units: $m_a + m_b = \frac{r}{G} v_{rel}^2$

$$m_a + m_b = \left[\frac{v_{rel} P}{2\pi}\right]\cdot\frac{v_{rel}^2}{G} = \frac{v_{rel}^3\cdot P}{2\pi G}$$

But radial component of v_{rel} is:

$$(v_{rel})_r = v_{rel}\cdot \sin i$$

therefore: $$m_a + m_b = \frac{(v_{rel})_r^3\cdot P}{2\pi G \sin^3 i}$$

$$\sin^3 i\,(m_a+m_b) = \frac{\left[(v_a)_r + (v_b)_r\right]^3\cdot P}{2\pi G}$$

But write: $m_a + m_b = m_a\left(1 + \frac{m_b}{m_a}\right)$

So $$m_a \sin^3(i) = \frac{[(v_a)_r + (v_b)_r]^3 P}{2\pi G \left(1 + \frac{m_b}{m_a}\right)}$$

But $$\frac{m_b}{m_a} = \frac{(v_a)_r}{(v_b)_r}$$

Final Result $$\boxed{m_a \sin^3(i) = \frac{[(v_a)_r + (v_b)_r]^3 \cdot P}{2\pi G \left(1 + \frac{(v_a)_r}{(v_b)_r}\right)}}$$

So mass of m_a, then m_b can be obtained if

(a) inclination of orbit known

(b) double-line spectroscopic binary, i.e., if both stars are observed.

If sin(i) is not known, last equation will give a lower limit to the masses since $\sin(i) < 1$

Example

Stars	P(days)	$(V_a)_r^{max}$ km/s	$(V_b)_r^{max}$ km/s	Masses m_a	m_b
β Phe	1.67	127	247	>6.08	>3.0
ν End	5	64	65	>0.56	>0.55

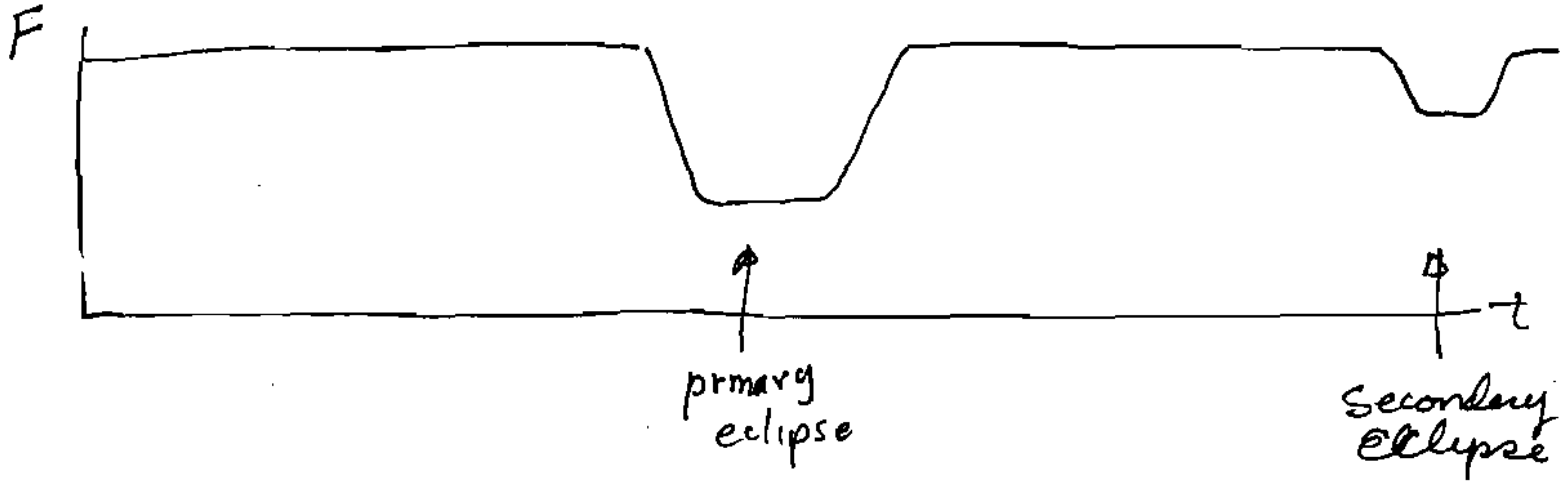
r_a(AU)	r_b
.02	.038
.03	.03

ρ

Eclipsing Binaries:

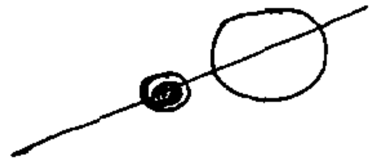
Best hope of getting i for spectroscopic binaries is when binary shows evidence of eclipse

what observer sees:



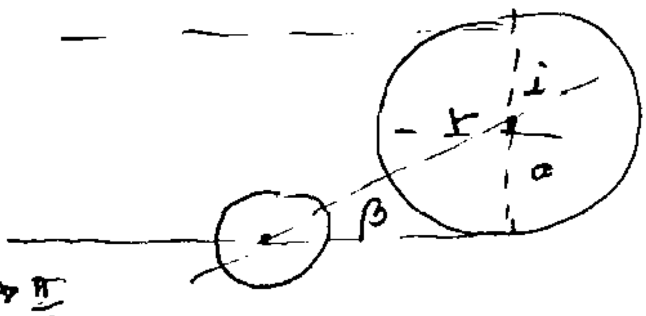
Duration of eclipse: $\Delta t_{eclipse} \simeq \frac{2a}{v_{rel}}$: a = primary radius

where $\Delta t_{eclipse} \ll P \simeq \frac{2\pi R}{v_{rel}}$ (since $R \gg a$)



$i \rightarrow 90^\circ$ for

eclipse Condition



$\sin\beta \simeq \frac{a}{r} \ll 1$

$i = \frac{\pi}{2} - \beta \Rightarrow i \to \frac{\pi}{2}$

Recap Masses

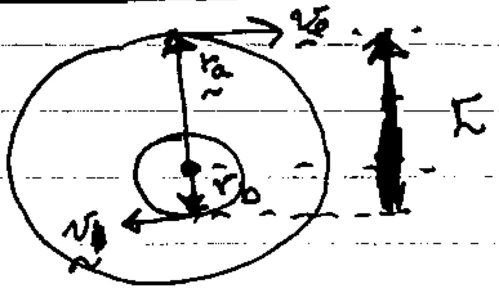
(1) Circular Orbits

$$m_a + m_b = \frac{r}{G} v_{rel}^2$$

(2) Relate r to v_{rel}:

$\underset{\sim}{r} = \underset{\sim}{r_a} - \underset{\sim}{r_b}$

$\therefore r = r_a + r_b$



For circular orbits: $v_a = \frac{2\pi r_a}{P}$; $v_b = \frac{2\pi r_b}{P}$

Recall $v_{rel} = v_a + v_b$; $r = r_a + r_b$

$$v_a + v_b = \frac{2\pi}{P}(r_a + r_b)$$

$$\Rightarrow v_{rel} = \frac{2\pi}{P} \cdot r$$

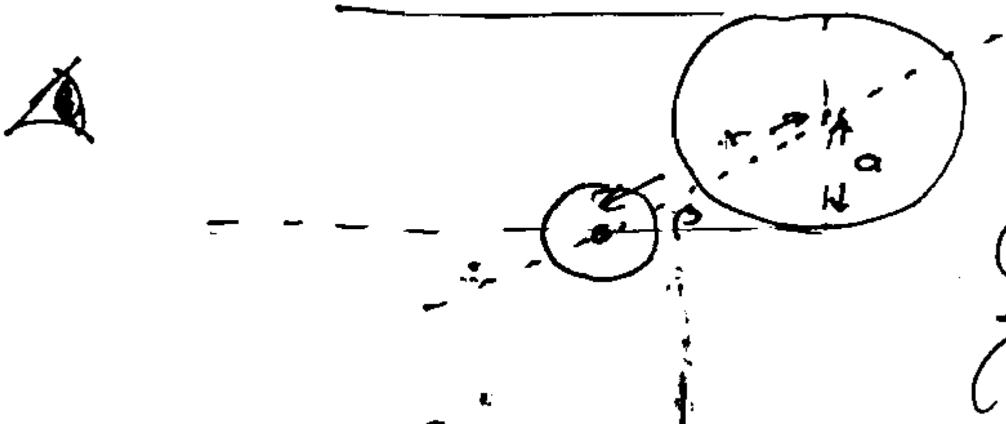
$$\therefore m_a + m_b = \left[\frac{v_{rel} \cdot P}{2\pi G}\right] v_{rel}^2$$

$$\boxed{m_a + m_b = \frac{v_{rel}^3 \cdot P}{2\pi G}}$$

(3) $$m_a = \frac{[(v_a)_r + (v_b)_r]^3 \cdot P}{2\pi G\left[1 + \frac{(v_a)_r}{(v_b)_r}\right] \sin^3(i)}$$

(4) We can get m_a, m_b if we know $\sin(i)$, $(v_a)_r$, $(v_b)_r$, $\underline{P}$

Eclipsing Binary criterion:



{Side View

$\frac{1}{2}$ cool star area in front of hot star

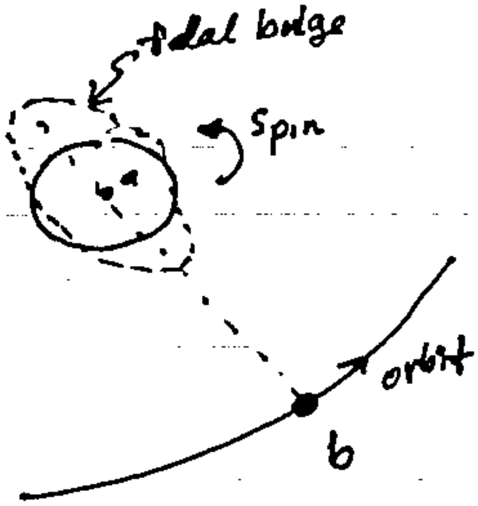
$\sin(\beta) = \frac{a}{r}$ is threshold condition for eclipse.

If $\sin\beta < \frac{a}{r}$, > $\frac{1}{2}$ cool star area in front of hot star

(5) Circular orbit Assumption

two gravitationally bound point masses are normally in elliptical orbits. However, stars are not point masses. For spectroscopic binaries with short periods, tidal forces result in energy dissipation that ultimately gets ~~?~~ radiated away from the star.

No Lag:

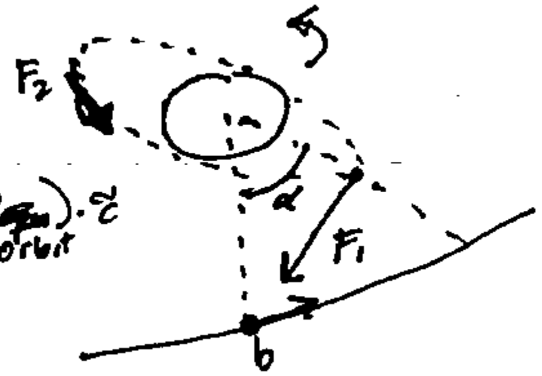


Tidal force by b on a causes instant reaction, so bulge points to star b

Lag: $\Omega_{spin} > \Omega_{orbit}$:

$\alpha \simeq (\Omega_{spin} - \Omega_{orbit}) \cdot \tau$

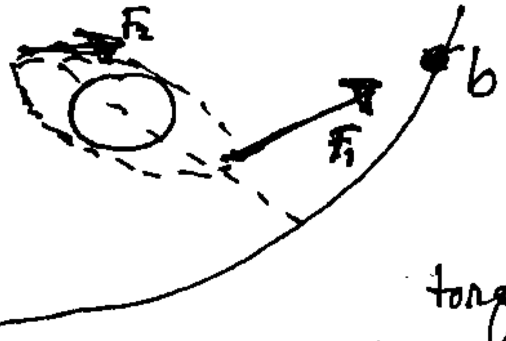
$\alpha > 0$



Bulge responds in time τ to tidal force. Since $\Omega_{spin} > \Omega_{orbit}$, bulge points ahead of b. Since $F_1 > F_2$, resultant torque cause Ω_{spin} to decrease. But total angular momentum conserved, so Ω_{orbit} increases

Lead: $\Omega_{spin} < \Omega_{orbit}$

$\alpha < 0$



In this case star b moves ahead of bulge axis. Since again $F_1 > F_2$, resultant torque cause Ω_{spin} to increase and Ω_{orbit} to decrease

Real elliptical orbit:

Tidal forces weak, so to zeroth order approx.

$J_{orbit} = r^2 \Omega_{orbit} = \text{const}$

(1) at ophelion ① r is lg. so $\Omega_{orbit} < \Omega_{spin}$

(2) At perihelion ②, v is small so Ω_{orbit} increases such that $\Omega_{orbit} > \Omega_{spin}$.

Characteristic of elliptical orbits: Ω_{orbit} varies with θ.

But now include tidal effects:

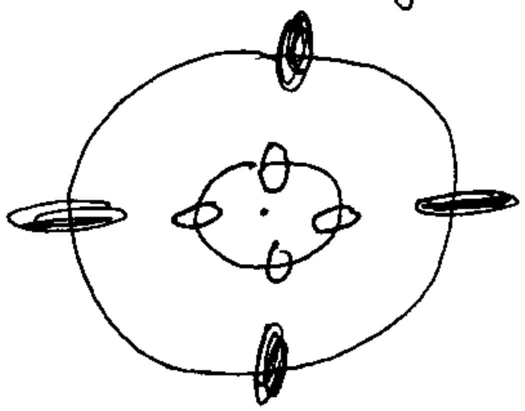
(A) At aphelion where $\Omega_{orbit} < \Omega_{spin}$, effect of bulge is to <u>increase</u> Ω_{orbit} from its initially low value

(B) At perihelion where $\Omega_{orbit} > \Omega_{spin}$, effect of bulge is to decrease Ω_{orbit} from its initial high value.

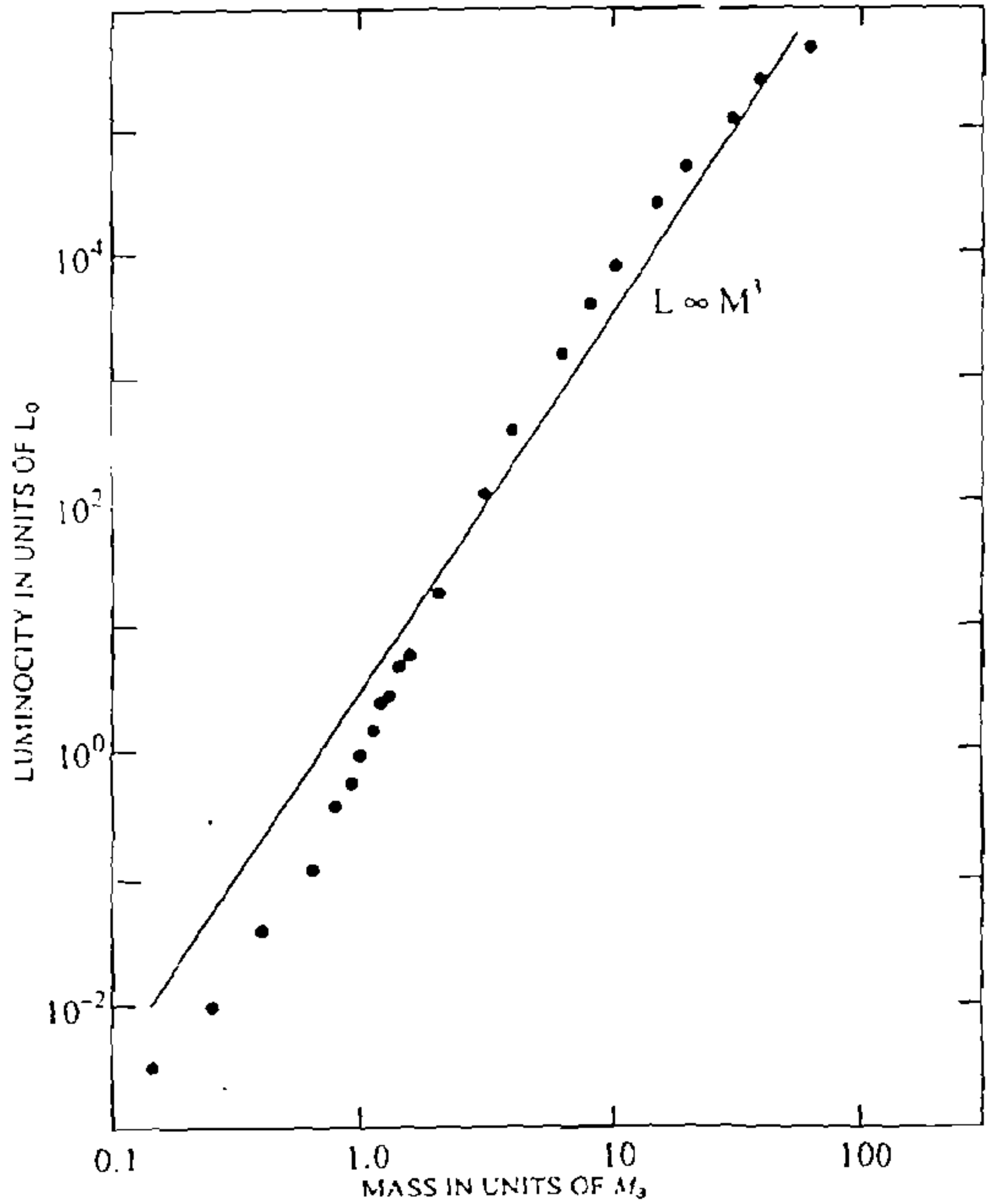
Net effect: Detailed calculations show that final state of system is that $\Omega_{orbit} = \text{const.}$ for all θ.

Final state: tidally locked circular orbit

$\Omega_{spin} = \Omega_{orbit}$



<u>Final Goal</u>: Mass determination lead to $L = L(M)$



.4 The mass-luminosity relation for hydrogen burning stars with a chemical

Spectral Classifications

During the late 19th and early 20th centuries, astronomers at Harvard Observatory began classifying stars according to patterns of appearance of absorption lines that ~~appeared~~ exhibited in their spectra.

Recall:

I'll explain how these absorption lines are formed later on. But for now, take it on faith that they form in the ~~thes~~ photospheric outer layers of the star.

Basic Finding: Stars with higher values of T_e (i.e., bluer stars) exhibited absorption spectra that differed systematically from stars with lower values of T_e (redder stars). The stars were arranged in sequence of spectral types or decreasing T_e

5 → 0 → 9 0 → 9 sp →

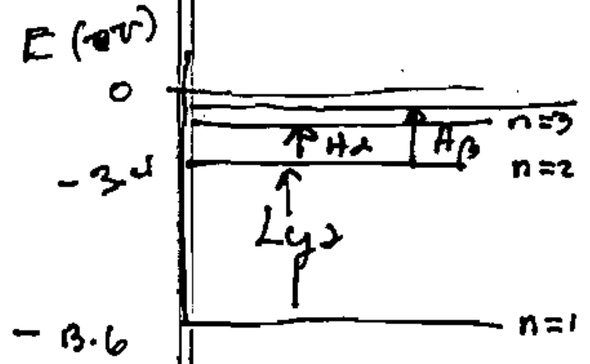
O B A F G K M

Hot ← T_e cool

~~The~~ the first breakthrough in understanding came with the identification of these lines as arising from transitions between bound energy levels of various elements:

Examples

Balmer Series of neutral hydrogen. (HI)



Balmer series arises from transitions arising from excited n=2 level

Hα (n=2→3) λ6563 Å

Hβ (n=2→4) λ4861 Å

CaII doublet, transitions arising in singly ionized Ca; i.e., Ca with one e^- missing CaII λλ 3934, 3969

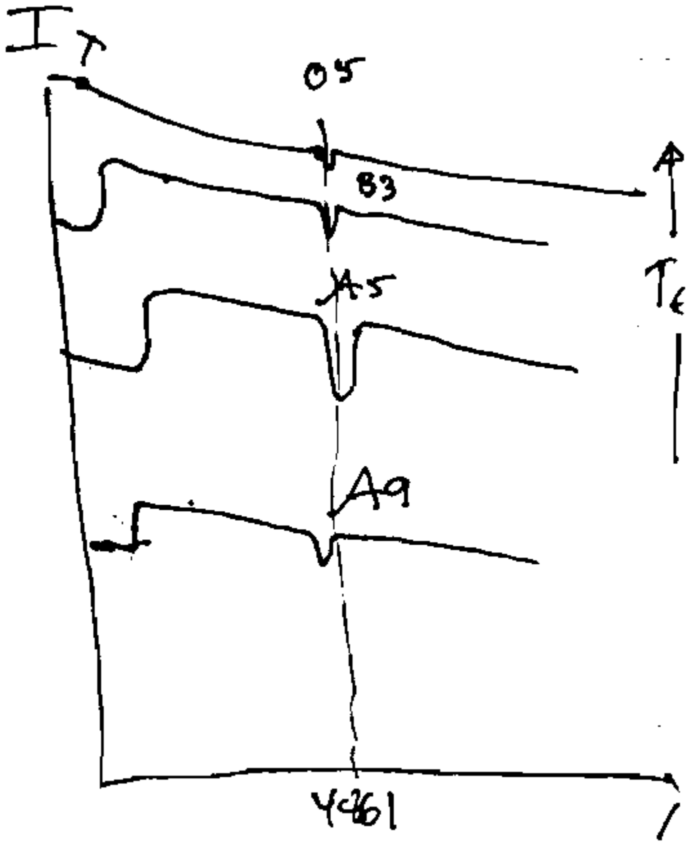
Basic Findings

(1) Wavelength of BB peak shifts redward as Te decrease

(2) H lines grow as O → A

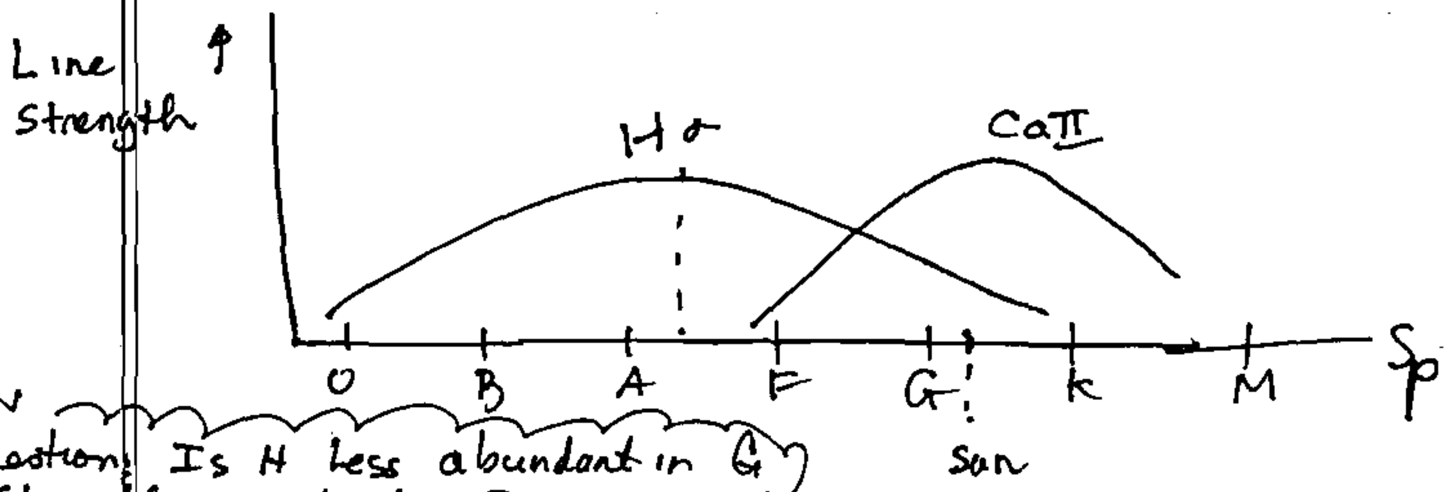
(3) H lines decrease as > A

(4) Metal lines & molecules appear in coolest stars where H is weak



Type	Te (K)	Color	M/M☉	R/R☉	L/L☉	fraction	Notes
O	$(6\to3)\times10^4$ O5→O9	Blue	≲60	~15	10^6	3×10^{-7}	• Not many lines • Strong HeII • HeI increases O5→O9
B	$(3\to1)\times10^4$ B0→B9	Blue/white	~18	~7	2×10^4	1.3×10^{-3}	• HeI increases to max at B2 • H lines appear, increase in later types • HeII lines weak
A	$(1\to.75)\times10^4$ A0→A9	white	~3	~2	80	6×10^{-3}	• Hα,β strongest in A0, weaker in A0→A9 • No HeII lines • CaII lines stronger A1→
F	$(7.5\to6.0)\times10^3$ F0→F9	Yellow/white	~1.7	1.3	6	3×10^{-2}	• CaII lines stronger: F5→ • Hα,β lines weaken
G	$(6\to5)\times10^3$ G0→G9	Yellow	~1.1	~1.1	1.2	8×10^{-2}	• Sun is G2 • CaII strongest at G2 • FeI appears • H weak
K	$(5\to3.5)\times10^3$ K0→K9	Orange	~0.8	~0.9	0.4	0.13	• CaII strongest at K0 • Neutral metal • molecular bands (TiO)
M	$(3.5\to2.2)\times10^3$	Red	~0.3	0.4	0.04	>0.80	• Most common star • Red dwarfs

Summary

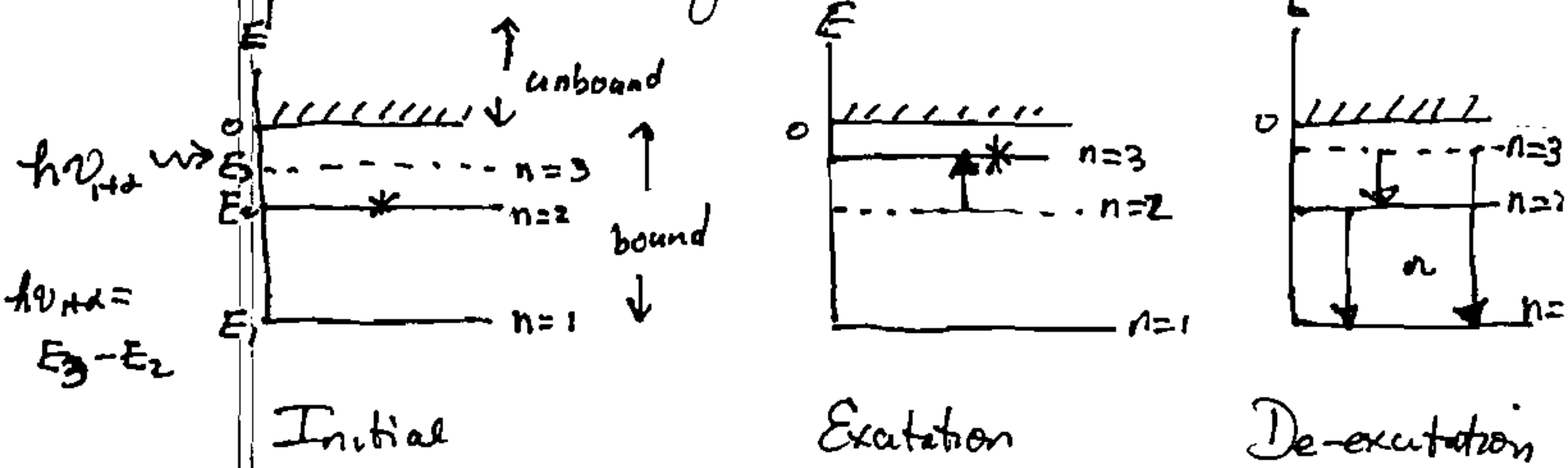


Question: Is H less abundant in G stars than in A stars?

Physics of Absorption line Formation

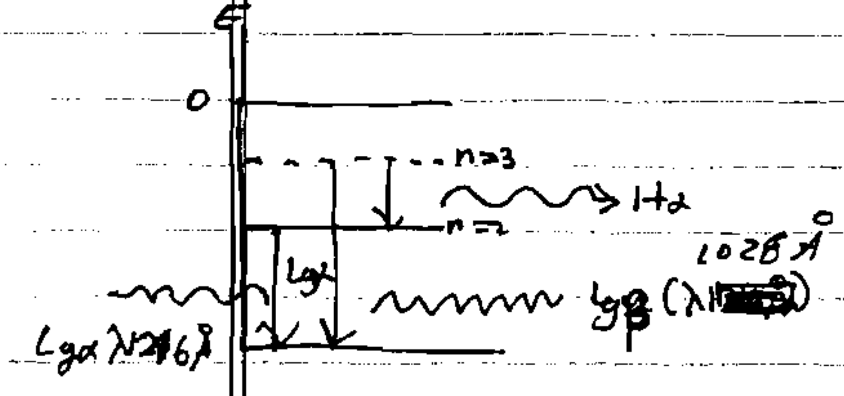
Absorption lines form when incident photons transfer energy, $h\nu$, to atom by exciting atomic states with energies higher than energies of initial state.

Example : Time Sequence ⟶

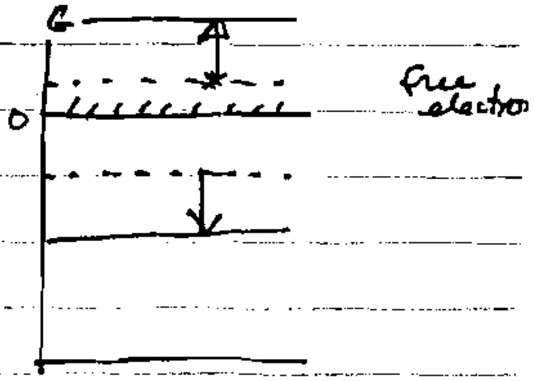


De-excitation to [redacted] $n=1$ state can occur by two different paths

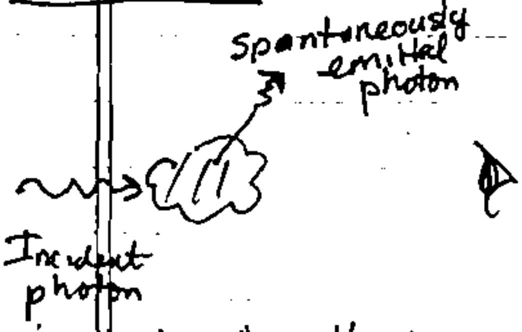
(A) Spontaneous photon emission



(B) Collisional de-excitation



Net Effect is an absorption line

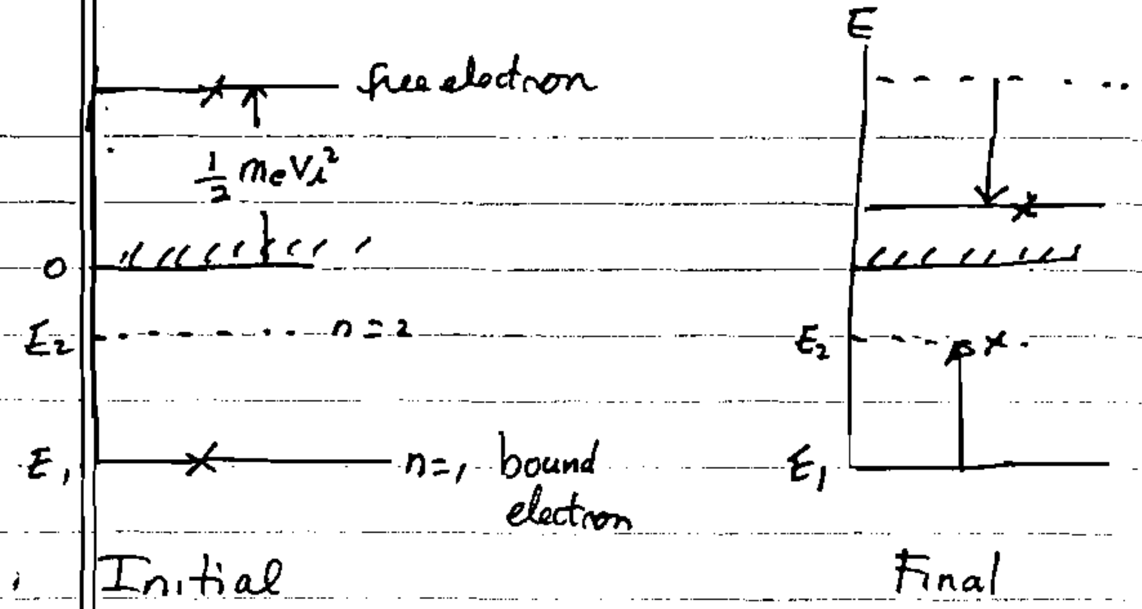


photons "scattered" out of incident beam

photon's energy converted into thermal energy of the gas

Question: How do bound electrons in HI get into excited n=2 state?

Answer: By inverse of collisional de-excitation, i.e., by collisional excitation due to free electrons!



Free electron transfers portion of its kinetic energy to excite n=2 state

Energetics

Initial $\quad E_{TOT}^{i} = \frac{1}{2} m_e V_i^2 + E_1 \qquad (E_1 = -13.6\,eV)$

Final $\quad E_{TOT}^{f} = \frac{1}{2} m_e V_f^2 + E_2 \qquad (E_2 = -3.4\,eV)$

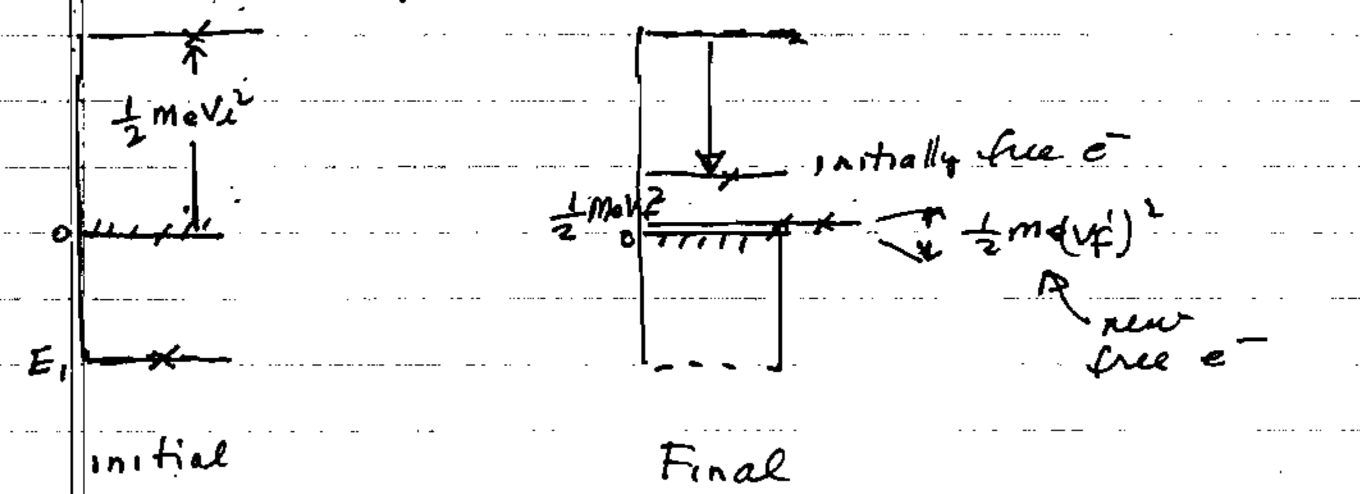
Energy conservation $\Rightarrow \quad E_{TOT}^{f} = E_{TOT}^{i}$

$$\frac{1}{2} m_e V_f^2 + E_2 = \frac{1}{2} m_e V_i^2 + E_1$$

$$\Delta E = \underbrace{E_2 - E_1}_{\text{gain in internal excitation energy}} = \underbrace{\frac{1}{2} m_e V_i^2 - \frac{1}{2} m_e V_f^2}_{\text{loss in kinetic energy of free } e^-} = 10.2\,eV$$

Schematically $\quad e^- + H^0 \rightarrow (H^0)^* + e^-$

Ionization: It is also possible for final energy of initially bound e^- to be positive. Thus bound electron is liberated from atom and becomes a free e^-.



In this case:

$$\frac{1}{2} m v_i^2 + E_1 = \frac{1}{2} m v_f^2 + \frac{1}{2} m \, (v_f')^2$$

Reaction: $e^- + H^0 \longrightarrow H^+ + e^- + (e^-{}')$

Can also ionize atoms when initial electrons are in excited bound states.

Basic Question: To understand physics underlying Sp distribution, we need to do the following:

(a) Compute probability for finding fraction of electrons in given interval of kinetic energies ΔE or speeds ΔV.

(b) Compute probability or fraction of atoms (or ions) with electrons in given bound states

(c) Compute fraction of elements in a given state of ionization.

Step a) First let's compute distribution of kinetic energies of free particles. This gives us fraction of e^- with kinetic energies able to excite bound states and ionize atoms or ions

Maxwellian Velocity Distribution

In TE

- all species coupled by Coulomb electrostatic interactions
- Rapid energy transfer back and forth results in unique distribution of kinetic energies that is independent of particle mass

Let $n_v(v)\,dv$ = no. of particles per unit volume with speeds in interval $(v, v+dv)$.

$$n_v(v)\,dv = n\left(\frac{m}{2\pi k T}\right)^{3/2} \exp\left(-\frac{\frac{1}{2}mv^2}{kT}\right) 4\pi v^2 dv$$

where Total density: $n = \int n_v(v)\,dv$

m = particle mass

T = kinetic temperature of gas describing translational deg. of freedom.

k = Boltzmann const

$k = 1.38\times10^{-16}$ erg/K or $k = 1.38\times10^{-23}$ J/K

Energy distribution

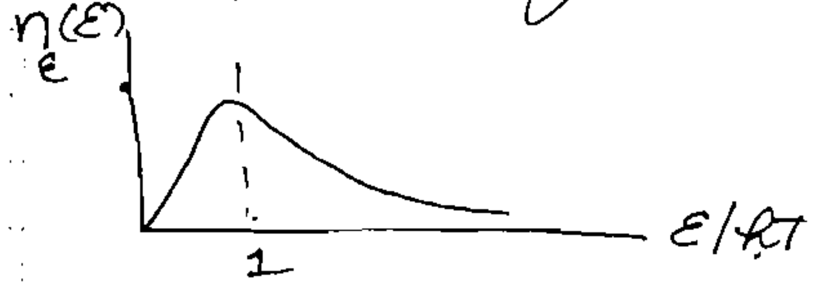
Let kinetic energy $\mathcal{E} = \frac{1}{2} m v^2$

Energy distribution $n_\mathcal{E}(\mathcal{E})\,d\mathcal{E} = n_v(v)\,dv$

$$n_\mathcal{E}(\mathcal{E}) = \frac{n_v[v(\mathcal{E})]}{|d\mathcal{E}/dv|}$$

$$\therefore\ n_\mathcal{E}(\mathcal{E}) \propto \frac{e^{-\mathcal{E}/kT} \cdot v^2}{v} \propto \mathcal{E}^{1/2} e^{-\mathcal{E}/kT}$$

independent of mass



- Characteristic energy $= kT$. Not many particles with $\mathcal{E} \gg kT$
- Distribution $n_v(v)$ peaks at most probable speed: $v_{mp} = \sqrt{\frac{2kT}{m}}$

(which does depend on m: more massive particles move slower).

- Mean Square Velocity: $\langle v^2 \rangle = \frac{\int_0^\infty v^2 n_v\,dv}{\int_0^\infty n_v\,dv} = \frac{3kT}{m}$