1. Consider the Lagrangian density

\[ \mathcal{L} = \frac{\hbar^2}{2m} \nabla \psi \cdot \nabla \psi^* + V \psi \psi^* - \frac{i\hbar}{2}(\psi^* \dot{\psi} - \psi \dot{\psi}^*) \]

where \( \psi \) and \( \psi^* \) are to be treated as independent variables. Show that this leads to the time-dependent Schrödinger equation for \( \psi \) and it’s complex conjugate.

2. We have seen that the Euler-Lagrange equations applied to the Lagrangian density

\[ \mathcal{L} = \frac{1}{2} \left[ (\partial_\mu \varphi)(\partial^\mu \varphi) - m^2 \varphi^2 \right] \]

leads to the Klein-Gordon equation \((\Box + m^2)\varphi = 0\).

(a) Recall that the Lagrangian in classical point particle mechanics is of the form \( L = T - V = m\dot{x}^2/2 - V(x) \). Observing that

\[ \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i = p_i \]

we define the canonical momentum in general by

\[ p_i = \frac{\partial L}{\partial \dot{x}_i} \].

In a (relativistic) field theory, we similarly define the field momentum by

\[ \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(x)} \].

What is the canonical momentum of the K-G field?

(b) The Hamiltonian density is defined by \( \mathcal{H} = \pi \dot{\varphi} - \mathcal{L} \). What is the Hamiltonian \( H \) of the field \( \varphi \)?

(c) The K-G equation has plane wave solutions \( e^{\pm ikx} \) where \( k^2 = k_0^2 - \mathbf{k}^2 = m^2 \). If we define \( \omega_k = +\sqrt{k^2 + m^2} \), then we may write a general real solution as a linear superposition of plane waves in the form

\[ \varphi(x) = \int \frac{d^4k}{(2\pi)^4} \left( a_k e^{-ikx} + a_k^* e^{ikx} \right) 2\pi \delta(k^2 - m^2) \theta(k_0) \]
where the step function $\theta(k_0)$ is required to avoid double counting in the integral $\int d^4k$ over all space, and $a_k, a_k^\dagger$ are hermitian conjugate expansion coefficients. This form is really telling you that in the exponentials we have $k_0 = \omega_k$.

Consider the factor

$$ \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 - m^2) \theta(k_0). $$

Since Lorentz transformations are orthogonal (i.e., $\Lambda^T g \Lambda = g$ so that $\det \Lambda = +1$ for proper transformations), it follows that $d^4k$ is unchanged under a Lorentz transformation (i.e., the Jacobian of the transformation is just 1). It is also obvious that $k^2 - m^2$ is unchanged under a Lorentz transformation since both $k^2$ and $m^2$ are Lorentz scalars. Furthermore, a proper orthochronous Lorentz transformation can’t change the sign of $k_0$. This is easiest to see from the basic equation $\Delta t' = \gamma(\Delta t - \beta \Delta x)$. Since $\beta < 1$ and $\Delta x < \Delta t$, we must have $\Delta t' > 0$ if and only if $\Delta t > 0$. Therefore the above factor is manifestly Lorentz invariant. Show that

$$ \frac{d^3k}{(2\pi)^3 2\omega_k} = \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2 - m^2) \theta(k_0). $$

Because of this, the term

$$ \frac{d^3k}{(2\pi)^3 2\omega_k} $$

is referred to as the invariant volume element. Note that the numerical factors aren’t necessary to maintain invariance.

(d) By considering the integral $f(p) = \int d^3q \delta(p - q) f(q)$ (where $p^2 = q^2 = m^2$), argue that

$$ (2\pi)^3 2\omega_p \delta(p - q) $$

is also Lorentz invariant. (This is called the invariant delta function.)

(e) We may now write

$$ \varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left( a_k e^{-ikx} + a_k^\dagger e^{ikx} \right). $$

If we let

$$ f_k(x) = \frac{e^{-ikx}}{[(2\pi)^3 2\omega_k]^{1/2}} $$

then we have

$$ \varphi(x) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} [f_k(x) a_k + f_k^*(x) a_k^\dagger]. $$

2
Define an “inner product” on functions $f(x), g(x)$ by

$$\langle f(x), g(x) \rangle := \int d^3 x f^*(x) \overrightarrow{i \partial_0} g(x)$$

$$:= \int d^3 x f^*(x) [i \partial_0 g(x)] - [i \partial_0 f^*(x)] g(x)$$

where $\partial_0 = \partial/\partial x^0 = \partial/\partial t$. Show that

$$\int d^3 x f_k^*(x) \overrightarrow{i \partial_0} f_{k'}(x) = \delta(k - k')$$

while

$$\int d^3 x f_k^*(x) \overrightarrow{i \partial_0} f_{k'}(x) = \int d^3 x f_k(x) \overrightarrow{i \partial_0} f_{k'}(x) = 0.$$ 

In other words, the functions $f_k(x)$ form an orthonormal set with respect to the above inner product.

(f) Show that

$$a_k = \int d^3 x [(2\pi)^3 2\omega_k]^{1/2} f_k^*(x) \overrightarrow{i \partial_0} \varphi(x)$$

and

$$a_k^\dagger = \int d^3 x [(2\pi)^3 2\omega_k]^{1/2} \varphi(x) \overrightarrow{i \partial_0} f_k(x).$$

(g) So far all of this has applied to a classical field. We go over to a quantum field by analogy with ordinary quantum mechanics where we have (with $\hbar = 1$) the operator commutation relations $[x_i, p_j] = i \delta_{ij}$. Thus we require the equal time canonical commutation relations

$$[\varphi(x, t), \pi(y, t)] = i \delta(x - y)$$

and

$$[\varphi(x, t), \varphi(y, t)] = [\pi(x, t), \pi(y, t)] = 0.$$ 

Show that these relations imply

$$[a_k, a_{k'}^\dagger] = (2\pi)^3 2\omega_k \delta(k - k')$$

and

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0.$$ 

This shows that the coefficients $a_k$ and $a_k^\dagger$ become the annihilation and creation operators of a harmonic oscillator. Thus the field $\varphi(x)$ acts to annihilate and create quanta with 4-momentum $k^\mu$. 

3
(h) Let us now write
\[ \phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} (a_k e^{-ikx} + a_k^\dagger e^{ikx}) \]
where
\[ \varphi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k e^{-ikx} \]
and
\[ \varphi^-(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k^\dagger e^{ikx} \]
are referred to as the positive and negative frequency parts of \( \phi(x) \). Show that
\[ [\varphi^+(x), \varphi^-(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x-y)} := i\Delta^+(x-y) \]
where \( \Delta^+(x-y) \) is the same function defined in Exercise 15 of Problem Set 1. This shows how the Green’s function arises in a natural way in quantum field theory.

3. We have seen that the rotation operator for a vector in \( \mathbb{R}^3 \) can be written in the form
\[ R(\theta) = e^{\theta \cdot J}. \]
Now let’s take a look at how the spatial rotation operator is defined in quantum mechanics. If we rotate a vector \( \mathbf{x} \) in \( \mathbb{R}^3 \), then we obtain a new vector \( \mathbf{x}' = R(\theta)\mathbf{x} \) where \( R(\theta) \) is the matrix that represents the rotation. In two dimensions this is
\[ \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \]
If we have a scalar wavefunction \( \psi(\mathbf{x}) \), then under rotation we obtain a new wavefunction \( \psi_R(\mathbf{x}') \), where \( \psi(\mathbf{x}) = \psi_R(\mathbf{x}') = \psi_R(R(\theta)\mathbf{x}) \). (See the figure below. This is for an active transformation.)

Alternatively, we can write
\[ \psi_R(\mathbf{x}) = \psi(R^{-1}(\theta)\mathbf{x}). \]
Since $R$ is an orthogonal transformation (it preserves the length of $x$) we know that $R^{-1}(\theta) = R^T(\theta)$, and in the case where $\theta \ll 1$ we then have

$$R^{-1}(\theta)x = \begin{bmatrix} x + \theta y \\ -\theta x + y \end{bmatrix}.$$  

Expanding $\psi(R^{-1}(\theta)x)$ with these values for $x$ and $y$ we have

$$\psi_R(x) = \psi(x + \theta y, y, \theta x) = \psi(x) - \theta[x\partial_y - y\partial_x]\psi(x)$$

or, using $p^i = -i\partial_i$, this is

$$\psi_R(x) = \psi(x) - i\theta[xp_y - yp_x]\psi(x) = [1 - i\theta L_z]\psi(x).$$

For finite $\theta$ we exponentiate this to write $\psi_R(x) = e^{-i\theta L_z}\psi(x)$, and in the case of an arbitrary angle $\theta$ in $\mathbb{R}^3$ this becomes

$$\psi_R(x) = e^{-i\theta \cdot L}\psi(x).$$

This was for spatial rotations. In general, for a particle with total angular momentum $J$ (where $[J_i, J_j] = i\varepsilon_{ijk}J_k$) we define the rotation operator to be

$$R(\theta) = e^{-i\theta \cdot J}.$$  

In the particular case of a spin one-half particle we have $J = S = \sigma/2$ where the Pauli matrices $\sigma$ are defined by

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and obey the commutation relations

$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k.$$  

(a) Show that the Pauli matrices obey the relations

$$\sigma_i\sigma_j = i\varepsilon_{ijk}\sigma_k \quad \text{for } i \neq j$$

and

$$[\sigma_i, \sigma_j]_+ := \sigma_i\sigma_j + \sigma_j\sigma_i = 2I\delta_{ij}$$

and hence that we also have the very useful result

$$\sigma_i\sigma_j = I\delta_{ij} + i\varepsilon_{ijk}\sigma_k.$$  

(b) Show that for any vectors $a$ and $b$ we have

$$(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b)I + i(a \times b) \cdot \sigma.$$
(c) Show that the rotation operator for spin one-half particles is given by
\[
e^{-i\theta \cdot \sigma / 2} = I \cos \frac{\theta}{2} - i(\sigma \cdot \hat{\theta}) \sin \frac{\theta}{2}
\]
\[
= \begin{bmatrix}
\cos \theta/2 - i\hat{\theta}_3 \sin \theta/2 & -i\hat{\theta}_- \sin \theta/2 \\
-i\hat{\theta}_+ \sin \theta/2 & \cos \theta/2 + i\hat{\theta}_3 \sin \theta/2
\end{bmatrix}
\]
where \( \hat{\theta}_\pm = \hat{\theta}_1 \pm i\hat{\theta}_2 \).

4. In Problem 2 you looked at some properties of the real Klein-Gordon field which describes neutral scalar particles (i.e., spinless neutral particles). In this problem you will look at some properties of the complex Klein-Gordon field which, as you will see, describes charged spinless particles. So, consider the Lagrangian density
\[
\mathcal{L} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi
\]
where I have written \( \varphi^\dagger \) rather than \( \varphi^* \) because we will be treating the fields \( \varphi \) as quantum mechanical operators rather than simply complex classical fields.

(a) Treating \( \varphi \) and \( \varphi^\dagger \) as independent fields, what are the corresponding equations of motion for this Lagrangian density? What are the canonical momenta \( \pi \) and \( \pi^\dagger \)?

(b) Now observe that this Lagrangian density is invariant under the global phase transformation
\[
\varphi(x) \rightarrow \varphi'(x) = e^{-i\Lambda \varphi(x)} \quad \text{and} \quad \varphi^\dagger(x) \rightarrow \varphi'^\dagger(x) = e^{i\Lambda \varphi^\dagger(x)}
\]
where \( \Lambda \ll 1 \) is a constant. Then we know that there exists a conserved current \( j^\mu \) with \( \partial_\mu j^\mu = 0 \) and where
\[
j^\mu = \frac{\partial \mathcal{L}}{\partial \varphi^r,\mu} \Delta \varphi^r - T^{\mu}_{\alpha} \delta x^\alpha
\]
with
\[
\Delta \varphi^r(x) = \delta \varphi^r(x) + \partial_\mu \varphi^r(x) \delta x^\mu
\]
and
\[
T^{\mu}_{\alpha} = \frac{\partial \mathcal{L}}{\partial \varphi^r,\mu} \varphi^r,\alpha - \delta^\mu_{\alpha} \mathcal{L}.
\]
What is the conserved charge
\[
Q = \int j^0(x) d^3 x ?
\]

Note that any overall factor of the constant \( \Lambda \) can be dropped from \( Q \) since \( d(\Lambda Q)/dt = 0 \) implies that \( dQ/dt = 0 \) also.
Following exactly the same approach as we did for the real K-G field, we expand the complex fields as
\[
\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k^{1/2}} \left\{ a_k f_k(x) + b_k^\dagger f_k^\ast(x) \right\}
\]
and
\[
\varphi^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k^{1/2}} \left\{ b_k f_k(x) + a_k^\dagger f_k^\ast(x) \right\}
\]
where we write \( f_k^\ast(x) \) for the complex conjugate of the ordinary function \( f_k(x) \), and \( a_k^\dagger \) for the “complex conjugate” (i.e., the adjoint) of the quantum mechanical operator \( a_k \). We have labelled the expansion coefficients of the positive and negative frequency parts by \( a_k \) (resp. \( b_k^\dagger \)) and \( b_k \) (resp. \( a_k^\dagger \)) because the fields are complex so there is no requirement that \( a_k \) and \( b_k^\dagger \) be complex conjugates of each other. We have added the dagger on \( b_k \) because with hindsight we know that it behaves as a creation operator.

We now impose the canonical equal time commutation relations (ETCR’s) on the fields:
\[
[\varphi(x, t), \pi(y, t)] = [\varphi^\dagger(x, t), \varphi(y, t)] = i\delta(x - y)
\]
and
\[
[\varphi^\dagger(x, t), \pi^\dagger(y, t)] = [\varphi(x, t), \varphi^\dagger(y, t)] = i\delta(x - y)
\]
and where all other commutators vanish.

Show that
\[
[Q, \varphi(x)] = -\varphi(x) \quad \text{and} \quad [Q, \varphi^\dagger(x)] = +\varphi^\dagger(x).
\]

Hint: Recall the commutator identity \([ab, c] = a[b, c] + [a, c]b\).

To help understand what these mean, let \(|q\rangle\) be an eigenstate of \( Q \), so that \( Q|q\rangle = q|q\rangle \). Then \( Q\varphi|q\rangle = \varphi(Q - 1)|q\rangle = (q - 1)\varphi|q\rangle \) and hence \( \varphi \) decreases the eigenvalue of \(|q\rangle\) by one. Similarly, \( Q\varphi^\dagger|q\rangle = \varphi^\dagger(Q + 1)|q\rangle = (q + 1)\varphi^\dagger|q\rangle \) and therefore \( \varphi^\dagger \) increases the eigenvalue of \(|q\rangle\) by one.

(d) Solving for the coefficients \( a_k, a_k^\dagger, b_k, b_k^\dagger \), as we did in the real case, we can then use the ETCR’s to finally show that (after a lot of algebra which wouldn’t hurt you to work out for yourself)
\[
[a_k, a_k^\dagger] = [b_k, b_k^\dagger] = (2\pi)^2 2\omega_k \delta(k - k')
\]
and
\[
[a_k, b_k^\dagger] = [a_k, b_k^\dagger] = 0.
\]
Thus we see that both $a_k^\dagger$ and $b_k^\dagger$ create harmonic oscillator quanta.

Show that

$$Q = \int \frac{d^3 k}{(2\pi)^3 2\omega_k} [a_k^\dagger a_k - b_k^\dagger b_k - (2\pi)^3 2\omega_k \delta(0)] .$$

Dropping the infinite constant $(2\pi)^3 2\omega_k \delta(0)$, we see that $Q$ represents the total number of $a$ quanta minus the total number of $b$ quanta, and hence $Q$ is essentially just the total charge operator.