3.27 Monopole: $Q = \sum_i q_i = 0.$

Dipole: $\vec{d} = \sum_i q_i r_i = 2qa\,\hat{z}$

$\vec{d} \neq 0$, so at $r \to \infty$, leading order is dipolar.

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\vec{d}\cdot\hat{r}}{r^2} = \frac{qa\cos\theta}{2\pi\varepsilon_0 r^2} \qquad (\cos\theta = \hat{r}\cdot\hat{z})$$

3.28. By def., $\vec{d} = \int \sigma \cdot \underbrace{R\hat{r}}_{\vec{r}}\, 2\pi R^2\, d(\cos\theta)$, where $\hat{r}$ is θ dependent.

Now notice that σ is φ-independent, thus $\vec{d}$ must be along $\hat{z}$, i.e., $\vec{d} = (\vec{d}\cdot\hat{z})\,\hat{z}$

Apply this decomposition to the RHS,

$\Rightarrow$

(a) $$\vec{d} = R^3 \int \sigma \cdot \underbrace{(\hat{r}\cdot\hat{z})}_{\cos\theta}\,\hat{z} \cdot 2\pi\, d(\cos\theta)$$

$$= R^3 \hat{z} \int k\cos^2\theta \cdot 2\pi\, d(\cos\theta)$$

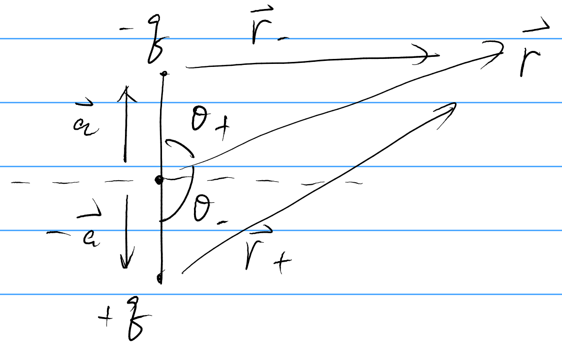
$$= R^3 \hat{z}\, k \cdot 2\pi\, \frac{1}{3}\cos^3\theta \Big|_{\cos\theta = -1}^{1}$$

$$= \frac{4}{3}\pi k R^3\, \hat{z}$$ (so one may call k the "dipole" density)

(b) $$V = \frac{1}{4\pi\varepsilon_0} \frac{\vec{d}\cdot\hat{r}}{r^2} = \frac{kR^3\cos\theta}{3\varepsilon_0 r^2}$$

Same as exact result $\Rightarrow$ Higher moments all 0.

3.29



As they stand,

$\vec{r}_\pm = \vec{r} \pm \vec{a}$ where $a = \frac{d}{2}$

$$\Rightarrow \frac{1}{r_\pm} = \frac{1}{|\vec{r} \pm \vec{a}|} = \sum_\ell P_\ell(\cos\theta_\pm)\frac{a^\ell}{r^{\ell+1}} \qquad (r > a)$$

Note that $\cos\theta_+ = -\cos\theta_-$ $(\because \theta_+ + \theta_- = \pi)$

$$\Rightarrow \frac{1}{r_+} - \frac{1}{r_-} = \sum_{\underline{\underline{\ell \text{ odd}}}} 2P_\ell(\cos\theta)\frac{a^\ell}{r^{\ell+1}}$$

$$\Rightarrow V = \frac{1}{4\pi\varepsilon_0}\cdot q \cdot \left(\frac{1}{r_+} - \frac{1}{r_-}\right) \equiv \sum_\ell V_\ell(a)$$

where $V_\ell(a) = \begin{cases} \frac{2q}{4\pi\varepsilon_0} P_\ell(\cos\theta)\frac{a^\ell}{r^{\ell+1}} & \ell \text{ odd} \\ 0 & \ell \text{ even} \end{cases}$

$\Rightarrow$ dipole: $\ell = 1$, $V_\ell = \frac{q}{2\pi\varepsilon_0}\frac{a}{r^2}P_1 = \frac{qd}{4\pi\varepsilon_0 r^2}\cos\theta$

recall $a \equiv \frac{d}{2}$

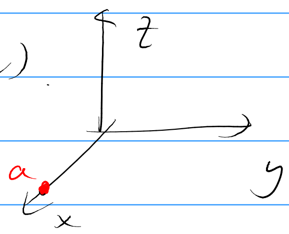
quadrupole: $\ell = 2$, $V_\ell = 0$.

octapole: $\ell = 3$, $V_\ell = \frac{q}{2\pi\varepsilon_0}\cdot\frac{a^3}{r^4}P_3$

$$= \frac{q}{4\pi\varepsilon_0}\frac{d^3}{8r^4}(5\cos^3\theta - 3\cos\theta)$$

3.31 $E(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$

(a)

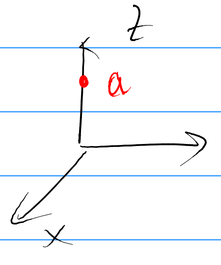


$(a,0,0)$: $r = a$, $\theta = \frac{\pi}{2}$,

$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{p}{a^3}\cdot\hat{\theta}$, $\hat{\theta} = -\hat{z}$ locally.

$\Rightarrow F = qE = -\frac{qp}{4\pi\varepsilon_0 a^3}\hat{z}$

b)



$(0,0,a)$: $r = a$, $\theta = 0$

$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{p}{a^3}\cdot 2\hat{r}$, $\hat{r} = \hat{z}$ locally

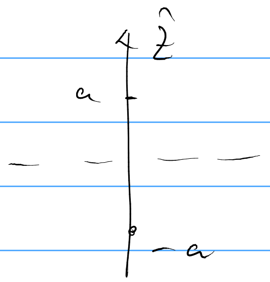
$$\Rightarrow F = qE = \frac{qp}{2\pi \varepsilon_0 a^3}\,\hat{z}$$

(c) $V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \Rightarrow V_{(a)} = 0 \quad (\hat{z}\cdot\hat{x} = 0)$

$$V(b) = \frac{p}{4\pi\varepsilon_0 a^2} \quad (\hat{z}\cdot\hat{z} = 1)$$

$$\Rightarrow W = q\left(V_{(b)} - V_{(a)}\right) = \frac{qp}{4\pi\varepsilon_0 a^2}$$

3.40



(a). $\lambda = k\cdot\cos(\pi z/2a)$

$$\Rightarrow Q = \int \lambda\, dz = \frac{2a}{\pi}\cdot k\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(\frac{\pi z}{2a}\right) d\left(\frac{\pi z}{2a}\right)$$

$$= \frac{4ka}{\pi} \neq 0$$

$\Rightarrow$ leading term is

$$V^{(0)}_{(r)} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r} = \frac{ka}{\pi^2\varepsilon_0 r}$$

(b). $\lambda = k\sin(\pi z/a)$

$$\Rightarrow Q = \int \lambda\, dz = \frac{ak}{\pi}\int_{-\pi}^{\pi} \sin\left(\frac{\pi z}{a}\right) d\left(\frac{\pi z}{a}\right) = 0.$$

$$\vec{d} = \hat{z}\int \lambda z\, dz = \hat{z}\,\frac{a^2 k}{\pi^2}\int_{-\pi}^{\pi} t\sin t\, dt$$

Now, $I \equiv \int_{-\pi}^{\pi} t\sin t\, dt$, $\quad t\sin t = -\frac{\partial}{\partial x}\cos(tx)\Big|_{x=1}$

$$\Rightarrow I = \left\{-\frac{\partial}{\partial x}\left[\frac{1}{x}\underbrace{\int_{-\pi x}^{\pi x}\cos(tx)\, d(tx)}_{=2\sin(\pi x)}\right]\right\}_{x=1}$$

$$= \left[\frac{1}{x^2} 2\sin(\pi x) - 2\cos(\pi x) \cdot \frac{\pi}{x} \right]_{x=1}$$

$$= 2(\underbrace{\sin\pi}_{0} - \pi\cos\pi) = 2\pi$$

Alternatively, $I = \int_{-\pi}^{\pi} t\sin t\,dt = -\int t\,d(\cos t)$

by part $= t\cos t\Big|_{\pi}^{-\pi} + \int_{-\pi}^{\pi} \cos t\,dt$

$= 2\pi$

$$\Rightarrow \vec{d} = \frac{2ka^2}{\pi}\hat{z}, \quad V = \frac{1}{4\pi\epsilon_0}\cdot\frac{1}{r^2}\underbrace{\frac{2ka^2}{\pi}}_{|d|}\underbrace{\cos\theta}_{\hat{d}\cdot\hat{r}}$$

(c). $\lambda = k\cos(\pi z/a)$

$$\Rightarrow Q = \int \lambda\,dz = \frac{ka}{\pi}\int_{-\pi}^{\pi}\cos t\,dt = 0.$$

$$\vec{d} = \hat{z}\int \lambda z\,dz = \hat{z}\frac{ka^2}{\pi^2}\int_{-\pi}^{\pi}\underbrace{t\cos t}_{\text{odd.}}\,dt = 0$$

$\Rightarrow$ Need higher moments, so resort back to:

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{\lambda(z)\,dz}{|\vec{r} - z\hat{z}|}, \quad \text{where}$$

$$\frac{1}{|\vec{r} - z\hat{z}|} = \sum_{l} P_l(\hat{r}\cdot\hat{z})\frac{z^l}{r^{l+1}}$$

$$\Rightarrow V_l = \frac{1}{4\pi\epsilon_0}P_l(\cos\theta)\cdot\frac{k}{r^{l+1}}\underbrace{\int z^l\lambda(z)\,dz}_{\equiv I^{(l)}}$$

(such that $V = \sum_l V_l$)

Quadrupole: $l=2$,

$$I^{(l)} = \left(\frac{a}{z}\right)^3 \int_{-z}^{z} t^2 \cos t \, dt$$

$$= -\left(\frac{a}{z}\right)^3 \left\{ \frac{\partial^2}{\partial x^2}\left[\frac{1}{x}\underbrace{\int_{-zx}^{zx} \cos(tx)\, d(tx)}_{2\sin(zx)}\right]\right\}_{x=1}$$

$$= \left(\frac{a}{z}\right)^3 \cdot \left\{\frac{\partial}{\partial x}\left[\frac{1}{x^2} 2\sin(zx) - 2\cos(zx)\cdot\frac{z}{x}\right]\right\}_{x=1}$$

with the intention of $x \to 1$, drop resulting $\sin(zx)$ in $\frac{\partial}{\partial x}$,

$$\Rightarrow \quad = \left(\frac{a}{z}\right)^3 \cdot 2\left[\frac{\cos(zx)}{x^2}\cdot z + \frac{\cos(zx)\cdot z}{x^2}\right]$$

$$= -4\left(\frac{a}{z}\right)^3 z = -4\frac{a^3}{z^2}$$

$\uparrow$ $\cos z$

(Alternatively by part twice).

$$\Rightarrow V_2 = \frac{1}{4\pi\varepsilon_0} P_2(\cos\theta)\cdot\frac{k}{r^3} I^{(2)}$$

$$= -\frac{k}{\varepsilon_0 \pi^3}\left(\frac{a}{r}\right)^3 \cdot P_2(\cos\theta)$$