

7 Lecture 10-9

7.1 Chapter 4 Energy (con)

7.1.1 Force as the Gradient of Potential Energy

Consider a particle acted on by a conservative force $\vec{F}(\vec{r})$ with a corresponding potential energy $U(\vec{r})$. The work done by $\vec{F}(\vec{r})$ in a small displacement from $\vec{r} \rightarrow \vec{r} + d\vec{r}$ is by definition

$$W(\vec{r} \rightarrow \vec{r} + d\vec{r}) = \vec{F}(\vec{r}) \cdot d\vec{r} = F_x dx + F_y dy + F_z dz. \quad (1)$$

On the other hand the work $W(\vec{r} \rightarrow \vec{r} + d\vec{r})$ results in a change in the potential energy given by

$$\begin{aligned} W(\vec{r} \rightarrow \vec{r} + d\vec{r}) &= -[U(\vec{r} + d\vec{r}) - U(\vec{r})] \\ W(\vec{r} \rightarrow \vec{r} + d\vec{r}) &= -[U(x + dx, y + dy, z + dz) - U(x, y, z)]. \end{aligned} \quad (2)$$

Expanding $U(x + dx, y + dy, z + dz)$ to first order in a Taylor's series results in

$$W(\vec{r} \rightarrow \vec{r} + d\vec{r}) = -\frac{\partial U}{\partial x} dx - \frac{\partial U}{\partial y} dy - \frac{\partial U}{\partial z} dz. \quad (3)$$

Equating equations (1) and (2) and recognizing that the infinitesimal displacements are independent of each other we find

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}. \quad (4)$$

A slightly more compact way to write this is

$$\vec{F}(\vec{r}) = -\hat{x} \frac{\partial U}{\partial x} - \hat{y} \frac{\partial U}{\partial y} - \hat{z} \frac{\partial U}{\partial z}. \quad (5)$$

From vector calculus the definition of the gradient ∇U or "grad U " is

$$\nabla U = \hat{x} \frac{\partial U}{\partial x} + \hat{y} \frac{\partial U}{\partial y} + \hat{z} \frac{\partial U}{\partial z}. \quad (6)$$

With this notation we have the simple relation

$$\boxed{\vec{F} = -\nabla U} \quad (7)$$

This important relation gives us the force \vec{F} in terms of derivatives of U , just as U was an integral of \vec{F} . Thus we have shown that any conservative force is derivable from a potential energy. It should be noted (as an exercise for the student) that the curl of the gradient of a scalar always vanishes, or

$$\nabla \times \nabla U = 0. \quad (8)$$

This is consistent with our proof that the curl of a conservative force must vanish.

At this point a simple example is in order. Consider a uniform electric field, E_o , pointing in the x direction. If we place a charge q into the field then this charge experiences a force $\vec{F} = q\vec{E} = qE_o\hat{x}$. The work done by \vec{F} between two points 1 and 2 along any path is

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r} = qE_o \int_1^2 \hat{x} \cdot d\vec{r} = qE_o \int_1^2 dx = qE_o(x_2 - x_1). \quad (9)$$

This work depends only on the endpoints and thus independent of the path. This means that this force must be conservative. To find the potential energy, $U(x)$, requires that we define a reference point at which $U(x_o) = 0$. A natural choice is the origin and we find

$$U(\vec{r}) = U(x) = -qE_o x. \quad (10)$$

There may be other reasons that it is beneficial to be able to find the potential energy function of a conservative force, other than finding the mechanical energy of a system, as it is often easier to work with a scalar field rather than a vector field. To see an example of this we will again consider the gravitational force. We wish to find the potential energy, $U_{\text{grav}}(\vec{r})$, such that

$$\vec{F}_{\text{grav}}(\vec{r}) = -\frac{GmM}{r^2}\hat{r} = -\nabla U_{\text{grav}}(\vec{r}). \quad (11)$$

We shall soon derive the form for the gradient operator in spherical coordinates. For now we simply state that the radial component is $\hat{r}\partial U/\partial r$. Thus the potential energy for the gravitational force is

$$U_{\text{grav}}(\vec{r}) = -\frac{GmM}{r}, \quad (12)$$

where we have defined the reference point, \vec{r}_o , to be infinitely removed the source of the field so that $U_{\text{grav}}(\vec{r}_o) = 0$.

We will now examine the gravitational field, \vec{F}_{grav} , due to a spherical shell of uniform mass density. Consider Figure 4-4, where a point mass m is located at P which is a distance R from the center of a uniform shell of mass M and radius a . The gravitational potential energy at point P is found by summing

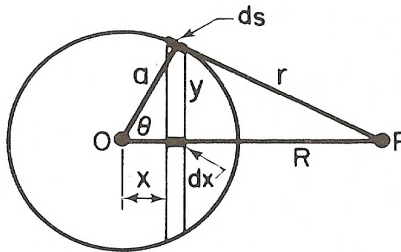


Figure 4-4 Thin spherical shell of uniform mass density with radius a .

the contribution to the potential of all the mass elements in the shell. This integral is expressed as

$$U_{\text{grav}}(\vec{r}) = -Gm \int \frac{dM}{r} = -Gm\sigma \int \frac{ds}{r}, \quad (13)$$

where σ is the uniform mass density of the shell given by $\sigma = M/(4\pi a^2)$ and ds is a differential surface element located a distance r from the point P . The angle θ is measured from the OP line with $y = a \sin \theta$ and $x = a \cos \theta$. Due to rotational symmetry about OP , any mass element at the same angle θ is the same distance r from P . Recognizing that $r^2 = (R - x)^2 + y^2$ enables us to write the integral as a function of R , a , and θ as

$$\begin{aligned} U_{\text{grav}}(\vec{r}) &= -\frac{GmM}{4\pi a^2} \int_0^\pi \frac{2\pi a^2 \sin \theta d\theta}{\sqrt{(R - a \cos \theta)^2 - a^2 \sin^2 \theta}} \\ &= -\frac{GmM}{2} \int_0^\pi \frac{\sin \theta d\theta}{\sqrt{R^2 - 2aR \cos \theta + a^2}}. \end{aligned} \quad (14)$$

With the substitution $x = \cos \theta$ this integral becomes

$$\begin{aligned} U_{\text{grav}}(\vec{r}) &= -\frac{GmM}{2} \int_{-1}^1 \frac{dx}{\sqrt{R^2 + a^2 - 2aRx}} = \frac{GmM}{2aR} \left[\sqrt{R^2 + a^2 - 2aRx} \right]_{-1}^1, \\ U_{\text{grav}}(\vec{r}) &= \frac{GmM}{2aR} (|R - a| - (R + a)). \end{aligned} \quad (15)$$

If $R > a$ so that $|R - a| = R - a$, the gravitational potential energy due to the spherical shell is

$$U_{\text{grav}}(\vec{r}) = -\frac{GmM}{R}. \quad (16)$$

While if $R < a$ so that $|R - a| = a - R$, the potential due to the sphere is

$$U_{\text{grav}}(\vec{r}) = -\frac{GmM}{a}. \quad (17)$$

These results are very enlightening. They tell us that outside the shell the gravitational potential a distance R from the center of the sphere is the same as though all the mass were concentrated at the center of the shell. But notice the potential inside the shell. Here the potential energy is a constant and independent of location inside the shell. Taking the gradient we would find that there is no gravitational force inside the shell, and a force outside the shell that is the same as if all the mass were at the center. We could have obtained this same result integrating the vector force field due to the mass elements in the shell. However, it would have been necessary to recognize that due to symmetry only the radial components would contribute and proceed accordingly. Calculating the potential energy none of this was necessary.

7.1.2 Time Dependent Potential Energy

There are times when the applied force is time dependent, $\vec{F}(\vec{r}, t)$, yet it satisfies the condition $\nabla \times \vec{F} = 0$. In this case we can still define a potential energy $U(\vec{r}, t)$ with the property $\vec{F} = -\nabla U$. Now however, the total mechanical energy, $E = T + U$, is not conserved.

Consider the example of a Van de Graaff generator with a charge $Q(t)$ that is slowly leaking away. Because the charge is leaking away, the electric field and any force it will exert on a test charge q is time dependent. Nevertheless, the spatial dependence of the force still satisfies $\nabla \times \vec{F} = 0$. The time dependent potential energy satisfies the usual relation

$$U(\vec{r}, t) = - \int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}, t) \cdot d\vec{r} \quad (18)$$

The change in the kinetic energy along the particle's path is still given by

$$dT = \frac{dT}{dt} dt = m \vec{v} \cdot \frac{d\vec{v}}{dt} dt = m \frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} dt = \vec{F} \cdot d\vec{r}. \quad (19)$$

Meanwhile, $U(\vec{r}, t) = U(x, y, z, t)$ so that an incremental change in the potential function is

$$dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz + \frac{\partial U}{\partial t} dt = \nabla U \cdot d\vec{r} + \frac{\partial U}{\partial t} dt. \quad (20)$$

This means that

$$dU = -\vec{F} \cdot d\vec{r} + \frac{\partial U}{\partial t} dt, \quad (21)$$

and

$$d(T + U) = \frac{\partial U}{\partial t} dt. \quad (22)$$

Clearly it is only when U is independent of t that the mechanical energy $E = T + U$ is conserved. Again the analogy of this result with the first law of thermodynamics, $\Delta U = Q - W$, is relevant. Equivalent to a net heat Q being transferred to the system, we now have a time dependent potential. If we were to include the energy transfer to the source of the field responsible for U , the total energy would be conserved. In the example of the Van de Graaff generator with a charge $Q(t)$ that is slowly leaking away, we find the loss of mechanical energy of a point charge is compensated by the thermal energy that comes from the leakage current.

As a simple example consider the case where the gravitational field at the surface of the Earth is varying in time. The potential energy is still $U = mgz$ where z is the height but $g = g(t)$. The force due to gravity is still $\vec{F} = -\nabla U = -mg\hat{z}$. However now the change in the total energy is

$$\begin{aligned} \frac{dE}{dt} &= \frac{d}{dt} \left(\frac{1}{2} m \vec{v} \cdot \vec{v} + mgz \right) = m \dot{\vec{v}} \cdot \vec{v} + mg\dot{z} + mz\dot{g}, \\ \frac{dE}{dt} &= -mg\hat{z} \cdot \vec{v} + mg\dot{z} + mz\dot{g} = mz\dot{g} \neq 0. \end{aligned}$$

7.1.3 Energy for Linear One-Dimensional Systems

We will consider an object constrained to move along a perfectly straight track, which we take to be the x axis. The only component of a force that can do work on an object is the x component and we will ignore the other two components. Additionally we will assume that the force is independent of the velocity and time. Thus the work done between x_1 and x_2 by F is simply the one-dimensional integral

$$W(x_1 \rightarrow x_2) = \int_{x_1}^{x_2} F dx. \quad (23)$$

Clearly this integral is path independent as it only depends on the antiderivative of F evaluated at x_1 and x_2 . Physically we might experience the situation where the particle travels from $x_1 \rightarrow A$, then reverse itself and travel from $A \rightarrow B$, where upon it reverses itself one more time and travel from $B \rightarrow x_2$. We could express this as the sum of integrals

$$W(x_1 \rightarrow x_2) = \int_{x_1}^A F dx + \int_A^B F dx + \int_B^{x_2} F dx. \quad (24)$$

From our knowledge of elementary one dimensional calculus, the sum of these integrals is independent of both A and B . It only depends on the endpoints. In any one-dimensional system the work integral is always path independent and the force is conservative as long as it is independent of velocity and time.

Graphs of the Potential Energy In one dimension the force is expressed as $F = -dU/dx$. If we plot the potential energy versus x as in Figure 4-5, it is clear, qualitatively, how an object will behave. By our definition the force is proportional to the negative of the slope of the potential. Thus the force always accelerates an object in the “downhill” direction. This is similar to the motion of a roller coaster which also always accelerates downhill. This analogy is not an accident. For a roller coaster the gravitational potential is proportional to the height above the ground. As long as $U(x)$ is plotted along a linear axis then its shape will be identical to that of its height, h , versus x , $h(x)$. Thus for any one-dimensional system we can always think about the plot of $U(x)$ as a picture of a roller coaster. Then common sense will tell us what kind of motion is possible.

At points such as x_3 and x_4 , where $dU/dx = 0$, the net force is zero. Thus the object is in equilibrium there. However, there is a significant difference between x_3 and x_4 . At x_3 where $d^2U/dx^2 > 0$, and $U(x)$ is a minimum, a small displacement from equilibrium results in force that returns the object back to x_3 . This is a position of *stable* equilibrium. At x_4 where $d^2U/dx^2 < 0$, and $U(x)$ is a maximum, a small displacement from equilibrium results in force that pushes the object away from x_4 . This is a position of *unstable* equilibrium.

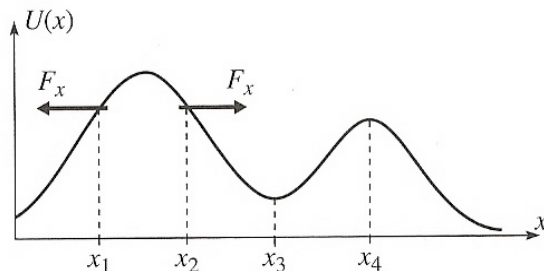


Figure 4-5. $U(x)$ for any 1-D system can be thought of as a roller coaster track. The force $F_x = -dU/dx$ tends to push the object down the "downhill" as at x_1 and x_2 . The points at x_3 and x_4 , where $U(x)$ is a minimum or maximum, $dU/dx = 0$, and the force is zero. x_3 is a position of stable equilibrium while x_4 is a position of unstable equilibrium.

Now consider Figure 4-6. Since the kinetic energy is positive definite, if an object is moving then its energy must be greater than the potential energy. This means an object with energy E (as shown in Figure 4-6) in the region $a < x < b$ will be confined to this region. To see this, assume that the object is traveling to the left. When it reaches $x = a$ its kinetic energy must vanish. Additionally the force on the object is in the positive x direction, and the object will be accelerated from $v = 0$ in the positive x direction. Thus, this is a turning point

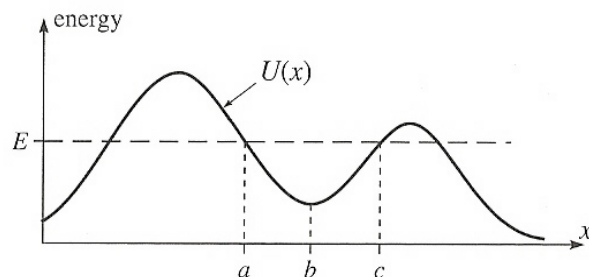


Figure 4-6. If an object with energy E starts out near $x = b$, it is trapped in the "potential well" between the two hills and oscillates between the turning points at $x = a$ and $x = c$ where $U(x) = E$ and the kinetic energy is zero.

for the object. Similarly for any object traveling to the right, its kinetic energy vanishes when it reaches $x = c$. At this point the force on the object is in the negative x direction. So $x = c$ is also a turning point.

Returning to the example of the roller coaster, if our roller coaster starts at $x = a$ it will accelerate down the "hill" until it reaches $x = b$. At this point it will decelerate up the "hill" until it reaches $x = c$. Ignoring any loss of mechanical energy to friction, the roller coaster will then return back to $x = a$. Our roller

coaster will oscillate indefinitely between $x = a$ and $x = c$. If however, the total mechanical energy, E , is greater than the maximum of U for $x > b$, then the roller coaster can start at $x = a$ and coast over the second “hill” never to return.

As a further example consider the potential shown in Figure 4.7.

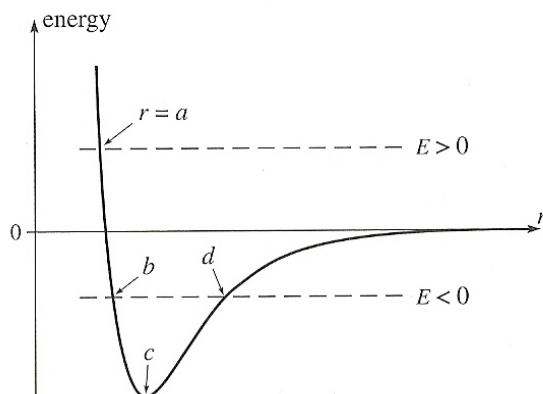


Figure 4.7. The potential energy for a typical diatomic molecule such as HCl plotted as a function of r the distance between the two atoms. If $E > 0$ the two atoms cannot approach closer than the turning point $r=a$ but they can move apart to infinity and hence the atoms are unbound. If $E < 0$ the atoms are trapped between the turning points b and d and form a bound molecule.

The equilibrium separation is $r = c$.

Figure 4.7 shows the potential energy of a typical diatomic molecule, such as HCl, as a function of the separation distance between the two atoms. For this example this potential energy function governs the radial motion of the hydrogen atom as it vibrates in and out from the much heavier chlorine atom. The $E = 0$ level has been chosen so that the kinetic energy of the hydrogen atom vanishes as the separation distance $r \rightarrow \infty$. As $r \rightarrow 0$ the potential energy becomes very large due to the electrostatic repulsion of the two nuclei. For $E > 0$ a hydrogen atom can approach from infinity until it reaches a minimum radial distance from the chlorine nucleus and then return to $r = \infty$, albeit for the general case, in a different direction. So this is a one dimensional system in the sense that it only depends on one parameter, r , the distance of separation, however it is not a linear one-dimensional system. But non-the-less a very useful example. For $E < 0$, the hydrogen is trapped and there are two turning points, $r = b$ and $r = d$. The equilibrium separation is shown as $r = c$. To form a molecule, a hydrogen and chlorine atom must come together to a separation somewhere near $r = c$, and then through a process such as the emission of light remove enough energy to form a bound molecule.

Complete Solution of Motion Another feature of one-dimensional conservative systems is we can, at least in principle, obtain a complete solution of

the motion. That is we can find the position as a function of time, $x(t)$. Since $E = T + U$ is conserved we can solve for v ,

$$v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m} \sqrt{E - U(x)}}. \quad (25)$$

Simply multiplying this first order differential equation by dt and separating variables results in (assuming that \dot{x} is positive)

$$t - t_i = \sqrt{\frac{m}{2}} \int_{x_i}^x \frac{dx}{\sqrt{E - U(x)}}. \quad (26)$$

Here the lower limit t_i is the initial time at which the object is located at x_i . Both t and x are arbitrary times and locations respectively. Performing this integral enables us to find $t(x)$ (at the very least we can perform this integral numerically for various final locations to numerically determine $t(x)$). Finally we invert $t(x)$ to find $x(t)$.

To illustrate this process we will consider the simple problem of an object falling from rest starting at $t = 0$ and $x = 0$. If we measure x as positive measuring downward then the potential energy is

$$U(x) = -mgx. \quad (27)$$

The total mechanical energy for this object at $t = 0$ is $E = 0$. Since this energy is conserved, we have from equation (26), the elapsed time, t , to fall a distance x is

$$t = \sqrt{\frac{m}{2}} \int_0^x \frac{dx}{\sqrt{mgx}} = \frac{1}{\sqrt{2g}} \int_0^x \frac{dx}{\sqrt{x}} = \sqrt{\frac{2x}{g}}. \quad (28)$$

Inverting this expression we find the familiar result

$$x = \frac{1}{2}gt^2. \quad (29)$$

As a further illustration consider a particle of mass m on the end of a spring with force constant k , i.e. $F = -kx$. The potential energy of the spring is $U(x) = kx^2/2$. If at time $t = 0$ the mass, which was in static equilibrium at the origin, is given an impulse so that it moves out to an amplitude A . The expression for the conservation of mechanical energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2. \quad (30)$$

When the particle reaches $x = A$, it comes to a stop after which it proceeds back toward its origin. With this constraint, the total mechanical energy of the system is $E = kA^2/2$. From equation (9) we can write

$$t = \sqrt{\frac{m}{k}} \int_0^x \frac{dx}{\sqrt{A^2 - x^2}} = \frac{1}{\omega} \int_0^x \frac{dx}{\sqrt{A^2 - x^2}}, \quad (31)$$

where we have defined $\omega^2 = k/m$. To perform this integration we make the substitution $x = A \sin \theta$ and we find

$$\omega t = \int_0^{\sin^{-1} x/A} \frac{A \cos \theta d\theta}{A \cos \theta} = \sin^{-1} x/A. \quad (32)$$

Simply taking the sine of both sides yields the result

$$x = A \sin \omega t. \quad (33)$$

For some problems it is necessary to do the integration numerically. But the point here is that the expression for the conservation of energy in one dimension is the first integration of the equation of motion and hence only one remaining integral remains to be done to determine the complete solution to the problem.