Shortest Distance Between Two Points on the Surface of a Cone Given a cone with a half angle α whose axis of symmetry is the z axis, using polar cylindrical coordinates, the distance between two points that are differentially separated is

$$ds = \sqrt{dz^2 + d\rho^2 + \rho^2 d\phi^2}.$$

z and ρ are related via $z = \rho/\tan\alpha$, which implies $dz = d\rho/\tan\alpha = \cot\alpha d\rho$. Hence

$$\begin{aligned} ds &= \sqrt{\cot^2\alpha d\rho^2 + d\rho^2 + \rho^2 d\phi^2} = \sqrt{d\rho^2/\sin^2\alpha + \rho^2 d\phi^2}, \\ ds &= \frac{1}{\sin\alpha}\sqrt{d\rho^2 + \sin^2\alpha\rho^2 d\phi^2}. \end{aligned}$$

The distance between two points can be represented in two different ways.

(1) In this method the total distance is given by

$$L = \frac{1}{\sin\alpha}\int_1^2 \sqrt{1 + \sin^2\alpha\rho^2\phi'^2}d\rho,$$

where $\phi' = d\phi/d\rho$. This integrand, f, is independent of ϕ. Hence the Euler-Lagrange equation is

$$\frac{\partial f}{\partial\phi} = \frac{d}{d\rho}\frac{\partial f}{\partial\phi'} = \frac{d}{d\rho}\frac{\sin\alpha\rho^2\phi'}{\sqrt{1 + \sin^2\alpha\rho^2\phi'^2}} = 0.$$

This implies

$$\frac{\sin\alpha\rho^2\phi'}{\sqrt{1 + \sin^2\alpha\rho^2\phi'^2}} = \rho_o,$$

where ρ_o is a constant. Solving for ϕ' results in

$$\begin{aligned} &\sin^2\alpha\rho^4\phi'^2 = \rho_o^2\left(1 + \sin^2\alpha\rho^2\phi'^2\right), \\ &\left(\rho^2 - \rho_o^2\right)\sin^2\alpha\rho^2\phi'^2 = \rho_o^2, \\ &\frac{d\phi}{d\rho} = \frac{\rho_o}{\sin\alpha\rho\sqrt{\rho^2 - \rho_o^2}}. \end{aligned}$$

This leads to the integral

$$\sin\alpha\int d\phi = (\phi - \phi_o)\sin\alpha = \rho_o\int\frac{d\rho}{\rho\sqrt{\rho^2 - \rho_o^2}}.$$

This integral is easily performed with the substitution

$$\rho = \rho_o/\cos\theta \to \rho^2 - \rho_o^2 = \rho_o^2\tan^2\theta, \text{ and } d\rho/d\theta = \rho_o\tan\theta/\cos\theta.$$

The integral now becomes

$$\begin{aligned}(\phi - \phi_o)\sin\alpha &= \rho_o \int (\rho_o \cos\theta)^{-1} \frac{\rho_o \tan\theta}{\cos\theta \rho_o \tan\theta} d\theta = \int d\theta = \theta \\ (\phi - \phi_o)\sin\alpha &= \cos^{-1}(\rho_o/\rho).\end{aligned}$$

Choosing the initial point to lie at $\phi_o = 0$, the curve for the shortest distance is

$$\rho \cos(\phi \sin\alpha) = \rho_o.$$

Some care must be taken here as it is necessary that the range in ϕ is less than π or else it is shorter to go the opposite way around the cone.

(2) In this method the total distance is given by

$$L = \frac{1}{\sin\alpha} \int_1^2 \sqrt{\rho'^2 + \sin^2\alpha\rho^2} d\phi,$$

where $\rho' = d\rho/d\phi$. Since the integrand, f, is independent of the independent variable ϕ the first integral of the Euler- Lagrange equation is

$$f - \rho' \frac{\partial f}{\partial \rho'} = \rho_o,$$

a constant. The expression becomes

$$\frac{1}{\sin\alpha}\sqrt{\rho'^2 + \sin^2\alpha\rho^2} - \frac{\rho'}{\sin\alpha}\frac{\rho'}{\sqrt{\rho'^2 + \sin^2\alpha\rho^2}} = \rho_o.$$

Multiplying by $\sqrt{\rho'^2 + \sin^2\alpha\rho^2}$ and squaring both sides of the equation leads to

$$\begin{aligned}\frac{1}{\sin\alpha}\left(\rho'^2 + \sin^2\alpha\rho^2\right) - \frac{\rho'^2}{\sin\alpha} &= \rho_o\sqrt{\rho'^2 + \sin^2\alpha\rho^2}, \\ \sin^2\alpha\rho^4 &= \rho_o^2\left(\rho'^2 + \sin^2\alpha\rho^2\right), \\ \left(\rho^2 - \rho_o^2\right)\sin^2\alpha\rho^2 &= \rho_o^2\rho'^2 \\ \sin\alpha\rho\sqrt{\rho^2 - \rho_o^2} &= \rho_o\rho' = \rho_o\frac{d\rho}{d\phi}.\end{aligned}$$

Separating and Integrating this expression yields

$$(\phi - \phi_o)\sin\alpha = \rho_o \int \frac{d\rho}{\rho\sqrt{\rho^2 - \rho_o^2}}.$$

This is the same integral as that obtained in method (1), hence the same curve,

$$\rho \cos(\phi \sin\alpha) = \rho_o.$$