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PHYSICS 140A
MIDTERM EXAM 1
Thursday October 15, 2009

1. Systems A, B, and C are gases with state variables p_A, V_A, p_B, V_B, p_C, V_C. When A and C are in thermal equilibrium, the equation

$$p_A V_A - n\ b\ p_A - p_C V_C = 0$$

is found to be satisfied. When B and C are in thermal equilibrium, the equation

$$p_B V_B - p_C V_C + \frac{n\ e\ p_C V_C}{V_B} = 0$$

is satisfied. In the above equations, n, b and e are constants.

(a) What are the three functions which are equal to one another at thermal equilibrium, each of which is equal to the empirical temperature T?

(b) What is the relation expressing thermal equilibrium between A and B?

(a)

$$p_A\,(V_A - n\ b) = p_C V_C$$

$$\frac{p_B V_B}{\left(1 + \frac{n\ e}{V_B}\right)} = p_C V_C$$

$$\phi_A(p_A, V_A) = p_A\,(V_A - n\ b)$$

$$\phi_B(p_B, V_B) = \frac{p_B V_B}{\left(1 + \frac{n\ e}{V_B}\right)}$$

$$\phi_C(p_C, V_C) = p_C V_C$$

(b)

$$p_A\,(V_A - n\ b) = \frac{p_B V_B}{\left(1 + \frac{n\ e}{V_B}\right)}$$

2. The internal energy U of a photon gas is related to its volume V and temperature T by

$$\frac{U}{V} = aT^4,$$

where a is a constant. The equation of state is given by

$$p = \frac{1}{3}\frac{U}{V} = \frac{a}{3}\, T^4,$$

so that the pressure depends only on temperature.

(a) Calculate how the temperature of a photon gas varies with volume during a quasi-static, reversible, adiabatic compression of the gas. [You should find a result of the form $T^{\alpha}V$ = constant, with a specific value for the constant α.

(b) What is C_V for a photon gas?

(c) What is the enthalpy H for a photon gas?

(a)

$$U = U(T, V)$$

$$dU = \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\frac{\partial U}{\partial V}\right)_T dV$$

$$đQ = dU + p\, dV$$

$$đQ = \left(\frac{\partial U}{\partial T}\right)_V dT + \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] dV$$

$$0 = \left(\frac{\partial U}{\partial T}\right)_V dT + \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] dV = 4a\; VT^3\; dT + \left[a\; T^4 + \frac{1}{3}a\; T^4\right]\; dV$$

$$0 = 4a\; VT^3\; dT + \left[a\; T^4 + \frac{1}{3}a\; T^4\right]\; dV = 4a\; VT^3\; dT + \frac{4}{3}a\; T^4\; dV$$

$$0 = 4a\; VT^3\; dT + \frac{4}{3}a\; T^4\; dV$$

$$3\; \frac{dT}{T} + \frac{dV}{V} = 0$$

$$T^3 V = \text{constant}$$

(b)

$$U(T, V) = a\; V\; T^4$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 4a\; VT^3$$

(c)

$$H = U + p\; V = \frac{4}{3}\; a\; VT^4 = \frac{4}{3}\; U = 4\; p\; V$$

3. During a quasi-static, reversible and adiabatic expansion of an ideal gas, the pressure at any given moment is given by $pV^\gamma = c$, where γ and c are constants.

(a) Show that the work done by the gas in expansion from volume V_A to volume V_B is

$$W(A \to B) = \frac{p_A V_A - p_B V_B}{\gamma - 1} \; .$$

(b) If the initial pressure and volume of the gas are $p_A = 10^6$ Pa and $V_A = 10^{-3}$ m^3, and the final volume is $V_B = 3.2 \times 10^{-2}$ m^3, what is the final pressure? Assume that the gas is diatomic, so $\gamma = \frac{7}{5} = 1.4$. How many joules of work is done by the gas in the adiabatic expansion?

(c) What is the work done by one mole of the gas if the expansion is isothermal rather than adiabatic? What is the final pressure?

(d) Sketch a picture of the adiabatic and isothermal expansion curves for the expansions of parts (b) and (c) in the $p - V$ plane. (x-axis is V, y-axis is p.) Be sure to label the curves as adiabatic and isothermal.

(a)

$$W = \int_{V_A}^{V_B} p(V)\; dV = c \int_{V_A}^{V_B} V^{-\gamma}\; dV = \frac{c}{(1-\gamma)} \; \left(V^{1-\gamma}\right)|_{V_A}^{V_B}$$

$$= \frac{c}{(1-\gamma)} \; \left[{V_B}^{1-\gamma} - {V_A}^{1-\gamma}\right] = \frac{1}{(1-\gamma)} \; \left[p_B \; V_B - p_A \; V_A\right]$$

(b)

$$p_A {V_A}^\gamma = p_B {V_B}^\gamma$$

$$p_B = p_A \left(\frac{V_A}{V_B}\right)^\gamma = \left(10^6 \text{ Pa}\right) \left(\frac{1}{32}\right)^{7/5} = \frac{1}{128} \times 10^6 \text{ Pa}$$

$$W = \frac{p_A V_A - p_B V_B}{\gamma - 1} = \frac{5}{2} \left[\left(10^6 \text{ Pa}\right)\left(10^{-3} \text{ m}^3\right)\right] \left(1 - \frac{1}{4}\right) = \frac{15}{8} \times 10^3 \text{ J}$$

(c)

$$W = \int_{V_A}^{V_B} p(V)\; dV = R\,T \int_{V_A}^{V_B} \frac{dV}{V} = R\,T\; \ln\left(\frac{V_B}{V_A}\right)$$

$$= \; p_A V_A \; \ln\left(\frac{V_B}{V_A}\right) = \ln(32) \times 10^3 \text{ J} \; \approx 3.5 \times 10^3 \text{ J}.$$

$$p_A V_A = p_B V_B$$

$$p_B = p_A \left(\frac{V_A}{V_B}\right) = \frac{1}{32} \times 10^6 \text{ Pa}$$

4. In this problem, you consider the expansivity β and the isothermal compressibility κ.

(a) Show that β and κ of a general system satisfy

$$\left(\frac{\partial \beta}{\partial p}\right)_T + \left(\frac{\partial \kappa}{\partial T}\right)_p = 0 \ .$$

(b) Consider a van der Waals gas with equation of state

$$p = \frac{RT}{(v-b)} - \frac{a}{v^2} \ .$$

Compute β and κ for a van der Waals gas.

(c) Show that

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{\beta}{\kappa}$$

is generally true.

(d) Compute $\left(\frac{\partial p}{\partial T}\right)_v$ from the van der Waals gas equation of state. Verify that

$$\left(\frac{\partial p}{\partial T}\right)_v = \frac{\beta}{\kappa}$$

is true for the van der Waals gas.

(a)

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p$$

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T$$

$$\left(\frac{\partial \beta}{\partial p}\right)_T = \frac{\partial}{\partial p}_T\left[\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p\right] = \frac{1}{v}\left(\frac{\partial^2 v}{\partial p \partial T}\right) - \frac{1}{v^2}\left(\frac{\partial v}{\partial T}\right)_p\left(\frac{\partial v}{\partial p}\right)_T$$

$$\left(\frac{\partial \kappa}{\partial T}\right)_p = \frac{\partial}{\partial T}_p\left[-\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T\right] = -\frac{1}{v}\left(\frac{\partial^2 v}{\partial T \partial p}\right) + \frac{1}{v^2}\left(\frac{\partial v}{\partial p}\right)_T\left(\frac{\partial v}{\partial T}\right)_p = -\left(\frac{\partial \beta}{\partial p}\right)_T$$

(b)

$$T = \frac{1}{R}\left(p + \frac{a}{v^2}\right)(v-b)$$

$$\left(\frac{\partial T}{\partial v}\right)_p = \frac{1}{R}\left(p - \frac{a}{v^2} + \frac{2ab}{v^3}\right)$$

$$v\left(\frac{\partial T}{\partial v}\right)_p = \frac{1}{R}\left(pv - \frac{a}{v} + \frac{2ab}{v^2}\right)$$

$$\beta = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p = \frac{R}{\left(pv - \frac{a}{v} + \frac{2ab}{v^2}\right)}$$

$$p = \frac{RT}{(v-b)} - \frac{a}{v^2}$$

$$\left(\frac{\partial p}{\partial v}\right)_T = -\frac{RT}{(v-b)^2} + \frac{2a}{v^3}$$

$$-v\left(\frac{\partial p}{\partial v}\right)_T = \frac{RTv}{(v-b)^2} - \frac{2a}{v^2}$$

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T = \frac{1}{\frac{RTv}{(v-b)^2} - \frac{2a}{v^2}} = \frac{(v-b)}{pv - \frac{a}{v} + \frac{2ab}{v^2}}$$

$$\kappa = \frac{(v-b)}{R}\beta = \frac{T}{\left(p + \frac{a}{v^2}\right)}\beta$$

(c)

$$f(p,v,T) = 0$$

$$\left(\frac{\partial p}{\partial T}\right)_v \left(\frac{\partial T}{\partial v}\right)_p \left(\frac{\partial v}{\partial p}\right)_T = -1$$

$$\left(\frac{\partial p}{\partial T}\right)_v = -\frac{1}{\left(\frac{\partial T}{\partial v}\right)_p \left(\frac{\partial v}{\partial p}\right)_T} = -\frac{\left(\frac{\partial v}{\partial T}\right)_p}{\left(\frac{\partial v}{\partial p}\right)_T} = \frac{\frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_p}{-\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T} = \frac{\beta}{\kappa}$$

(d)

$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial}{\partial T}\right)_v \left(\frac{RT}{(v-b)} - \frac{a}{v^2}\right) = \frac{R}{(v-b)} = \frac{\beta}{\kappa}$$

$$\kappa = \frac{1}{\frac{RTv}{(v-b)^2} - \frac{2a}{v^2}} = \frac{(v-b)}{pv - \frac{a}{v} + \frac{2ab}{v^2}} = \frac{(v-b)}{R}\beta$$