Homework Set 5

1. In class we showed that the lowest-order relativistic correction to the kinetic energy (i.e., to the Hamiltonian) is given by

\[ H' = H_{\text{rel}} = -\frac{p^4}{8m^3c^2}. \]

Using this, evaluate the first-order relativistic correction \( E_{\text{rel}}^{(1)} \) to the energy levels of the one-dimensional harmonic oscillator. \textit{Hints:} From \( H = E = \frac{p^2}{2m} + V \), show that

\[ E_{\text{rel}}^{(1)} = -\frac{1}{2m^2c^2} \left[ E^2 - 2E\langle V \rangle + \langle V^2 \rangle \right]. \]

Now write the harmonic oscillator potential in terms of the creation and annihilation operators

\[ a_\pm = \mp \frac{ip + m\omega x}{\sqrt{2\hbar\omega}}. \]

2. The most prominent feature of the visible hydrogen atom spectrum is the red Balmer line representing the \( n = 3 \) to \( n = 2 \) transition. Because of the fine structure Hamiltonian, this line splits into several closely spaced lines.

(a) Determine how many sublevels the \( n = 2 \) line splits into, and find \( E_{\text{fs}}^{(1)} \) for each of them, in eV.

(b) Repeat part (a) for the \( n = 3 \) level.

(c) Draw an energy level diagram that shows all possible transitions from the \( n = 3 \) to the \( n = 2 \) level.

(d) The energy of the emitted photon in each transition is given by \( E_3 - E_2 + \Delta E \) where the factor \( E_3 - E_2 \) is the same for all transitions, and the term \( \Delta E \) is due to \( H_{\text{fs}} \) and varies from one transition to another. Find \( \Delta E \) in eV for each transition.

(e) Convert \( \Delta E \) to photon frequency, and determine the spacing between adjacent spectral lines, in Hz. Be sure to clearly label each transition along with the frequency spacing relative to its adjacent lines.

3. Analyze the Zeeman effect for the \( n = 3 \) levels of hydrogen in the strong, weak and intermediate regimes as follows.

(a) First consider the strong-field regime. Construct a table of energies similar to the energies listed on pages 65-66 of the class notes. In the table, list the \( l \) value, the state \(|n l m_l m_s\rangle\), the value of \( m_l + 2m_s \), and the value of
the term in \{brackets\} in equation 2.72 of the notes. Write your answer for the total energy in terms of \(E_3^{(0)} = E_1^{(0)}/3^2 = -13.6/9 = -1.51\) eV, 
\(\beta = \mu_B B\) and the fine structure constant \(\alpha\).

(b) Repeat part (a) in the weak-field regime. Make up a similar table, but instead of the term \(m_l + 2m_s\) used in the strong-field case, list the value of \(g_J\). And instead of the term from equation 2.72, list the value of the term in \{brackets\} in equation 2.59.

(c) Construct a table of energies in the intermediate-field regime. You will find it convenient to write the energies \(E_{ls}^{(1)}\) in terms of the constant
\[
\gamma = \frac{13.6}{324} \alpha^2.
\]
Show that these results reduce to both the strong- and weak-field limits. Plot the energies as functions of the external \(B\) field as in Figure 8 of the notes. If you explain what you are doing, you may plot only half of the energies. For each of the energy corrections you plot, label each line by its slope at both very small \(B\) and at very large \(B\).

4. A hydrogen atom is in the ground state at \(t = -\infty\). An electric field \(E(t) = \mathcal{E} e^{-t^2/\tau^2} \mathbf{\hat{z}}\) is applied until \(t = \infty\). Show that the probability the atom ends up in any of the \(n = 2\) states is, to first order,
\[
\mathcal{P}_{n=1 \rightarrow n=2}(t \rightarrow \infty) = \left(\frac{e\mathcal{E}}{\hbar}\right)^2 \left(\frac{215 \mathcal{E}_0^2}{310}\right) \pi \gamma^2 e^{-\omega_{21}^2 \tau^2/2}
\]
where \(\omega_{21} = (E_{2l(m) - E_{100})}/\hbar\). Does the answer depend on whether or not we include spin? Hint: You can find the spherical harmonics you will need in Griffiths, Table 4.3, and the radial functions in Table 4.7.

5. Suppose you have a spin one-half particle at rest in a static magnetic field \(B = B_0 \mathbf{\hat{z}}\). From Section 4.2 of the Angular Momentum notes we know that the particle will precess at the Larmor frequency
\[
\omega_0 = \frac{gqB_0}{2mc} := \gamma B_0
\]
where \(\gamma\) is the gyromagnetic ratio. Now suppose that we turn on an additional transverse rf (radio frequency) magnetic field so that the total field becomes
\[
\mathbf{B} = B_{rf} \cos \omega t \mathbf{\hat{x}} - B_{rf} \sin \omega t \mathbf{\hat{y}} + B_0 \mathbf{\hat{z}}.
\]

(a) What is the \(2 \times 2\) Hamiltonian matrix for this system?

(b) Let the spin state at time \(t\) be
\[
\chi(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}.
\]
Defining $\Omega = \gamma B_{\text{rf}}$, show that

$$\dot{a} = \frac{i}{2} (\Omega e^{i\omega t} b + \omega_0 a) \quad \text{and} \quad \dot{b} = \frac{i}{2} (\Omega e^{-i\omega t} a - \omega_0 b).$$

(c) Solve these coupled differential equations by hand, and show that their solution is

$$a(t) = \left\{ a_0 \cos(\omega' t/2) + \frac{i}{\omega} [a_0 (\omega_0 - \omega) + b_0 \Omega \sin(\omega' t/2)] \right\} e^{i\omega t/2}$$

$$b(t) = \left\{ b_0 \cos(\omega' t/2) + \frac{i}{\omega} [b_0 (\omega - \omega_0) + a_0 \Omega \sin(\omega' t/2)] \right\} e^{-i\omega t/2}$$

where $a_0 = a(0)$, $b_0 = b(0)$ and

$$\omega' = \sqrt{(\omega - \omega_0)^2 + \Omega^2}.$$

Be sure to clearly show your work, and don’t turn in scratch paper.

(d) If the particle is initially in the spin up state, find the probability as a function of time that it makes a transition to the spin down state.

(e) Sketch the resonance curve

$$P(\omega) = \frac{\Omega^2}{(\omega - \omega_0)^2 + \Omega^2}$$

as a function of the driving frequency $\omega$ (i.e., where $\omega_0$ and $\Omega$ are fixed). Find $\Delta \omega$, the width at half maximum.

(f) Note that the maximum of $P(\omega)$ occurs at $\omega = \omega_0$. Since $\omega_0 = \gamma B_0$, this resonance can be used to determine the magnetic dipole moment of the particle. Suppose that in an nmr (nuclear magnetic resonance) experiment, we wish to measure the $g$-factor of the proton. If the static magnetic field strength is 10,000 gauss, and the rf field has an amplitude of 0.01 gauss, what will the resonant frequency be (in Hz)? Find the width of the resonant curve (in Hz).