

Lagrange Multiplier Problems

Problem 7.52 A mass m is supported by a string that is wrapped many times about a cylinder with a radius R and a moment of inertia I . The cylinder is supported by a frictionless horizontal axis so that the cylinder can rotate freely about its axis. Here will develop the equation of motion for the mass and cylinder using the method of Lagrange multipliers. The generalized coordinates are ϕ which defines the configuration of the cylinder and x (measured downward) which defines the configuration of the mass m . These two coordinates are constrained via the nonslip condition between the string and the cylinder.

The constraint condition is

$$R\phi - x = \text{const.}$$

The modified Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx + \lambda(R\phi - x).$$

The two equations of motion are

$$\begin{aligned} x &: m\ddot{x} = mg - \lambda \\ \phi &: I\ddot{\phi} = \lambda R \end{aligned}$$

Eliminating ϕ via the constraint we find

$$\begin{aligned} m\ddot{x} &= mg - \lambda, \\ I\ddot{x}/R^2 &= \lambda. \end{aligned}$$

Adding these two expressions yields

$$(m + I/R^2)\ddot{x} = mg \rightarrow \ddot{x} = \frac{m}{m + I/R^2}g,$$

which is the result obtained without Lagrange multipliers or via Newton's second law combined with free body diagrams.

The constraint force is given by

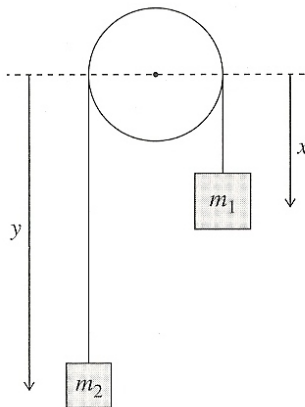
$$F_x^{cstr} = \lambda \frac{\partial (R\phi - x)}{\partial x} = -\lambda.$$

To determine the relation between the Lagrange multiplier and the tension in the string we consider the equations of motion obtained from the two free body diagrams:

$$\begin{aligned} m\ddot{x} &= mg - T \\ I\ddot{\phi} &= I\ddot{x}/R = \lambda R \rightarrow I\ddot{x}/R^2 = \lambda. \end{aligned}$$

Here it is clear that the Lagrange multiplier is the negative of the tension in the string. Again this is due to defining x to positive in the downward direction.

Atwood Machine with a pulley of radius R and moment of inertia I . Consider an Atwood machine with three generalized coordinates, x , y , and ϕ . The first two coordinates define the downward distance to the masses m_1 and m_2 respectively. The ϕ coordinate defines the configuration of the pulley.



Atwood machine with three generalized coordinates, x , y , and ϕ .

The kinetic energy of all the objects in this system is

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2.$$

Since x and y are measured downward the potential energy is

$$U = -m_1gx - m_2gy.$$

For our constraint equations we note that the length of the string is fixed and there is a no slip condition between the string and pulley on the x side,

$$x + y = \text{const.} \quad \text{and} \quad R\phi - x = \text{const.}$$

The modified Lagrangian including these two constraints via two Lagrange multipliers, λ_1 and λ_2 , is

$$\mathcal{L} = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{y}^2 + \frac{1}{2}I\dot{\phi}^2 + m_1gx + m_2gy + \lambda_1(x + y) + \lambda_2(R\phi - x).$$

This leads to three equations of motion

$$\begin{aligned} x &: m_1\ddot{x} = m_1g + \lambda_1 - \lambda_2 \\ y &: m_2\ddot{y} = m_2g + \lambda_1 \\ \phi &: I\ddot{\phi} = \lambda_2R. \end{aligned}$$

Using the constraint conditions to form three equations with x as the single generalized coordinate yields

$$\begin{aligned} m_1 \ddot{x} &= m_1 g + \lambda_1 - \lambda_2 \\ m_2 \ddot{x} &= -m_2 g - \lambda_1 \\ I \ddot{x} / R^2 &= \lambda_2 R \end{aligned}$$

Summing these three equations eliminates the Lagrange multipliers with the result

$$(m_1 + m_2 + I/R^2) \ddot{x} = (m_1 - m_2) g \rightarrow \ddot{x} = \frac{m_1 - m_2}{m_1 + m_2 + I/R^2} g.$$

which is the standard result obtained using Newton's second law.

To find the constraint forces we need to evaluate the partial derivatives of the constraint functions or

$$\begin{aligned} F_x^{cstr} &= \lambda_1 \frac{\partial(x+y)}{\partial x} + \lambda_2 \frac{\partial(R\phi-x)}{\partial x} = \lambda_1 - \lambda_2, \\ F_y^{cstr} &= \lambda_1 \frac{\partial(x+y)}{\partial y} = \lambda_1. \end{aligned}$$

To determine the physical significance of the Lagrange multipliers we note that the three equations of motion obtained using free body diagrams for m_1 , m_2 , and the pulley are:

$$\begin{aligned} m_1 \ddot{x} &= m_1 g - T_1, \\ m_2 \ddot{x} &= T_2 - m_2 g, \\ I \ddot{x} / R &= (T_1 - T_2) R. \end{aligned}$$

Here we have made use of the two constraints to form the equations of motion. We now have three equations of motion with three unknowns, T_1 , T_2 , and of course \ddot{x} . Comparing these two set of equations that

$$T_1 = -(\lambda_1 - \lambda_2) = -F_x^{cstr}, \quad \text{and} \quad T_2 = -\lambda_1 = -F_y^{cstr}.$$

Hence the constraint forces are just the negative of the tensions in the string. This negative sign is the result of defining the downward direction as positive.

Atwood Machine with a massless pulley of radius R that is supported by a spring of spring constant k . There are 3 generalized coordinates, z being the downward distance from the unstretched position of the spring while x and y define the position of the masses relative to the center of the pulley as shown in the figure above. The constraint is the length of the string is fixed. The kinetic energy of this system is

$$T = \frac{1}{2} m_1 (\dot{x} + \dot{z})^2 + \frac{1}{2} m_2 (\dot{y} + \dot{z})^2.$$

The potential energy of the system is

$$U = \frac{1}{2}kz^2 - m_1g(x+z) - m_2g(y+z).$$

The constraint condition is

$$x + y = L.$$

The modified Lagrangian is

$$\begin{aligned}\mathcal{L}' &= T - U + \lambda(t)(x + y), \\ \mathcal{L}' &= \frac{1}{2}m_1(\dot{x} + \dot{z})^2 + \frac{1}{2}m_2(\dot{y} + \dot{z})^2 - \frac{1}{2}kz^2 + m_1g(x+z) + m_2g(y+z) + \lambda(x+y).\end{aligned}$$

The modified Lagrange equations of motion are

$$\begin{aligned}x &: \frac{\partial \mathcal{L}'}{\partial x} = m_1g + \lambda = \frac{d}{dt} \frac{\partial \mathcal{L}'}{\partial \dot{x}} = m_1(\ddot{x} + \ddot{z}), \\ y &: \frac{\partial \mathcal{L}'}{\partial y} = m_2g + \lambda = \frac{d}{dt} \frac{\partial \mathcal{L}'}{\partial \dot{y}} = m_2(\ddot{y} + \ddot{z}), \\ z &: \frac{\partial \mathcal{L}'}{\partial z} = -kz + (m_1 + m_2)g = \frac{d}{dt} \frac{\partial \mathcal{L}'}{\partial \dot{z}} = m_1(\ddot{x} + \ddot{z}) + m_2(\ddot{y} + \ddot{z}).\end{aligned}$$

Making use of the constraints we eliminate y ($\ddot{y} = -\ddot{x}$) with the result

$$m_1g + \lambda = m_1(\ddot{x} + \ddot{z}) \quad (1a)$$

$$-m_2g - \lambda = m_2(\ddot{x} - \ddot{z}), \quad (1b)$$

$$-kz + (m_1 + m_2)g = m_1(\ddot{x} + \ddot{z}) + m_2(-\ddot{x} + \ddot{z}). \quad (1c)$$

Summing equations 1(a) and 1(b) eliminates the Lagrange multiplier and we are left with two equations of motion:

$$\begin{aligned}(m_1 - m_2)g &= (m_1 + m_2)\ddot{x} + (m_1 - m_2)\ddot{z} \\ -kz + (m_1 + m_2)g &= (m_1 - m_2)\ddot{x} + (m_1 + m_2)\ddot{z}\end{aligned}$$

To solve for \ddot{z} we consider the determinates

$$\begin{aligned}\ddot{z} &= \frac{\det \begin{bmatrix} m_1 + m_2 & (m_1 - m_2)g \\ m_1 - m_2 & -kz + (m_1 + m_2)g \end{bmatrix}}{\det \begin{bmatrix} m_1 + m_2 & m_1 - m_2 \\ m_1 - m_2 & m_1 + m_2 \end{bmatrix}} = \frac{4m_1m_2g - (m_1 + m_2)kz}{4m_1m_2} \\ \ddot{z} &= g - \frac{(m_1 + m_2)kz}{4m_1m_2} \rightarrow 4\mu\ddot{z} + kz = 4\mu g,\end{aligned}$$

where $\mu = m_1m_2/(m_1 + m_2)$. At equilibrium $\ddot{z} = 0$. Thus the new equilibrium position is

$$z_o = \frac{4m_1m_2}{m_1 + m_2} \frac{g}{k} = 4 \frac{\mu g}{k},$$

The pulley oscillates about this position at an angular frequency of

$$\omega^2 = \frac{m_1 + m_2}{4m_1m_2}k = 4k/\mu.$$

The general solution for z is

$$z = 4\frac{\mu g}{k} + A \cos(\omega t - \delta).$$

To determine the Lagrange multiplier we substitute equations 1(a) and 1(b) into the right hand side of 1(c) with the result

$$-kz = 2\lambda \rightarrow \lambda = -kz/2,$$

or

$$\lambda = -k \left(2\frac{\mu g}{k} + \frac{A}{2} \cos(\omega t - \delta) \right)$$

The tension in the string is given by

$$F_x^{cstr} = \lambda \frac{\partial(x+y)}{\partial x} = \lambda.$$

Here we see that λ is the negative of the tension in the string as we measured x and y positive in the downward direction. It should be noted that In general the Lagrange multiplier may be time dependent and in this case it is.