10 Lecture 10-16

10.0.1 Virial Theorem

Before we begin our discussion of oscillating systems we will have a short expose of the Virial Theorem for a single particle. Consider the quantity (usually called the virial G) as $G = \overrightarrow{p} \cdot \overrightarrow{r}$. The time rate of change of G is given by

$$\frac{dG}{dt} = \dot{\overrightarrow{p}} \cdot \overrightarrow{r} + \overrightarrow{p} \cdot \dot{\overrightarrow{r}} = \overrightarrow{F} \cdot \overrightarrow{r} + 2T,$$

where T is the kinetic energy. Now if the force is a central conservative force (usually the case) then it is given by

$$\overrightarrow{F} = -\nabla U(r),$$

where U is the potential energy which is a function only of the distance between the object source of the force. Additionally if the potential is of the form

$$U = kr^n,$$

then

$$-\nabla U(r) = -\frac{\partial U}{\partial r}\widehat{r} = -nkr^{n-1}\widehat{r}.$$

The time derivative of G can now be written

$$\frac{dG}{dt} = 2T - nkr^{n-1}\widehat{r} \cdot \overrightarrow{r} = 2T - nkr^{n-1}\widehat{r} \cdot r\widehat{r} = 2T - nU.$$

Taking the average of this equation results in

$$\left\langle \frac{dG}{dt} \right\rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{dG}{dt} dt = 2\langle T \rangle - n\langle U \rangle$$
$$\lim_{\tau \to \infty} \frac{1}{\tau} (G(\tau) - G(0)) = 2\langle T \rangle - n\langle U \rangle.$$

For any object in a bound orbit G itself is *bounded*, i.e. it has a maximum value as it oscillates between its minimum and maximum values. This means its average over a long time period must vanish. This results in the virial theorem for a single particle, which is stated as

$$\langle T \rangle = \frac{n}{2} \langle U \rangle.$$

We have already verified this for perfectly circular orbits. In that case $\langle T \rangle = T$ a constant and $\langle U \rangle = U$ a constant. Taking the time average we see that this expression is more generally true.

10.1 Chapter 5 Oscillations

Almost any system that is displaced from a position of stable equilibrium exhibits *oscillations*. If the displacement is small the oscillations are almost always of the type called simple harmonic. Oscillations and particularly simple harmonic oscillations, are therefore extremely widespread. Their importance to society ranges from a simple pendulum clock to atomic oscillations. It is for these reasons that we shall study simple harmonic oscillations (SHO) and then go on to damped and driven oscillations.

10.1.1 Hooke's Law

Hooke's law asserts that the force exerted by a spring has the form

$$F_x(x) = -kx, \tag{1}$$

where x is the displacement of the spring from equilibrium and k (a positive number) is the force constant. The fact that k is positive means that the equilibrium at $x = 0$ is stable. This is easily seen from the form of the potential energy

$$U(x) = \frac{1}{2}kx^2. \tag{2}$$

The second derivative of this parabola is simply k and for $k > 0$ we have a position of stable equilibrium.

Now consider an arbitrary conservative one-dimensional system with a potential energy $U(x)$. Further suppose that the system has a position of stable equilibrium at $x = x_o$. A Taylor's series about this position is

$$U(x) = U(x_o) + U'(x_o)(x - x_o) + \frac{1}{2}U''(x_o)(x - x_o)^2 + \cdots \tag{3}$$

As long as x near x_o then these three terms should be a good approximation to the potential. Since the system is at equilibrium at $x = x_o$, $U'(x_o)$ vanishes. For convenience we can almost always choose the origin to coincide with x_o. Additionally we can redefine the reference point for the potential energy so that $U(0) = 0$. To a good approximation the arbitrary potential takes the form Hooke's law

$$U(x) = \frac{1}{2}kx^2, \tag{4}$$

at least for small displacements from equilibrium. Note that if $U''(x_o)$ were less than zero, then the point of equilibrium would be unstable. For the time being we will such systems.

Hooke's law comes up in many situations as it is not necessary for the coordinate to be a rectangular coordinate such as x. Consider again the problem of a cube balanced on a cylinder. The potential energy was given by

$$U(\theta) = mg\left[(r + b)\cos\theta + r\theta\sin\theta\right]. \tag{5}$$

If θ is small then we can make the approximations $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1-\theta^2/2$, so that

$$U(\theta) \simeq mg\left[(r+b)\left(1-\theta^2/2\right)+r\theta^2\right] = mg(r+b)+\frac{1}{2}mg(r-b)\theta^2. \qquad (6)$$

Apart from the constant this potential has the form of $k\theta^2/2$ with an effective "spring constant" $k = mg(r-b)$. Notice that the equilibrium is stable only when $k = mg(r-b) > 0$, which is what we found previously.

From our earlier plots of the potential energies, the general features of any one dimensional system can be understood from Figure 5.1 which plots a parabolic potential as a function of x. If a particle has a total energy $E > 0$,

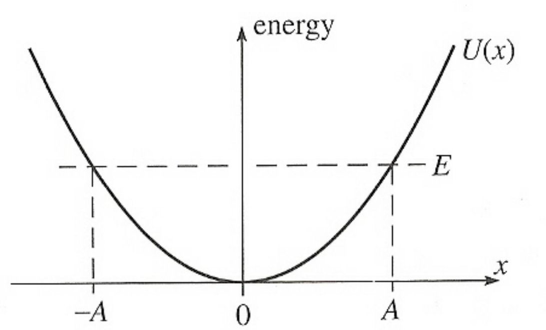

Figure 5.1 Potential energy $U(x) = \frac{1}{2}kx^2$. A particle with energy $E = \frac{1}{2}kA^2$ oscillates between two turning points at $x = \pm A$.

it is trapped and oscillates between $x = \pm A$. Since the kinetic energy is zero at the turning points the total energy of the particle is $E = \frac{1}{2}kA^2$. Note that due to the symmetry of the potential the two turning points are equidistant from the origin.

10.1.2 Simple Harmonic Motion - SHO

Consider a mass m attached to spring with spring constant k, From Newton's equation of motion we have

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x, \qquad (7)$$

where ω^2 is defined to be $\omega^2 = k/m$. Replacing x with ϕ, this is the same equation that governed the skateboard in a trough (at least for small values of ϕ). There are several ways to write these solutions. They are have their own advantages and you should be familiar with all of them.

Exponential Solutions Equation (7) is a second order, linear, homogeneous differential equation and has two independent solutions. We will consider the solutions of the form

$$x(t) = e^{i\omega t} \text{ and } x(t) = e^{-i\omega t}. \tag{8}$$

A simple substitution readily verifies that both of these functions satisfy equation (7). Since the EOM is linear and homogeneous the most general solution is a linear superposition of these two independent solutions, i.e.

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}. \tag{9}$$

Sine and Cosine Solutions The exponential solutions are so easily handled, they are often the solution of choice. But they do have one drawback. They are complex and we know that the displacement of the particle is real. The initial conditions have to force a real solution in a totally natural way. However before we discuss that let's consider the sine and cosine solutions. From Euler's formula we know that

$$e^{\pm i\omega t} = \cos\omega t \pm i \sin\omega t. \tag{10}$$

Substituting these solutions into equation (9) and grouping terms yields

$$\begin{aligned} x(t) &= (C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t, \\ x(t) &= B_1 \cos\omega t + B_2 \sin\omega t, \end{aligned} \tag{11}$$

where $B_1 = C_1 + C_2$ and $B_2 = i(C_1 - C_2)$. This form of the solution makes it clear why the differential equation is that of a simple harmonic oscillator. Additionally, since B_1 and B_2 are both real, we see that the constraints on C_1 and C_2, so that $x(t)$ is real, is that they must be the complex conjugate of each other, namely $C_1 = (B_1 - iB_2)/2$ and $C_2 = C_1^* = (B_1 + iB_2)/2$.

The coefficients B_1 and B_2 can both be easily determined from the initial conditions. If initially the mass is stationary at $x = A$ then we find

$$x(t) = A\cos\omega t. \tag{12}$$

If instead, we launch the cart at $x = 0$ with a velocity v_o, we find

$$\dot{x}(t) = v_o \cos\omega t \rightarrow x(t) = \frac{v_o}{\omega}\sin\omega t.$$

As seen in figure 5.2, both of these solutions oscillate at the same frequency with a period of $\tau = 2\pi/\omega = 2\pi\sqrt{m/k}$ and are out of phase with each other by $\pi/2$.

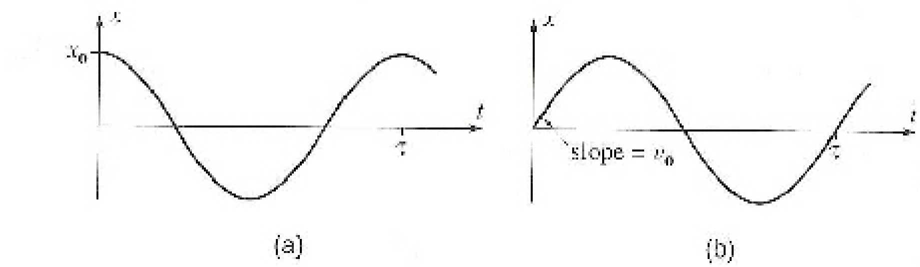

Figure 5.2 (a) Oscillations in which the cart is released from from rest x_o at $t = 0$ obey a cosine curve. (b) Oscillations that at equilibrium have an initial velocity $v = v_o$ obey a sine curve. Both curves have the same frequency.

Phase Shifted Cosine Solution The general solution in equation (11) can be rewritten as

$$x(t) = A\cos(\omega t - \delta). \tag{13}$$

To see this we expand the cosine function and find

$$x(t) = A\cos\delta\cos\omega t + A\sin\delta\sin\omega t. \tag{14}$$

Equating $B_1 = A\cos\delta$ and $B_2 = A\sin\delta$ makes it clear that these are identical solutions. The advantage of this form is that the amplitude of the oscillations is

$$A = \sqrt{B_1^2 + B_2^2}, \tag{15}$$

with a phase shift given by

$$\delta = \tan^{-1} B_2/B_1. \tag{16}$$

Solution as the Real Part of a Complex Exponential As we already noted C_1 and C_2 must be the complex conjugate of each other,

$$C_2 = C_1^*. \tag{17}$$

Thus our original solution, equation (9), can be written as

$$x(t) = C_1 e^{i\omega t} + C_1^* e^{-i\omega t}, \tag{18}$$

where we note that the second term on the right is the complex conjugate of the first term on the right. For any complex number $z = x + iy$, we have

$$z + z^* = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}(z), \tag{19}$$

where $\mathrm{Re}(z)$ denotes the real part of z. Thus equation (18) can be written as

$$x(t) = 2\,\mathrm{Re}\left(C_1 e^{i\omega t}\right) = \mathrm{Re}\left(2C_1 e^{i\omega t}\right) = \mathrm{Re}\left(Ce^{i\omega t}\right), \tag{20}$$

where we have defined $C = 2C_1$. In terms of the real numbers B_1 and B_2 we know that

$$2C_1 = C = B_1 - iB_2 = A\cos\delta - iA\sin\delta = Ae^{-i\delta}. \tag{21}$$

This allows to write the answer as simply

$$x(t) = \mathrm{Re}\left(Ae^{i(\omega t-\delta)}\right) \tag{22}$$

This result is shown in figure 5.3 which is in agreement with equation (13).

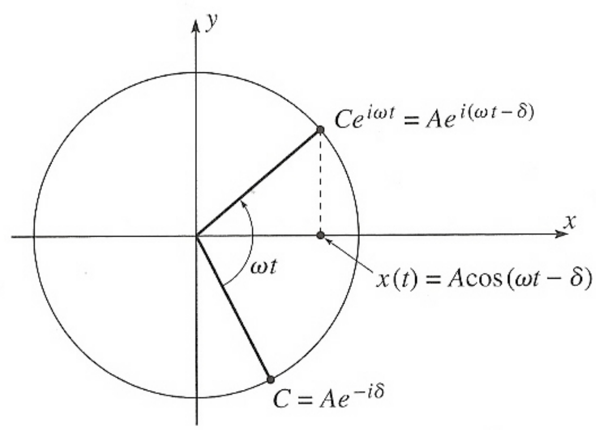

Figure 5.3 The position $x(t)$ of the cart is the real part of the complex number $Ae^{i(\omega t-\delta)}$. As the complex number moves around the circle the position of the cart oscillates back and forth on the x axis with amplitude A.

The complex number moves counterclockwise with angular velocity ω around a circle of radius A. It real part is a projection onto the real axis. While the complex number rotates around the circle the real part oscillates back and forth with angular frequency ω and amplitude A. Specifically

$$x(t) = A\cos(\omega t - \delta), \tag{23}$$

As an example of simple harmonic motion consider a bottle partially filled with sand floating in water. At equilibrium it is submerged to a depth of d_o. From the principle of Archimedes we know that at equilibrium

$$mg = \rho g A d_o, \tag{24}$$

where ρ is the density of water and A is the cross sectional area of the bottle. If we push it to a depth of $d = d_o + x$, the bottle experiences a restoring force of

$$F = -\rho g A(d - d_o) = -\rho g A x, \tag{25}$$

where we are measuring x positive in the downward direction. Newton's EOM becomes

$$\begin{aligned} m\ddot{x} &= \rho A d_o \ddot{x} = -\rho g A x, \\ \ddot{x} &= -\frac{g}{d_o} x. \end{aligned} \tag{26}$$

Which is our equation for simple harmonic motion. If $d_o = 20cm$ then the period of oscillation is

$$\tau = 2\pi\sqrt{d_o/g} = 2\pi\sqrt{.2/9.8} = .9\,\mathrm{sec}\,. \tag{27}$$

Energy Considerations Before we move on from this discussion it is time to summarize the energy of an oscillator. Consider Figure 5.1 once again. (now Figure 5.4). As we have seen a general solution for the motion of the particle is

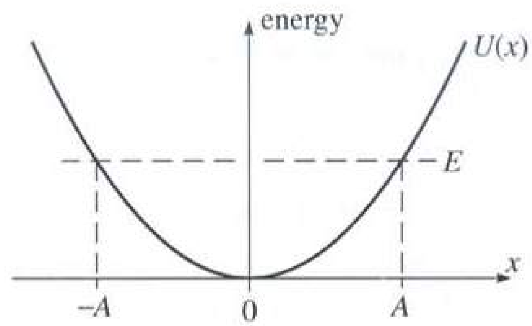

Figure 1: Potential energy $U(x) = \frac{1}{2}kx^2$. A particle with energy $E = \frac{1}{2}kA^2$ oscillates between two turning points at $x = \pm A$.

$x(t) = A\cos(\omega t - \delta)$ with the velocity being given by $\dot{x}(t) = -A\omega\sin(\omega t - \delta)$. Thus the potential energy is simply

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t - \delta), \tag{28}$$

and the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t - \delta), \\ &= \frac{1}{2}kA^2\sin^2(\omega t - \delta). \end{aligned} \tag{29}$$

So both T and U oscillate between 0 and $\frac{1}{2}kA^2$, with their oscillations perfectly out of step. When T is a maximum U is zero and vice-versa. The sum of T and U is simply

$$E = T + U = \frac{1}{2}kA^2, \tag{30}$$

which is a constant as it had to be.

Expectation Values and the Virial Theorem For an SHO of period the average value of a variable f is expressed as

$$\langle f \rangle = \frac{1}{\tau} \int_0^\tau f(t)\, dt.$$

Alternatively for a nonoscillating system we could use the same expression while letting $\tau \to \infty$. With this definition we find that the average value of the potential energy for an SHO is

$$\langle U \rangle = \frac{1}{2} k \left\langle x^2(t) \right\rangle = \frac{1}{2} k A^2 \left\langle \cos^2(\omega t - \delta) \right\rangle = \frac{1}{2} k A^2 \frac{1}{\tau} \int_0^\tau \left(\frac{1}{2} + \frac{1}{2} \cos 2(\omega t - \delta) \right) dt. \tag{31}$$

The integral over the cosine function vanishes leading to the result

$$\langle U \rangle = \frac{1}{2} \left(\frac{1}{2} k A^2 \right) = \frac{1}{2} E. \tag{32}$$

Meanwhile the average value of the kinetic energy is

$$\langle T \rangle = \frac{1}{2} m \left\langle \dot{x}^2(t) \right\rangle = \frac{1}{2} m A^2 \omega^2 \left\langle \sin^2(\omega t - \delta) \right\rangle = \frac{1}{2} k A^2 \left\langle \frac{1}{2} - \frac{1}{2} \cos 2(\omega t - \delta) \right\rangle. \tag{33}$$

Using the same argument that was used in evaluating the average potential energy we find

$$\langle T \rangle = \frac{1}{2} \left(\frac{1}{2} k A^2 \right) = \langle U \rangle = \frac{1}{2} E. \tag{34}$$

Hence

$$\langle T + U \rangle = \langle E \rangle = E,$$

as it had to be. Remembering the virial theorem that we proved for circular orbits, $T = nU/2$, we notice that on average for an SHO that

$$\langle T \rangle = \frac{n}{2} \langle U \rangle = \langle U \rangle,$$

as $n = 2$ for an SHO. So on average our SHO satisfies the same relation.

10.1.3 Two-Dimensional Oscillators

In two dimensions the possibilities for oscillations are much richer than in one dimension. The simplest is that of an isotropic oscillator for which the restoring force is proportional to the displacement from equilibrium with the same constant in all directions:

$$\overrightarrow{F} = -k \overrightarrow{r}. \tag{35}$$

In component form this equation becomes $F_x = -kx$, $F_y = -ky$, and (for three dimensions $F_z = -kz$). In an exercise for the student, the four identical springs as shown in Figure 5.5 produce a restoring force resulting in an isotropic oscillator.

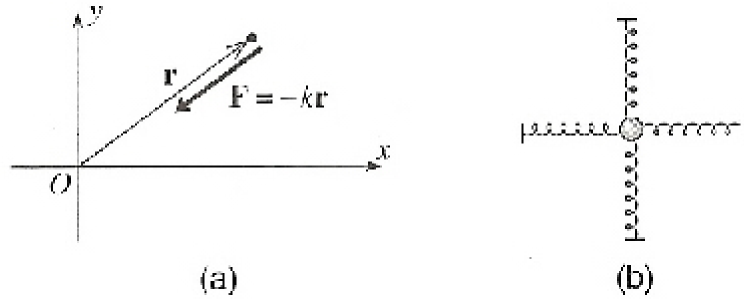

Figure 5.5 (a) A restoring force that is proportional to $\overrightarrow{r}$ defines an isotropic harmonic oscillator. (b) A mass at the center of 4 springs (in this arrangement) would experience a net force $\overrightarrow{F} = -k\overrightarrow{r}$ in the plane of the springs.

A particle that is subject to this kind of force in two dimensions satisfies the two independent equations

$$\ddot{x} = -\omega^2 x \text{ and } \ddot{y} = -\omega^2 y, \tag{36}$$

where as usual $\omega^2 = k/m$. The solutions for these two equations were discussed in the last section and are

$$x(t) = A_x \cos(\omega t - \delta_x), \tag{37a}$$
$$y(t) = A_y \cos(\omega t - \delta_y), \tag{37b}$$

where the four constants are determined by the initial conditions of the problem. By redefining the time origin we can eliminate one of the phases. Thus the simplest form for the general solution is

$$x(t) = A_x \cos \omega t, \tag{38a}$$
$$y(t) = A_y \cos(\omega t - \delta), \tag{38b}$$

where $\delta = \delta_y - \delta_x$ and is the relative phase of the x and y oscillations.

The behavior of the solutions 38a and 38b depends on the values of the three constants, A_x, A_y, and δ. If either A_x or A_y is zero, then the particle executes simple harmonic motion along one of the axes. If neither A_x nor A_y is zero, the motion depends critically on the relative phase δ. If $\delta = 0$, then both x and y rise and fall in step along a line passing through the origin with slope A_y/A_x as shown in Figure 5.6(a). If $\delta = \pi/2$, then x and y oscillate out of step. When x is at an extreme, y is zero and vice versa. The resulting curve is an ellipse with semimajor and semiminor axes A_x and A_y as shown in Figure 5.6(b). For other values of δ the curves determined by $y(t)$ and $x(t)$ are slanting ellipses as shown in Figure 5.6(c) for $\delta = \pi/4$. What would you expect for $\delta = \pi$?

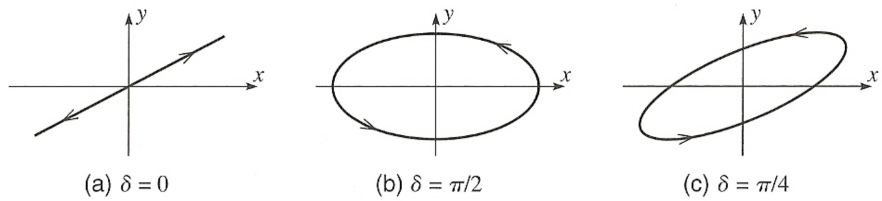

Figure 5.6 Motion of a two-dimensional isotropic oscillator as given by equations 5.38 (a) & (b) for relative phases (a) $\delta = 0$, (b) $\delta = \pi/2$, and (c) $\delta = \pi/4$.

In an anisotropic oscillator, the restoring force constants are different for the different directions:

$$F_x = -k_x x, \quad F_y = -k_y y, \quad \text{and} \quad F_z = -kz. \tag{39}$$

For simplicity we will again only consider this problem in two dimensions. The solutions to Newton's EOM are similar to the isotropic case and we have

$$x(t) = A_x \cos \omega_x t, \tag{40a}$$
$$y(t) = A_x \cos(\omega_y t - \delta). \tag{40b}$$

Because of the two different frequencies, there is a much richer variety of possible motions. If ω_x/ω_y is a rational number, it is fairly easy to see (as an exercise for the student) that the motion is periodic. The resulting path is a Lissajou figure and an example for $\omega_x/\omega_y = 2$ is shown in Figure 5.7(a). In that figure you can see that x goes back and forth twice for each time that y does so once. If ω_x/ω_y is an irrational number then the motion is more complicated and never repeats itself. This case is illustrated for $\omega_x/\omega_y = \sqrt{2}$ in figure 5.7(b).

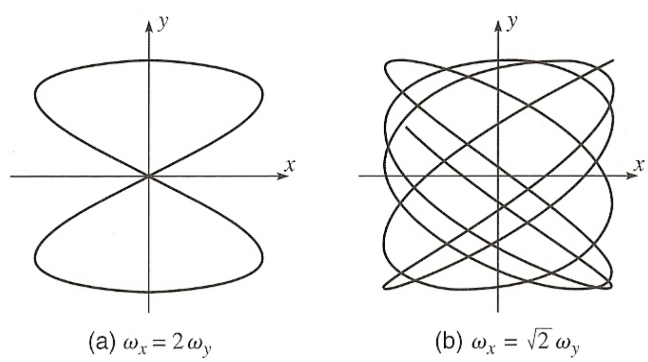

Figure 5.7 Possible paths for anisotropic oscillators with (a) $\omega_x = 2\omega_y$ and (b) $\omega_x = \sqrt{2}\omega_y$. The motion in (b) is called quasi-periodic as it is periodic in either x or y but $\overrightarrow{r}(t)$ is not periodic.