5 Lecture 10-5

5.1 Chapter 3 Momentum and Angular Momentum (con)

5.1.1 Center of Mass

Consider a group of N particles with masses m_α and positions $\overrightarrow{r}_\alpha$ measured relative to some origin O. The center of mass (CM) of this system is defined to be

$$\overrightarrow{R} = \frac{1}{M} \sum_{\alpha=1}^{N} m_\alpha \overrightarrow{r}_\alpha, \tag{1}$$

where M is the total mass of all of the particles. We can now write the total momentum $\overrightarrow{P}$ of any N-particle system in terms of the system's CM as

$$\overrightarrow{P} = \sum_{\alpha=1}^{N} \overrightarrow{p}_\alpha = \sum_{\alpha=1}^{N} m_\alpha \dot{\overrightarrow{r}}_\alpha = M \dot{\overrightarrow{R}}. \tag{2}$$

This important result tells us that the total momentum of the N particles is exactly the same as that of a single particle of mass M with the velocity equal to that of the CM.

Now if we take the time derivative of this expression we find

$$\overrightarrow{F}^{\text{ext}} = \dot{\overrightarrow{P}} = M \ddot{\overrightarrow{R}}. \tag{3}$$

This tells us that the center of mass moves exactly as if it were a single particle of mass M subject to the net external force on the system. This result is the main reason why we often treat extended bodies, such as baseballs, planets, and even galaxies, as if they were point particles. As long as the body is small compared to its trajectory, its CM is a good representative of its overall location.

What if we have two extended bodies of masses M_1 and M_2? Assume that M_1 and M_2 are both made up of a set of masses $\{m_{1\alpha}\}$ and $\{m_{2\alpha}\}$. Their locations are given by $\{\overrightarrow{r}_{1\alpha}\}$ and $\{\overrightarrow{r}_{2\alpha}\}$ respectively. Then the center of mass for M_1 and M_2 are found from

$$M_1 \overrightarrow{R}_1 = \sum_{\alpha=1}^{N_1} m_{1\alpha} \overrightarrow{r}_{1\alpha} \text{ and } M_2 \overrightarrow{R}_2 = \sum_{\alpha=1}^{N_2} m_{2\alpha} \overrightarrow{r}_{2\alpha}.$$

The center of mass of all of the particles is found from the sum of $m\overrightarrow{r}$ over all of the particles.

$$(M_1 + M_2) \overrightarrow{R} = \sum_{\alpha=1}^{N_1} m_{1\alpha} \overrightarrow{r}_{1\alpha} + \sum_{\alpha=1}^{N_2} m_{2\alpha} \overrightarrow{r}_{2\alpha} = M_1 \overrightarrow{R}_1 + M_2 \overrightarrow{R}_2.$$

Hence the center of mass of the whole system is

$$\overrightarrow{R} = \frac{M_1 \overrightarrow{R}_1 + M_2 \overrightarrow{R}_2}{M_1 + M_2}.$$

This rather easily obtained result is significant as it means that in finding the CM of a complicated system, you can treat its component parts just like point masses positioned at their respective centers of mass. This is true even when the component parts are themselves extended bodies.

Before we leave this subject it is useful to perform an example calculation to determine the center of mass If the mass is distributed continuously, then the sum goes over to an integral such that

$$\overrightarrow{R} = \frac{1}{M} \int \overrightarrow{r}_\alpha dm = \frac{1}{M} \int \overrightarrow{r}_\alpha \rho dV, \tag{4}$$

where ρ is the mass density for the object of interest. Now consider the solid cone in Figure 3-1. We will assume that it has a uniform density so that its mass is given by $M = \frac{1}{3}\rho\pi R^2 h$. From symmetry we note that the center of mass lies on the z axis. Thus we only have to determine the z component for the center of mass. This component is given by the integral

$$Z = \frac{1}{M} \int z dm = \frac{\rho}{M} \int z dV. \tag{5}$$

Since a cone has axial symmetry the differential volume element is given by $dV = \pi r^2 dz$, where r is the outer radius of the cone at a height z. Since the
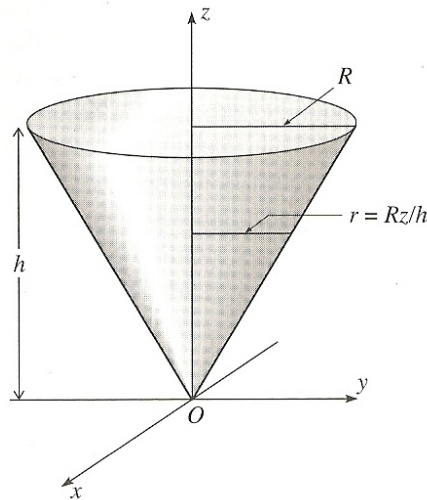

Figure 3-1. A uniform solid inverted cone centered on the z axis with its vertex at the origin. Its height is h and its base has a radius R.

radius at the top of the cone, $z = h$, is given by $r = R$, the radius must satisfy the expression (see figure) $r = Rz/h$. The z component for the center of mass

can now be found by performing the integral

$$
\begin{aligned}
Z &= \frac{\rho}{(1/3)\,\rho\pi R^2 h} \int_0^h z\pi r^2\,(z)\,dz = \frac{3}{R^2 h} \int_0^h z\left(\frac{Rz}{h}\right)^2 dz \\
Z &= \frac{3}{R^2 h}\frac{R^2}{h^2}\frac{h^4}{4} = \frac{3}{4}h.
\end{aligned}
\tag{6}
$$

As you would expect the center of mass lies well above the midpoint, $h/2$.

5.1.2 Angular Momentum of a Single Particle

First we will define the angular momentum of a single particle. This is a vector given by

$$
\overrightarrow{\ell} = \overrightarrow{r} \times \overrightarrow{p},
\tag{7}
$$

where $\overrightarrow{r}$ is the position vector for the particle as defined by a certain origin, O, and $\overrightarrow{p}$ is its momentum. It is important to note that the angular momentum depends on the choice of origin. So strictly speaking we should speak of the particle's angular momentum relative to O.

The time rate of change of $\overrightarrow{\ell}$ is

$$
\dot{\overrightarrow{\ell}} = \frac{d\overrightarrow{\ell}}{dt} = \dot{\overrightarrow{r}} \times \overrightarrow{p} + \overrightarrow{r} \times \dot{\overrightarrow{p}}.
\tag{8}
$$

Since $\dot{\overrightarrow{r}}$ is parallel to $\overrightarrow{p}$ (i.e. $\overrightarrow{p} = m\dot{\overrightarrow{r}}$) the cross product of these two quantities vanishes. Also we have $\dot{\overrightarrow{p}} = \overrightarrow{F}$ where $\overrightarrow{F}$ is the net force on the particle, so that this definition becomes

$$
\frac{d\overrightarrow{\ell}}{dt} = \overrightarrow{r} \times \overrightarrow{F} \equiv \overrightarrow{\Gamma},
\tag{9}
$$

where $\overrightarrow{\Gamma}$ denotes the net torque about O on the particle. Equation (17) is the rotational analog of Newton's second law and is often expressed as such.

In many one-particle problems one chooses the origin O so that the net torque about O vanishes. For example, consider a single planet (or any orbiting body) rotating the Sun. The only force on the planet is the gravitational force which is parallel to $\overrightarrow{r}$ (actually since the gravitational force is attractive it is antiparallel to $\overrightarrow{r}$). Hence this *central force* produces no torque on an orbiting body. So choosing the Sun as the origin greatly simplifies the analysis of the motion of an orbiting body. The fact that $\overrightarrow{r} \times \overrightarrow{p}$ is constant implies that both $\overrightarrow{r}$ and $\overrightarrow{p}$ remain in a fixed plane. In other words a planet's orbit is confined to a single plane, reducing the problem to one of two dimensions.

5.1.3 Kepler's Second Law

One of Newton's greatest triumphs was to explain Kepler's three laws of planetary motion. These laws are mathematical descriptions of the observed motion of the planets. They made no attempt to explain this motion in terms of more

fundamental ideas, NLT they were brilliant summaries. The first and third of
Kepler's laws will be derived in Chapter 8. However we are now in position to
explain Kepler's second law. Kepler's Second Law can be stated as

Kepler's Second Law

As each planet orbits the Sun, a line drawn from the
Sun to the planet sweeps out equal areas in equal times.

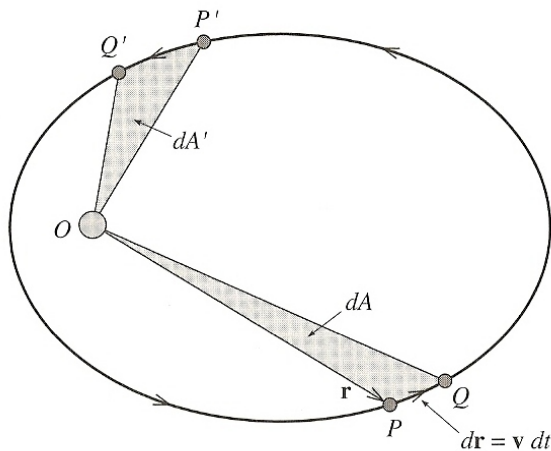To visualize this statement we will examine Figure 3-2.

Figure 3-2. The solid curve represents the orbit of a planet with the Sun fixed
at O. Kepler's second law states that if two pairs of points on the orbit, P, Q
and P', Q' are separated by equal times, $dt = dt'$, then the two areas dA and
dA' are equal.

To prove this assertion, we first calculate the area inside OPQ. We will
assume that PQ is small compared to the magnitude of $\overrightarrow{r}$ so that the problem
reduces to finding the area inside the triangle OPQ. This is given by

$$dA = \frac{1}{2} \left| \overrightarrow{r} \times \overrightarrow{v} \, dt \right|. \tag{10}$$

Replacing $\overrightarrow{v}$ by $\overrightarrow{p}/m$ and dividing by dt yields

$$\frac{dA}{dt} = \frac{1}{2m} \left| \overrightarrow{r} \times \overrightarrow{p} \right| = \frac{1}{2m} \ell, \tag{11}$$

where ℓ is defined to be the magnitude of the angular momentum $\overrightarrow{\ell}$. Since as
we have seen that the angular momentum is conserved for central forces, the
rate at which planets sweep out areas is also conserved.

It is interesting to note that our proof of Kepler's second law depended only
on the fact that the gravitational force is a central force and hence the planet's

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angular momentum about the Sun is constant. Thus Kepler's second law is true for an object that moves under the influence of any central force. By contrast, we shall see that the first and third laws depend on the inverse square nature of the gravitational force and are not true for other force laws.

5.1.4 Angular Momentum for Several Particles

We now want to discuss the angular for a system of N particles. We define the total angular momentum as

$$\vec{L} = \sum_{\alpha=1}^{N} \vec{\ell}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{p}_\alpha. \tag{12}$$

Taking the derivative with respect to t (and remembering that $\dot{\vec{r}}_\alpha$ is parallel to $\vec{p}_\alpha$) yields

$$\dot{\vec{L}} = \sum_{\alpha=1}^{N} \dot{\vec{\ell}}_\alpha = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha, \tag{13}$$

where as usual $\vec{F}_\alpha$ is the net force on particle α. So we see that the rate of change in the total angular momentum is just the net torque on the whole system. However we can do more with this expression. Again we can decompose $\vec{F}_\alpha$ into internal forces and external forces via

$$\vec{F}_\alpha = \sum_{\beta \neq \alpha}^{N} \vec{F}_{\alpha\beta} + \sum_{\alpha=1}^{N} \vec{F}_\alpha^{\text{ext}}. \tag{14}$$

Now the contribution to the rate of change to the total angular momentum from the internal forces is

$$\sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} + \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{r}_\beta \times \vec{F}_{\beta\alpha}, \tag{15}$$

where we have simply broken the sum in half and the relabeled the dummy summation indices in the second sum. Assuming that all of the internal forces obey Newton's third law, $\vec{F}_{\beta\alpha} = -\vec{F}_{\alpha\beta}$, this expression reduces to

$$\sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} \vec{r}_\alpha \times \vec{F}_{\alpha\beta} = \frac{1}{2} \sum_{\alpha=1}^{N} \sum_{\beta \neq \alpha}^{N} (\vec{r}_\alpha - \vec{r}_\beta) \times \vec{F}_{\alpha\beta}. \tag{16}$$

Now if in addition to all of the internal forces satisfying Newton's third law they are also central forces then $\vec{r}_\alpha - \vec{r}_\beta$ is parallel to $\vec{F}_{\alpha\beta}$ and this term vanishes. This allows us to express the rate of change in the total angular momentum as

$$\dot{\vec{L}} = \sum_{\alpha=1}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}} = \vec{\Gamma}^{\text{ext}}. \tag{17}$$

In particular if the net external torque is zero, then we can state

Principle of Conservation of Angular Momentum

If the external torque on a system of N particles is zero, then the system's total angular momentum, $\overrightarrow{L} = \sum_{\alpha=1}^{N} \overrightarrow{r}_\alpha \times \overrightarrow{p}_\alpha$, is constant.

Remember that the validity of this statement depends on two assumptions, (i) all of the internal forces satisfy Newton's third law and (ii) all of the internal forces are central forces. Since these assumptions are almost always valid, the "Principle of Conservation of Angular Momentum" is likewise.

5.1.5   Moment of Inertia

The calculation of angular momenta does not always rely on the basic definition. It can be shown (as an exercise for the student) that the magnitude of the angular momentum for a single particle can also be expressed as

$$\ell = mr^2\omega, \tag{18}$$

where $\omega = \dot{\phi}$, i.e. the angular velocity of a particle about an origin defined by r. If we consider only the component of the angular momentum about a given axis, then for a multiparticle system the angular momentum about this axis can be expressed as

$$L = \sum_{\alpha=1}^{N} m_\alpha r_\alpha^2 \omega_\alpha, \tag{19}$$

where r_α is the distance measured from the axis of rotation. Further, if all of the masses are incremental masses that form a single solid object then they are all rotating at the same angular velocity. The angular momentum of this solid object about a given axis, e.g. the z axis, can then be written as

$$L_z = \left(\int r_z^2 dm \right) \omega. \tag{20}$$

We then define the momentum of inertia about a given axis as

$$I = \int r^2 dm, \tag{21}$$

where r is the distance from that axis. The problem is actually somewhat more involved than is apparent in this short discussion, and consequently it will be discussed in more detail in Chapter 10. That being said this will suffice for the present.

As an example we are interested in the example of the collision of a lump of putty with a turntable. To proceed it is useful to determine the moment of inertia for the turntable. We will represent the turntable as a disk of mass M

and radius R. The mass per unit area of this disk is $\sigma = M/\pi R^2$. Writing the incremental mass element as $dm = \sigma dA$, the moment of inertia for the disk is given by the integral

$$
\begin{aligned}
I_{\text{disk}} &= \frac{M}{\pi R^2} \int r^2 dA = \frac{M}{\pi R^2} \int_0^R r^2 2\pi r dr \\
I_{\text{disk}} &= \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2} \frac{R^4}{4} = \frac{1}{2} M R^2.
\end{aligned} \tag{22}
$$

Now consider the situation where a lump of putty of mass m approaches the turntable (in the same plane as the turntable) with a velocity v and a distance
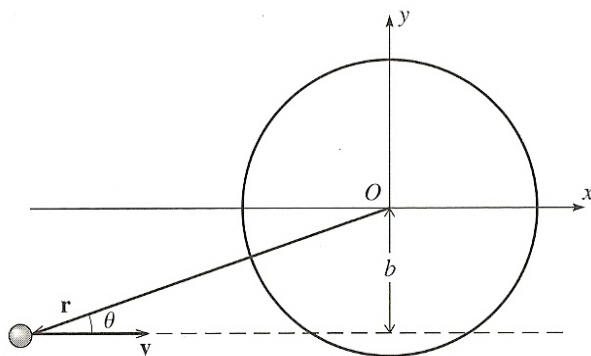

Figure 3-3. Lump of putty of mass m approaches a stationary turntable with velocity $\overrightarrow{v}$. The putty's line of approach passes within a distance b of the table's center.

of closest approach, usually designated as the impact parameter, of b. This geometry is shown in Figure 3. The turntable is initially at rest and we wish to determine the rate of angular rotation, ω, of this system after the lump of putty collides with and sticks to the turntable. The initial angular momentum about the center of the turntable is just that of the lump of putty,

$$
L_z^{\text{in}} = (\overrightarrow{r} \times \overrightarrow{p})_z = rmv \sin\theta. \tag{23}
$$

Since the quantity $r\sin\theta$ is equal to b, the impact parameter, this expression reduces to $L_z^{\text{in}} = mbv$. Knowing the moment of inertia of the turntable, the angular momentum after the collision is expressed as

$$
L_z^{\text{fin}} = mR^2\omega + \frac{M}{2} R^2 \omega \tag{24}
$$

From the conservation of angular momentum we can equate the initial and final values for the angular momentum and find

$$
\omega = \frac{m}{m + M/2} \frac{bv}{R^2}. \tag{25}
$$

What is of interest here is how easily we were able to determine the final answer. Conservation of angular momentum as well as linear momentum can be used to great effectiveness in many situations in physics.

5.1.6   Angular Momentum about the CM

The conservation of angular momentum and the more general result, $\dot{\overrightarrow{L}} = \overrightarrow{\Gamma}^{\text{ext}}$, were derived on the assumption that all quantities were measured in an inertial frame. This required that both $\overrightarrow{L}$ and $\overrightarrow{\Gamma}^{\text{ext}}$ were measured about an origin O fixed in some inertial frame. As it turns out the same results hold if $\overrightarrow{L}$ and $\overrightarrow{\Gamma}^{\text{ext}}$ are measured about the center of mass even if the CM is being accelerated and not fixed in an inertial frame. The proof of this is left as an exercise for the student and is stated as

$$\frac{d}{dt}\overrightarrow{L}\,(\text{about CM}) = \overrightarrow{\Gamma}^{\text{ext}}\,(\text{about CM}). \tag{26}$$

An example that demonstrates the ease that this result allows us to solve various problems is that of a dumbbell lying on a frictionless plane. The dumbbell consists of two equal masses on the ends of a rigid massless rod of length $2b$ that lies along the x axis centered at the origin as shown in Figure 4. At time $t = 0$, the left mass is given a sharp tap with a force F lasting for a short time Δt. We wish to find the initial motion immediately after the tap as well as the subsequent force free motion. The change in momentum of the dumbbell is determined by the impulse $F\Delta t$. Since the dumbbell is initially at rest the final momentum is

$$P = F\Delta t = Mv_{\text{CM}} = 2mv_{\text{CM}}, \tag{27}$$

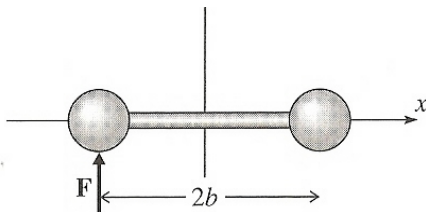where M is the mass of the system so that $M = 2m$.

Figure 3-4. The left mass of a dumbbell is given an implusive blow in the y direction.

Similarly the change in the angular momentum about its CM is given by $\overrightarrow{\Gamma}^{\text{ext}}\Delta t$. Thus

$$L = Fb\Delta t = I\omega = 2mb^2\omega. \tag{28}$$

This amounts to a clockwise rotation with frequency $\omega = F\Delta t/2mb$. Thus the mass on the left side of the dumbbell is moving with a velocity ωb relative to

the center of mass while the mass on the right side of the dumbbell is moving with a velocity of $-\omega b$ relative to the center of mass. This is expressed as

$$
\begin{align}
v_{\text{left}} &= F\Delta t/2m + \omega b = F\Delta t/2m + F\Delta t/2m = F\Delta t/m \tag{29}\\
v_{\text{right}} &= F\Delta t/2m - \omega b = F\Delta t/2m - F\Delta t/2m = 0. \tag{30}
\end{align}
$$

We see that initially the left mass carries all of the momentum of the system while the right mass is stationary. Physically this is what we should have expected.

The subsequent motion is very straightforward. Once the impulse has ceased the CM continues to move straight up the y axis with a velocity v_{CM} while the dumbbell continues to rotate with angular velocity ω about the center of mass.