6 Lecture 10-7

6.0.1 Angular Momentum about the CM

The conservation of angular momentum and the more general result, $\dot{\vec{L}} = \vec{\Gamma}^{ext}$, were derived on the assumption that all quantities were measured in an inertial frame. This required that both $\vec{L}$ and $\vec{\Gamma}^{ext}$ were measured about an origin O fixed in some inertial frame. As it turns out the same results hold if $\vec{L}$ and $\vec{\Gamma}^{ext}$ are measured about the center of mass even if the CM is being accelerated and not fixed in an inertial frame. The proof of this is left as an exercise for the student (problem 3.37) and is stated as

$$\frac{d}{dt}\vec{L}\,(\text{about CM}) = \vec{\Gamma}^{ext}\,(\text{about CM})\,. \tag{1}$$

An example that demonstrates the ease that this result allows us to solve various problems is that of a dumbbell lying on a frictionless plane. The dumbbell consists of two equal masses on the ends of a rigid massless rod of length $2b$ that lies along the x axis centered at the origin as shown in Figure 4. At time $t = 0$, the left mass is given a sharp tap with a force F lasting for a short time Δt. We wish to find the initial motion immediately after the tap as well as the subsequent force free motion. The change in momentum of the dumbbell is determined by the impulse $F\Delta t$. Since the dumbbell is initially at rest the final momentum is

$$P = F\Delta t = Mv_{\text{CM}} = 2mv_{\text{CM}}, \tag{2}$$

where M is the mass of the system so that $M = 2m$.

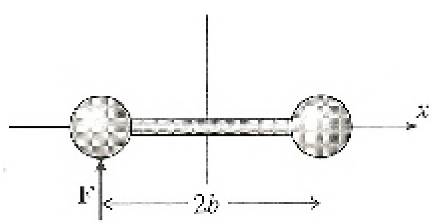

Figure 3-4. The left mass of a dumbbell is given an implusive blow in the y direction.

Similarly the change in the angular momentum about its CM is given by $\vec{\Gamma}^{\text{ext}}\Delta t$. Thus

$$L = Fb\Delta t = I\omega = 2mb^2\omega. \tag{3}$$

This amounts to a clockwise rotation with frequency $\omega = F\Delta t/2mb$. Thus the mass on the left side of the dumbbell is moving with a velocity ωb relative to the center of mass while the mass on the right side of the dumbbell is moving with a velocity of $-\omega b$ relative to the center of mass. This is expressed as

$$v_{\text{left}} = F\Delta t/2m + \omega b = F\Delta t/2m + F\Delta t/2m = 2v_{\text{CM}} \tag{4}$$

$$v_{\text{right}} = F\Delta t/2m - \omega b = F\Delta t/2m - F\Delta t/2m = 0. \tag{5}$$

We see that initially the left mass carries all of the momentum of the system and is moving at twice the velocity of the center of mass, while the right mass is stationary. Physically this is what we should have expected.

The subsequent motion is very straightforward. Once the impulse has ceased the CM continues to move straight up the y axis with a velocity v_{CM} while the dumbbell continues to rotate with angular velocity ω about the center of mass.

6.1 Chapter 4 Energy

Here we will examine the conservation of energy. This will require the use of some tools from vector calculus, namely the gradient and the curl. We shall describe these concepts as needed.

6.1.1 Kinetic Energy and Work

The kinetic energy, T, of a particle with mass m is defined to be

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\overrightarrow{v}\cdot\overrightarrow{v}. \tag{6}$$

We are interested in how the kinetic changes are the particle moves through space. To examine this we will start by taking the time derivative of the kinetic energy,

$$\frac{dT}{dt} = \frac{1}{2}m\left(\dot{\overrightarrow{v}}\cdot\overrightarrow{v} + \overrightarrow{v}\cdot\dot{\overrightarrow{v}}\right) = m\dot{\overrightarrow{v}}\cdot\overrightarrow{v}.$$

From Newton's second law this can be rewritten as

$$\frac{dT}{dt} = \overrightarrow{F}\cdot\overrightarrow{v}. \tag{7}$$

If we simply multiply through by dt and recognize that $\overrightarrow{v}dt = d\overrightarrow{r}$ then we find that the change in the kinetic energy as it moves through space from $\overrightarrow{r}$ to $\overrightarrow{r} + d\overrightarrow{r}$ is

$$dT = \overrightarrow{F}\cdot d\overrightarrow{r}. \tag{8}$$

The expression on the right, $\overrightarrow{F}\cdot d\overrightarrow{r}$, is defined to be the *work done by the force* $\overrightarrow{F}$ in the displacement $d\overrightarrow{r}$. Equation (8) is a statement of the *Work-KE theorem*. Namely that the change in the particle's kinetic energy between two neighboring points on its path is equal to the work done by the net force between the two points. It is of interest to note that this quantity may be negative as well as positive. If the force is in the opposite direction of the path then the kinetic energy is reduced.

If the two points are not differentially separated then equation (8) becomes

$$\boxed{\Delta T = T_2 - T_1 = \int_1^2 \overrightarrow{F}\cdot d\overrightarrow{r} = W\left(\overrightarrow{r}_1 \to \overrightarrow{r}_2\right).} \tag{9}$$

This is the *Work-KE theorem* for arbitrary displacement. The integral in this theorem is a line integral and is a generalization over the one dimensional integral

$\int f(x)\,dx$. As the name implies a line integral (for more than one dimension) in general is path dependent, i.e. it depends on the path that the particle takes getting from point 1 to point 2.

As an example consider the a line integral of the force $\overrightarrow{F} = y\widehat{x} + 2x\widehat{y}$ from points $O = (0,0)$ to $P = (1,1)$: along three different paths as shown in Figure 4-1 below.

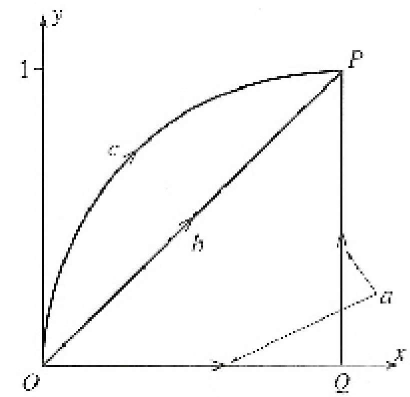

Figure 4-1. Multiple paths a, b, and c from the origin to the point $P : (1,1)$.

The integral along path a is given by

$$W_a = \int_a \overrightarrow{F} \cdot d\overrightarrow{r} = \int_0^1 y dx + \int_0^1 2x dy = \int_0^1 2dy = 2, \tag{10}$$

as $y = 0$ along the x axis and $x = 1$ along path a as y ranges from $0 \to 1$. The integral along path b is

$$\begin{aligned} W_b &= \int_b \overrightarrow{F} \cdot d\overrightarrow{r} = \int \overrightarrow{F} \cdot (\widehat{x}dx + \widehat{y}dy) = \int_0^1 x dx + \int_0^1 2x dx \\ W_b &= \frac{3}{2}, \end{aligned} \tag{11}$$

where we have noted that $y = x$ along path b. The path along c is a quarter circle with an origin at $Q = (1,0)$ and a unit radius. As a function of θ the x coordinate is given by $x = 1 + \cos\theta$ and the y coordinate is $y = \sin\theta$. This allows us to express the force $\overrightarrow{F}$ as

$$\overrightarrow{F} = y\widehat{x} + 2x\widehat{y} = \sin\theta\widehat{x} + 2(1 + \cos\theta)\widehat{y}. \tag{12}$$

Since $\overrightarrow{r} = x\widehat{x} + y\widehat{y}$, the incremental line element $d\overrightarrow{r}$ is given by

$$d\overrightarrow{r} = -\widehat{x}\sin\theta d\theta + \widehat{y}\cos\theta d\theta. \tag{13}$$

We can now write the line integral along path c as

$$\begin{aligned}W_c &= \int_c F_x dx + F_y dy = \int_{\pi}^{\pi/2} \left(-\sin^2\theta + 2\left(1+\cos\theta\right)\cos\theta\right) d\theta \\ W_c &= \frac{\pi}{4} + 2\left(1 - \frac{\pi}{4}\right) = 2 - \frac{\pi}{4} = 1.21. \end{aligned} \tag{14}$$

From these results it should be clear that in general, for greater than one dimension, the work getting from one point to another is path dependent.

It is important to remember that the force used in the Work-KE theorem is the net force on the particle. For example, the net force on a projectile is the sum of two forces, the gravitational weight plus the drag due to air resistance.

6.1.2 Potential Energy and Conservative Forces

The next step is to introduce the concept of potential energy, PE, corresponding to the forces on an object. Not every force lends itself to the definition of a corresponding potential energy. The forces that do have a corresponding potential energy are called *conservative forces*.

The first condition for a force, $\overrightarrow{F}$, to be conservative is that $\overrightarrow{F}$ depends only on the position $\overrightarrow{r}$ of the object on which it acts. It must not depend on the velocity, time, or any variables other than $\overrightarrow{r}$. Fortunately there are many forces that have this property. The gravitational force of the Sun on a planet (or any gravitational interaction) can be written as

$$\overrightarrow{F}_{\text{grav}}\left(\overrightarrow{r}\right) = -\frac{GmM}{r^2}\widehat{r} = -\frac{GmM}{r^3}\overrightarrow{r}, \tag{15}$$

and obviously only depends on the variable $\overrightarrow{r}$. Similarly the electrostatic force has this property. Forces that do not satisfy this condition include the force of air resistance (which depends on the velocity), friction in general which depends on the direction of motion, the magnetic force which depends on the velocity, and the force of a time varying electric field as it depends on time.

The second condition that a force must satisfy to be called conservative is that the work done by the force on an object between points 1 *and* 2*,*

$$W_{12} = \int_1^2 \overrightarrow{F} \cdot d\overrightarrow{r}, \tag{16}$$

is independent of the path connecting these two points. Now consider two different paths C_1 and C_2 between the two points. Since for a conservative force the work done on a object between these two points is independent of the path we can state

$$W_{12} = \int_{C_1[1,2]} \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{C_2[1,2]} \overrightarrow{F} \cdot d\overrightarrow{r}. \tag{17}$$

Now for a conservative force the work done only depends on the endpoints, thus the work done along C_2 in the opposite direction is $-W_{12}$ or

$$W_{12} = -\int_{C_2[2,1]} \overrightarrow{F} \cdot d\overrightarrow{r}. \tag{18}$$

We can now combine these two integrals and write

$$\int_{C_1[1,2]} \overrightarrow{F} \cdot d\overrightarrow{r} + \int_{C_2[2,1]} \overrightarrow{F} \cdot d\overrightarrow{r} = 0. \tag{19}$$

However, these two integrals define a closed curve, thus a conservative force satisfies the condition

$$\oint_{C_1+C_2} \overrightarrow{F} \cdot d\overrightarrow{r} = \oint \overrightarrow{F} \cdot d\overrightarrow{r} = 0. \tag{20}$$

Since C_1 and C_1 were chosen arbitrarily, we can conclude that for a conservative force $\oint \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for *any* closed loop.

To expand on this concept, we will now prove (although not a rigorous proof from a mathematical point of view) Stokes theorem, which is a well known theorem in vector calculus. Stokes theorem states that

$$\oint \overrightarrow{F} \cdot d\overrightarrow{r} = \int_{Area} \left(\nabla \times \overrightarrow{F}\right) \cdot \widehat{n} dA, \tag{21}$$

where $\widehat{n}$ is the unit normal to the area element dA and the subscript $Area$ denotes the area enclosed by the loop. Consider the closed loop in Figure 4-2. This loop encompasses a rectangular area in the $x-y$ plane extending in the x direction from x_1 to x_2 and in the y direction from y_1 to y_2. The unit normal to this area is $\widehat{z}$.

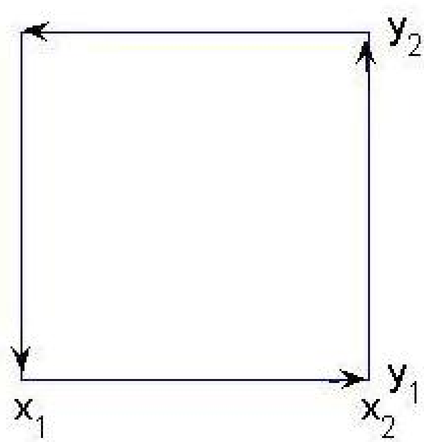

Figure 4-2 Closed rectangular loop with x varying between x_1 and x_2 while y is held constant, y_1 or y_2, and y ranging between y_1 and y_2 while x is held constant, x_1 or x_2.

With these constraints the integral of the curl of an arbitrary vector normal to this area is

$$\begin{aligned}
\int_{Area}\left(\nabla\times\overrightarrow{F}\right)\cdot\widehat{n}dA &= \int_{y_1}^{y_2}\int_{x_1}^{x_2}\left(\nabla\times\overrightarrow{F}\right)_z dxdy,\\
\int_{Area}\left(\nabla\times\overrightarrow{F}\right)\cdot\widehat{n}dA &= \int_{y_1}^{y_2}\int_{x_1}^{x_2}\left(\frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y}\right)dxdy. \qquad (22)
\end{aligned}$$

The first integral for both of the terms on the right hand side of this equation can be performed immediately and we find

$$\int_{Area}\left(\nabla\times\overrightarrow{F}\right)\cdot\widehat{n}dA = \int_{y_1}^{y_2}\left(F_y\left(x_2,y\right)-F_y\left(x_1,y\right)\right)dy+\int_{x_1}^{x_2}\left(F_x\left(x,y_1\right)-F_x\left(x,y_2\right)\right)dx. \qquad (23)$$

Simply regrouping terms and reordering the limits on the integrals when appropriate yields

$$\begin{aligned}
\int_{Area}\left(\nabla\times\overrightarrow{F}\right)\cdot\widehat{n}dA &= \int_{x_1}^{x_2}F_x\left(x,y_1\right)dx+\int_{y_1}^{y_2}F_y\left(x_2,y\right)dy\\
&\quad+\int_{x_2}^{x_1}F_x\left(x,y_2\right)dx+\int_{y_2}^{y_1}F_y\left(x_1,y\right)dy. \qquad (24)
\end{aligned}$$

This result is simply the line integral around the closed loop and for our rectangular loop we have

$$\int_{Area}\left(\nabla\times\overrightarrow{F}\right)_z dA = \oint\overrightarrow{F}\cdot d\overrightarrow{r}. \qquad (25)$$

To generalize this result we take the limit that this area is incrementally small. Then we consider additional adjacent loops that share one common border. All of the paths are taken to be counter clockwise, thus the path integrals along these common borders cancel identically. Since we can simulate any continuous surface, including curved surfaces, with a sum over incrementally small areas, it is clear that Stokes theorem is satisfied. Since the path integral over an arbitrary closed loop vanishes for a conservative force, we can conclude that

$$\nabla\times\overrightarrow{F}=0, \qquad (26)$$

for conservative forces. This allows us to state that the necessary and sufficient condition for the second criteria for a force to be conservative is that the curl of a such a force must vanish. As an example, consider the expression for the curl of a vector in spherical coordinates that is contained inside the back cover of Taylor's book,

$$\begin{aligned}
\nabla\times\overrightarrow{F} &= \widehat{r}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta F_\phi\right)-\frac{\partial}{\partial\phi}F_\theta\right]+\widehat{\theta}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}F_r-\frac{1}{r}\frac{\partial}{\partial r}\left(rF_\phi\right)\right]\\
&\quad+\widehat{\phi}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right)-\frac{\partial}{\partial\theta}F_r\right]. \qquad (27)
\end{aligned}$$

Since the gravitational force only has a radial component and depends only the radial coordinate, we see that the curl of the gravitational interaction, $\overrightarrow{F}_{\text{grav}}$, vanishes. Thus the gravitational force satisfies both conditions which means that the gravitational force is conservative.

Now for a conservative force the work done depends only on the endpoints and not on any particular path. This means we can define a potential energy $U(\overrightarrow{r})$, corresponding to a given conservative force, that only depends on position. We first choose a reference point $\overrightarrow{r}_0$ at which U is defined to be zero, basically this is now just a constant of integration. The definition for the potential energy is now

$$U(\overrightarrow{r}) = -W(\overrightarrow{r}_0 \to \overrightarrow{r}) = -\int_{\overrightarrow{r}_o}^{\overrightarrow{r}} \overrightarrow{F}(\overrightarrow{r}) \cdot d\overrightarrow{r}. \tag{28}$$

With this definition $U(\overrightarrow{r})$ is the negative of the work done by $\overrightarrow{F}(\overrightarrow{r})$ as the particle moves from $\overrightarrow{r}_o$ to the point of interest $\overrightarrow{r}$.

To understand the reason for the minus sign let us consider the work done by our conservative force $\overrightarrow{F}(\overrightarrow{r})$ going first from $\overrightarrow{r}_0 \to \overrightarrow{r}_1$ and then from $\overrightarrow{r}_1 \to \overrightarrow{r}_2$. This work is given by

$$W(\overrightarrow{r}_0 \to \overrightarrow{r}_2) = W(\overrightarrow{r}_0 \to \overrightarrow{r}_1) + W(\overrightarrow{r}_1 \to \overrightarrow{r}_2), \tag{29}$$

and hence

$$W(\overrightarrow{r}_1 \to \overrightarrow{r}_2) = W(\overrightarrow{r}_0 \to \overrightarrow{r}_2) - W(\overrightarrow{r}_0 \to \overrightarrow{r}_1). \tag{30}$$

From our definition of potential energy

$$W(\overrightarrow{r}_1 \to \overrightarrow{r}_2) = -\left[U(\overrightarrow{r}_2) - U(\overrightarrow{r}_1)\right] = -\Delta U. \tag{31}$$

However from the Work-KE theorem we know that $\Delta T = W(\overrightarrow{r}_1 \to \overrightarrow{r}_2)$, so that

$$\Delta T = -\Delta U \tag{32}$$

We now see that the minus sign that was used in the definition of the potential allows us to write

$$\Delta T + \Delta U = 0. \tag{33}$$

That is the *mechanical energy*

$$\boxed{E = T + U} \tag{34}$$

is conserved as a particle moves from $\overrightarrow{r}_1$ to $\overrightarrow{r}_2$. Since the points $\overrightarrow{r}_1$ and $\overrightarrow{r}_2$ were chosen arbitrarily, we have the important conclusion: If the force on a particle is conservative, then the particle's mechanical energy is conserved, hence the use of the adjective "conservative" when defining such a force.

Nonconservative Forces We will assume that some of the forces acting on our particle are nonconservative. If we divide the forces into those that are conservative and those that are nonconservative, then by the Work-KE theorem we can write

$$\Delta T = W = W_{\text{cons}} + W_{\text{nc}} = -\Delta U + W_{\text{nc}}. \tag{35}$$

Again defining the mechanical energy as $E = T + U$, we find

$$\Delta E = \Delta (T + U) = W_{\text{nc}} \tag{36}$$

The mechanical energy is no longer conserved, but we do have the next best thing. The mechanical energy changes by exactly the amount that the nonconservative forces do work on the particle. This is analogous to a more general law of the conservation of energy, the first law of thermodynamics, $\Delta U = Q - W$, where ΔU is the change in the internal energy, Q is the net heat transferred to the system, and W is the net work done by the system. Here ΔU plays the role of the mechanical energy. There is no heat transfer, but the work performed by the nonconservative force is represented by $-W$. So the mechanical energy may not be conserved but the total energy certainly is!

In many cases the nonconservative force is that of friction, which usually does negative work (i.e. The direction of the force is in the opposite direction of the motion, $\vec{F} \cdot d\vec{r} < 0$.). The object looses mechanical energy to the friction usually in the form of heat. We can illustrate these concepts by again considering the example of a block sliding down a plane shown in Figure 4-3.

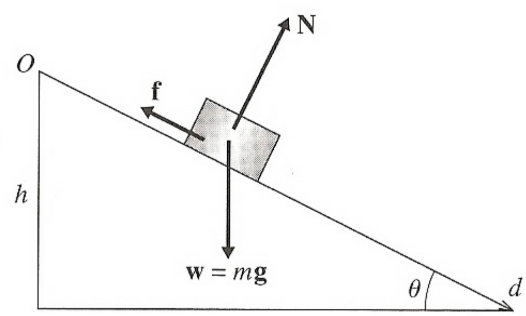

Figure 4-3. Block on an incline plane of angle θ. The length of the plane is d with a height of $h = d \sin\theta$.

The block starts from rest a distance d from the bottom as measured along the slide and then proceeds to slide to the bottom of the inclined plane. The gravitational force is conservative and its potential energy is given by $U = mgy$, where y is the height as measured from the bottom of the incline. The normal force does no work (it is normal to the motion of the block) and will not contribute to the energy balance. Over the entire length of the slide the frictional force does work

$$W_{\text{fric}} = \int \vec{f} \cdot d\vec{r} = -fd = -\mu mgd \cos\theta. \tag{37}$$

This means that the total energy changes via

$$\Delta E = \Delta T + \Delta U = W_{fric}, \tag{38}$$

or

$$\frac{1}{2}mv^2 - mgd\sin\theta = -\mu mgd\cos\theta. \tag{39}$$

Thus the velocity at the bottom of the slide is

$$v = \sqrt{2gd\,(\sin\theta - \mu\cos\theta)}. \tag{40}$$

At $\theta = \pi/2$ we obtain the usual free fall solution. However what happens when $\tan\theta < \mu$?

I now pose the following conundrum. Clearly, the frictional force for a block sliding down a plane is not conservative. However it is time independent and its curl vanishes as it is constant. So how is it that the frictional force for a block sliding down a plane is not conservative?