Solutions Assignment 7

7.16 The kinetic energy of a cylinder of mass m is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2,$$

where $\dot{x}$ is the velocity of the center of mass, I is the moment of inertia of the disk about its center of mass, and ω is its angular velocity. If x is the linear coordinate measured *down* an incline and the cylinder rolls without slipping then the kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\dot{x}^2/R^2 = \frac{3}{4}m\dot{x}^2,$$

where R is the radius of the disk. If the angle of the incline is α, then the potential energy of the disk is

$$U = -mgx\sin\alpha.$$

The Lagrangian is

$$\mathcal{L} = T - U = \frac{3}{4}m\dot{x}^2 + mgx\sin\alpha.$$

The Lagrange equation of motion is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \to \frac{d}{dt}\frac{3}{2}m\dot{x} = \frac{3}{2}m\ddot{x} = mg\sin\alpha.$$

Hence

$$\ddot{x} = \frac{2}{3}g\sin\alpha.$$

7.19 The acceleration of the block relative to the wedge was found to be

$$\ddot{q}_1 = \frac{M+m}{M+m-m\cos^2\alpha}g\sin\alpha,$$

while the acceleration of the wedge relative to the table is

$$\begin{aligned}
\ddot{q}_2 &= -\frac{m}{M+m}\ddot{q}_1\cos\alpha = -\frac{m}{M+m}\frac{M+m}{M+m-m\cos^2\alpha}g\sin\alpha\cos\alpha, \\
\ddot{q}_2 &= -\frac{m}{M+m-m\cos^2\alpha}g\sin\alpha\cos\alpha.
\end{aligned}$$

The acceleration of the block relative to the table, $\ddot{x}$, is

$$\begin{aligned}
\ddot{x} &= \ddot{q}_2 + \ddot{q}_1\cos\alpha = -\frac{m}{M+m-m\cos^2\alpha}g\sin\alpha\cos\alpha + \frac{M+m}{M+m-m\cos^2\alpha}g\sin\alpha\cos\alpha \\
\ddot{x} &= \frac{M}{M+m-m\cos^2\alpha}g\sin\alpha\cos\alpha.
\end{aligned}$$

In the limit that $M \to 0$ the acceleration of the block relative to the table is $\ddot{x} = 0$. This is as expected for if the wedge is massless, due to conservation of momentum, the block will not accelerate in the horizontal direction. Meanwhile a massless wedge will experience an acceleration (relative to the table) given by

$$\ddot{q}_2 = -\frac{1}{\sin^2\alpha} g \sin\alpha \cos\alpha = -g\frac{\cos\alpha}{\sin\alpha}.$$

This is also to be expected for the block will fall vertically with an acceleration of g. This will result in the wedge accelerating to the left with an acceleration of $\ddot{q}_2 = -g\cos\alpha/\sin\alpha$.

7.20 The kinetic energy for a bead on the helix is

$$T = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}mR^2\dot{\phi}^2,$$

where $z = \lambda\phi$. Since the potential energy due to the gravitational field is $U = mgz$ we will keep z as a generalized coordinate and replace $\dot{\phi}$ with $\dot{z}/\lambda$. The Lagrangian is then

$$\mathcal{L} = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\frac{R^2}{\lambda^2}\dot{z}^2 - mgz = \frac{1}{2}m\frac{\lambda^2 + R^2}{\lambda^2}\dot{z}^2 - mgz.$$

The Lagrange equation is

$$\frac{\partial\mathcal{L}}{\partial z} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{z}} \to -mg = m\frac{\lambda^2 + R^2}{\lambda^2}\ddot{z}.$$

The vertical acceleration is

$$\ddot{z} = -\frac{\lambda^2}{\lambda^2 + R^2}g.$$

In the limit that $R \to 0$, $\ddot{z} = -g$, or the helix has become a vertical wire and the bead is falling straight down.

7.22 The x and y coordinates for the pendulum bob are

$$x = l\sin\phi \text{ and } y = l(1 - \cos\phi) + \frac{1}{2}at^2.$$

Hence the velocities in these directions are

$$\dot{x} = l\cos\phi\dot{\phi} \text{ and } \dot{y} = l\sin\phi\dot{\phi} + at.$$

The kinetic energy of the bob is

$$\begin{aligned} \mathcal{T} &= \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}m\left(l^2\cos^2\phi\dot{\phi}^2 + l^2\sin^2\phi\dot{\phi}^2 + 2atl\sin\phi\dot{\phi} + a^2t^2\right) \\ T &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + 2atl\sin\phi\dot{\phi} + a^2t^2\right) \end{aligned}$$

The potential energy and Lagrangian of the bob are

$$\begin{aligned} U &= mgy = mg\left(l\left(1-\cos\phi\right)+\frac{1}{2}at^2\right) \\ \mathcal{L} &= \frac{1}{2}m\left(l^2\dot{\phi}^2+2atl\sin\phi\dot{\phi}+a^2t^2\right)-mg\left(l\left(1-\cos\phi\right)+\frac{1}{2}at^2\right). \end{aligned}$$

The equation of motion for the accelerating bob is

$$\begin{aligned} &\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\frac{d}{dt}m\left(l^2\dot{\phi}+atl\sin\phi\right)=\frac{\partial\mathcal{L}}{\partial\phi}=m\left(atl\cos\phi\dot{\phi}-gl\sin\phi\right) \\ &l^2\ddot{\phi}+al\sin\phi+atl\cos\phi\dot{\phi}=atl\cos\phi\dot{\phi}-gl\sin\phi \\ &l\ddot{\phi}=-\left(g+a\right)\sin\phi. \end{aligned}$$

Here we find the normal equation of motion for a pendulum except that g has been replaced with $g+a$, which is what would be expected from Einstein's equivalence principle. Hence the angular frequency of small oscillations is $\omega=\sqrt{\left(g+a\right)/l}$

7.27 For the double Atwood machine we will assume that the coordinate (pointed downward) for the mass $4m$ is y. The coordinate (again pointed downward) from the second pulley to the mass $3m$ is x. Using these coordinates the kinetic energy of the three masses is

$$\begin{aligned} T &= \frac{1}{2}4m\dot{y}^2+\frac{1}{2}3m\left(-\dot{y}+\dot{x}\right)^2+\frac{1}{2}m\left(-\dot{y}-\dot{x}\right)^2 \\ T &= \frac{1}{2}4m\dot{y}^2+\frac{1}{2}3m\left(\dot{x}^2-2\dot{y}\dot{x}+\dot{y}^2\right)+\frac{1}{2}m\left(\dot{x}^2+2\dot{y}\dot{x}+\dot{y}^2\right) \\ T &= \frac{m}{2}\left(4\dot{x}^2-4\dot{y}\dot{x}+8\dot{y}^2\right). \end{aligned}$$

The potential energy (to within a constant) is

$$U=-4mgy-3mg\left(x-y\right)-mg\left(-x-y\right)=-2mgx.$$

Hence the Lagrangian is

$$\mathcal{L}=\frac{m}{2}\left(4\dot{x}^2-4\dot{y}\dot{x}+8\dot{y}^2\right)+2mgx.$$

The equations of motion are

$$\begin{aligned} \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} &= \frac{d}{dt}\frac{m}{2}\left(8\dot{x}-4\dot{y}\right)=\frac{\partial\mathcal{L}}{\partial x}=2mg \\ 2\ddot{x}-\ddot{y} &= g \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} &= \frac{d}{dt}\frac{m}{2}\left(-4\dot{x}+16\dot{y}\right) = \frac{\partial \mathcal{L}}{\partial y} = 0 \\ \ddot{x} &= 4\ddot{y}\end{aligned}$$

Solving for $\ddot{y}$ we find

$$\ddot{y} = g/7.$$

The masses attached to the second pulley are accelerating, hence the tension in the string supporting this pulley is not equal to $4mg$.

7.29 Using the hint provided in the problem, the Cartesian coordinates for the bob whose support is attached to the edge of a wheel, as measured from the center of the wheel are

$$\begin{aligned} x &= R\cos\omega t + l\sin\phi, \\ y &= R\sin\omega t - l\cos\phi. \end{aligned}$$

Taking the time derivative of these expressions yields

$$\begin{aligned} \dot{x} &= -\omega R\sin\omega t + l\cos\phi\dot{\phi}, \\ \dot{y} &= \omega R\cos\omega t + l\sin\phi\dot{\phi}. \end{aligned}$$

We can now write the kinetic energy as

$$\begin{aligned} T &= \frac{1}{2}m\left(\omega^2R^2 - 2\omega Rl\left(\sin\omega t\cos\phi - \cos\omega t\sin\phi\right)\dot{\phi} + l^2\dot{\phi}^2\right), \\ T &= \frac{1}{2}m\left(l^2\dot{\phi}^2 + \omega^2R^2 + 2\omega Rl\sin\left(\phi-\omega t\right)\dot{\phi}\right). \end{aligned}$$

The potential energy is simply $U = mgy = mg\left(R\sin\omega t - l\cos\phi\right)$. Therefore the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(l^2\dot{\phi}^2 + \omega^2R^2 + 2\omega Rl\sin\left(\phi-\omega t\right)\dot{\phi}\right) + mg\left(l\cos\phi - R\sin\omega t\right).$$

For the Lagrange equations we need

$$\frac{\partial \mathcal{L}}{\partial \phi} = m\omega Rl\cos\left(\phi-\omega t\right)\dot{\phi} - mgl\sin\phi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = ml^2\dot{\phi} + m\omega Rl\sin\left(\phi-\omega t\right).$$

Thus the Lagrange equation is

$$\begin{aligned} \omega R\cos\left(\phi-\omega t\right)\dot{\phi} - g\sin\phi &= \frac{d}{dt}\left(l\dot{\phi} + \omega R\sin\left(\phi-\omega t\right)\right), \\ \omega R\cos\left(\phi-\omega t\right)\dot{\phi} - g\sin\phi &= l\ddot{\phi} + \omega R\cos\left(\phi-\omega t\right)\dot{\phi} - \omega^2R\cos\left(\phi-\omega t\right), \\ l\ddot{\phi} &= -g\sin\phi + \omega^2R\cos\left(\phi-\omega t\right). \end{aligned}$$

Note that as $\omega \to 0$ this equation of motion becomes that for a simple pendulum.

7.30 (a) The Cartesian coordinates of the bob are inside an accelerating railroad car are

$$x = \frac{1}{2}at^2 + l\sin\phi \quad \text{and} \quad y = l\cos\phi,$$

where we are measuring y as positive in the downward direction. Taking the time derivative of these expressions yields

$$\dot{x} = at + l\cos\phi\dot{\phi} \quad \text{and} \quad \dot{y} = -l\sin\phi\dot{\phi}.$$

Thus the kinetic energy of the bob is

$$T = \frac{1}{2}m\left(a^2t^2 + l^2\dot{\phi}^2 + 2atl\cos\phi\dot{\phi}\right),$$

and the potential energy is $U = -mgy = -mgl\cos\phi$. Thus the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m\left(a^2t^2 + l^2\dot{\phi}^2 + 2atl\cos\phi\dot{\phi}\right) + mgl\cos\phi.$$

The quantities required for the Lagrange equation for the ϕ coordinate are

$$\frac{\partial\mathcal{L}}{\partial\phi} = -matl\sin\phi\dot{\phi} - mgl\sin\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = ml^2\dot{\phi} + matl\cos\phi.$$

Thus the Lagrange equation is

$$l\ddot{\phi} = -g\sin\phi - a\cos\phi.$$

Using the suggestion in the problem we factor out $\sqrt{a^2+g^2}$ and find

$$\begin{aligned}
l\ddot{\phi} &= -\sqrt{a^2+g^2}\left(\frac{g}{\sqrt{a^2+g^2}}\sin\phi + \frac{a}{\sqrt{a^2+g^2}}\cos\phi\right)\\
l\ddot{\phi} &= -\sqrt{a^2+g^2}\left(\cos\phi_o\sin\phi + \sin|\phi_o|\cos\phi\right)\\
l\ddot{\phi} &= -\sqrt{a^2+g^2}\sin\left(\phi+\phi_o\right).
\end{aligned}$$

(b) At equilibrium $\ddot{\phi} = 0$, so that ϕ at equilibrium is given by

$$\sin\left(\phi+\phi_o\right) = 0 \rightarrow \phi = -\phi_o = -\tan^{-1} a/g.$$

If the bob is slightly displaced from from equilibrium so that $\phi = -\phi_o + \delta\phi$, where $\delta\phi << 1$, then the equation of motion becomes

$$l\ddot{\delta\phi} = -\sqrt{g^2+a^2}\sin\delta\phi \simeq -\sqrt{g^2+a^2}\delta\phi.$$

The minus sign denotes that this is a restoring force. So this is a position of stable equilibrium with a frequency given by $\omega^2 = \sqrt{g^2+a^2}/l$.

7.33 The kinetic energy of the soap bar is

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2,$$

where x is the distance of the soap from the edge about which the plate pivots. The potential energy is $U = mgx\sin\omega t$. Hence the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mx^2\omega^2 - mgx\sin\omega t.$$

The Lagrange equation of motion for the soap bar is

$$\frac{\partial\mathcal{L}}{\partial x} = m\omega^2 x - mg\sin\omega t = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = m\ddot{x}.$$

The equation of motion is

$$\ddot{x} - \omega^2 x = -g\sin\omega t.$$

Which is the required result

(b) The homogeneous and particular solutions are of the form

$$\begin{aligned} x_h(t) &= A\sinh\omega t + B\cosh\omega t, \\ x_p(t) &= C\sin\omega t. \end{aligned}$$

For the particular solution we must have

$$-2\omega^2 C = -g \rightarrow C = g/2\omega^2.$$

From the initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$ we find

$$x_0 = B, \text{ and } 0 = A\omega + C\omega \rightarrow A = -g/2\omega^2.$$

Hence the solution is

$$x(t) = x_0\cosh\omega t - g/2\omega^2\sinh\omega t + g/2\omega^2\sin\omega t.$$

7.35 If the radius of the hoop is R then the x and y coordinates of the bead are

$$x = R\cos\omega t + R\cos(\phi + \omega t) \quad \text{and} \quad y = R\sin\omega t + R\sin(\phi + \omega t).$$

The velocities in the x and y directions are

$$\begin{aligned} \dot{x} &= -\omega R\sin\omega t - \left(\omega + \dot{\phi}\right)R\sin(\phi + \omega t), \\ \dot{y} &= \omega R\cos\omega t + \left(\omega + \dot{\phi}\right)R\cos(\phi + \omega t). \end{aligned}$$

The kinetic energy is then

$$
\begin{aligned}
T &= \frac{1}{2}m\left(\omega^2R^2+\left(\omega+\dot{\phi}\right)^2R^2\right) \\
&\quad +\frac{1}{2}m\left(2\omega\left(\omega+\dot{\phi}\right)R^2\left(\sin\omega t\sin\left(\phi+\omega t\right)+\cos\omega t\cos\left(\phi+\omega t\right)\right)\right), \\
T &= \frac{1}{2}m\left(\omega^2R^2+\left(\omega+\dot{\phi}\right)^2R^2+2\omega\left(\omega+\dot{\phi}\right)R^2\cos\phi\right).
\end{aligned}
$$

Since there is no potential energy this is also the Lagrangian. The Lagrange equation of motion is

$$
\begin{aligned}
\frac{\partial\mathcal{L}}{\partial\phi} &= -m\omega\left(\omega+\dot{\phi}\right)R^2\sin\phi \\
&= \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}=\frac{d}{dt}\left(mR^2\left(\omega+\dot{\phi}\right)+m\omega R^2\cos\phi\right), \\
-\omega\left(\omega+\dot{\phi}\right)\sin\phi &= \frac{d}{dt}\left(\omega+\dot{\phi}+\omega\cos\phi\right)=\ddot{\phi}-\omega\sin\phi\dot{\phi}, \\
\ddot{\phi} &= -\omega^2\sin\phi.
\end{aligned}
$$

This the same equation as that for a pendulum with g/l replaced by ω^2. Clearly the frequency of oscillations for small amplitudes is ω.

7.37 (a) The polar coordinates of the first mass are (r,ϕ) and the coordinates of the second mass is $z=l-r$, where z is measured downward. The total kinetic energy of both masses is

$$T=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2+\dot{z}^2\right)=\frac{1}{2}m\left(2\dot{r}^2+r^2\dot{\phi}^2\right),$$

while the potential energy is

$$U=-mgz=-mg\left(L-r\right)=mgr+const.$$

Thus the Lagrangian is

$$\mathcal{L}=\frac{1}{2}m\left(2\dot{r}^2+r^2\dot{\phi}^2\right)-mgr.$$

(b) The Lagrange equation for the ϕ coordinate is

$$\frac{d}{dt}mr^2\dot{\phi}=0\rightarrow L=mr^2\dot{\phi},$$

where L is the angular momentum which is conserved. The Lagrange equation for the radial coordinate is

$$\frac{\partial\mathcal{L}}{\partial r}=mr\dot{\phi}^2-mg=\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}}=\frac{d}{dt}\left(2m\dot{r}\right)=2m\ddot{r}.$$

(c) Eliminating $\dot{\phi}$ from the radial equation in terms of L the angular momentum we find

$$\frac{L^2}{mr^3} - mg = 2m\ddot{r}.$$

For a circular orbit at $r = r_o$ we have

$$\frac{L^2}{mr_o^3} = mg \rightarrow r_o^3 = \frac{L^2}{m^2 g}.$$

In Newtonian terms, this is the equilibrium that occurs when the centripetal acceleration $L^2/m^2 r_o^3$ equals the acceleration due to the gravitational field, g.

(c) If the particle on the table is given a small radial nudge, $r = r_o + \epsilon\left(t\right)$, then the radial equation becomes

$$\frac{L^2}{m\left(r_o + \epsilon\right)^3} - mg = \frac{L^2}{mr_o^3} - 3\frac{L^2\epsilon}{mr_o^4} - mg = -3\frac{L^2\epsilon}{mr_o^4} = 2m\ddot{\epsilon},$$

or

$$\ddot{\epsilon} = -\frac{3}{2}\frac{L^2}{m^2 r_o^4}\epsilon.$$

This is a stable oscillation with a frequency of $\omega = \sqrt{3/2}L/mr_o^2$.