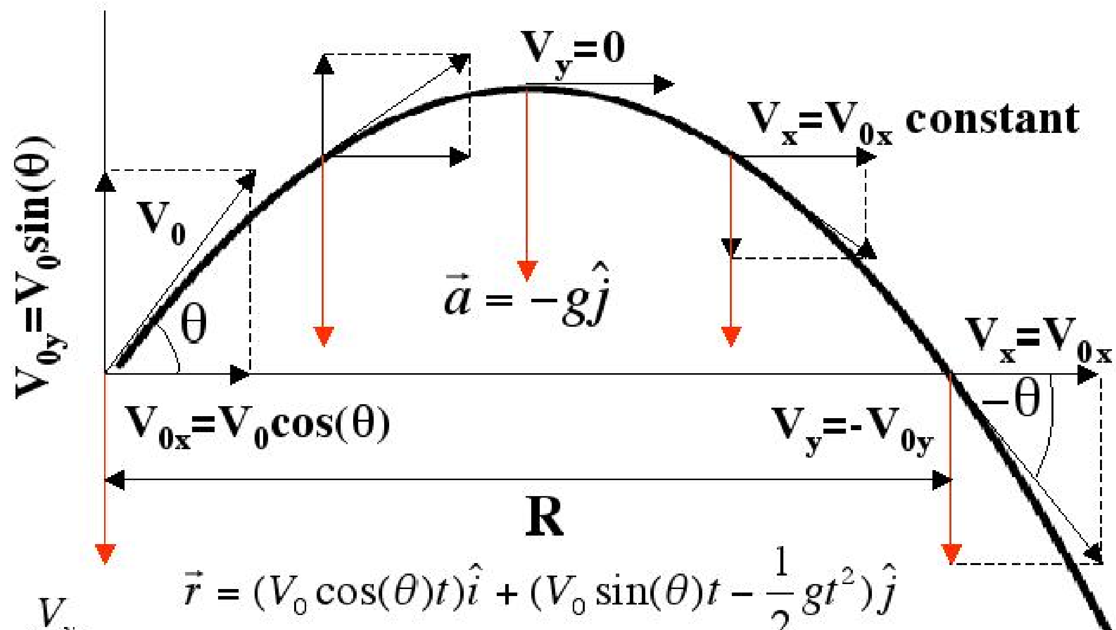
Today's Lecture

Motion in More Than One-D
Circular Motion 1

Review: Projectile Motion

The equations of motion apply to a projectile launched at some angle to the ground.



$$\vec{r} = (V_0\cos(\theta)t)\hat{i} + (V_0\sin(\theta)t - \frac{1}{2}gt^2)\hat{j}$$

$$\theta = \arctan(\frac{V_y}{V_x})$$

$$\vec{V} = (V_0\cos(\theta))\hat{i} + (V_0\sin(\theta) - gt)\hat{j}$$

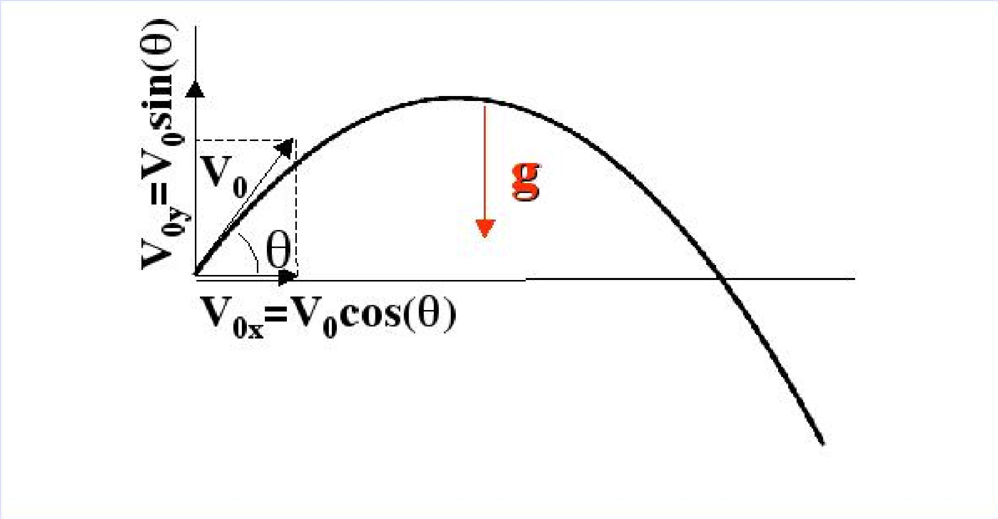
The motion in x has constant velocity because there is no acceleration in x.

Projectile Trajectory is Parabolic

We can show this by eliminating time from the problem and deriving the function y(x). First, solve for the time from the position equation in x

$$x = V_0 \cos(\theta) t$$

$$t = \frac{x}{V_0 \cos(\theta)}$$



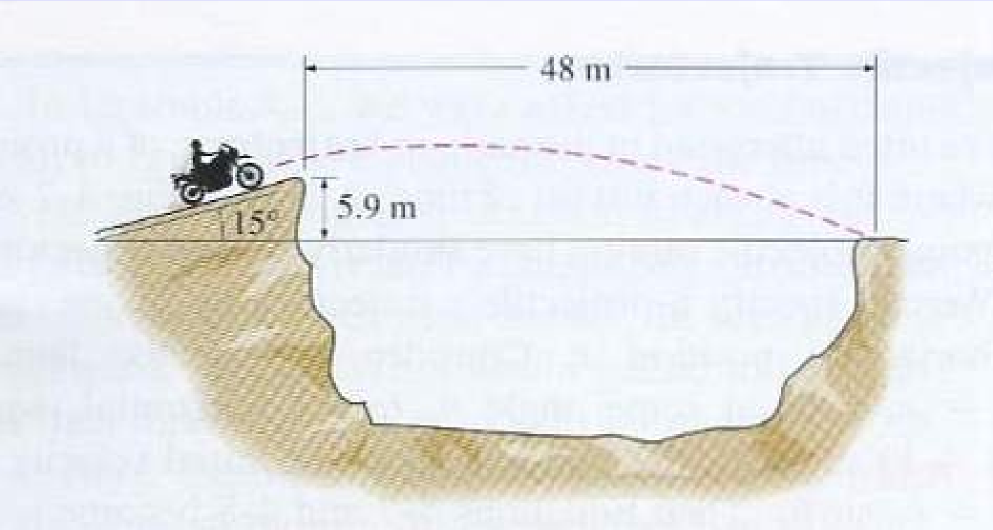
Then substitute in for t in position equation for y

$$y = V_0 \sin(\theta)\left(\frac{x}{V_0 \cos(\theta)}\right) - \frac{1}{2} g \left(\frac{x}{V_0 \cos(\theta)}\right)^2$$

$$y = \tan(\theta) x - \frac{g}{2V_0^2 \cos^2(\theta)} x^2$$

Trajectory is always parabolic in x

Example: Daredevil Motorcyclist



(a) What is the minimum speed for the cyclist to make the jump?

This problem can be easily solved by making use of the parabolic trajectory equation.

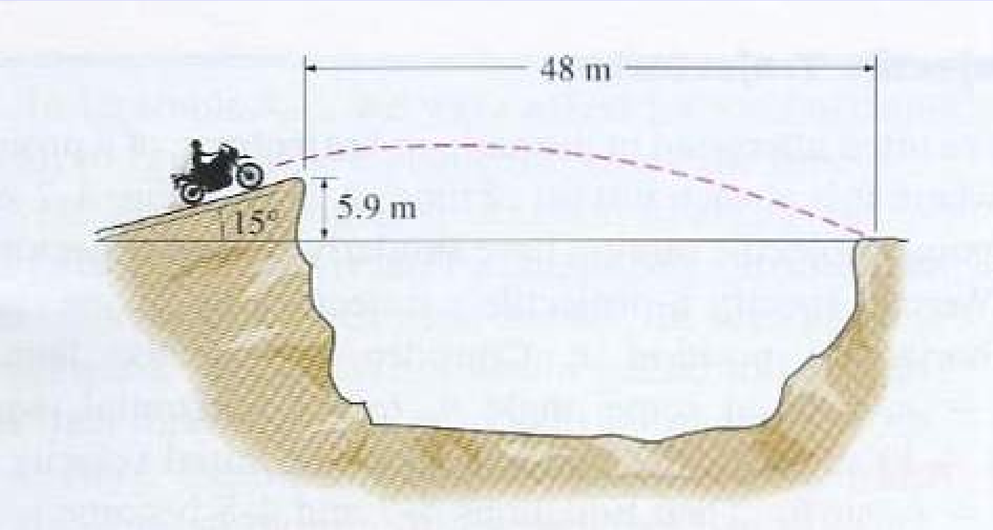
$$y = \tan(\theta)x - \frac{g}{2V_0^2\cos^2(\theta)}x^2 \quad \text{with} \quad y = -5.9m,\ x = 48m,\ \theta = 15^\circ$$

Solving for the initial velocity we find

$$2V_0^2\cos^2(\theta) = \frac{gx^2}{\tan(\theta)x - y} \rightarrow V_0 = \sqrt{\frac{1}{2\cos^2(\theta)}\frac{gx^2}{\tan(\theta)x - y}}$$

$$V_0 = \frac{x}{\cos(\theta)}\sqrt{\frac{g/2}{\tan(\theta)x - y}} = 25.4m/s = 91km/h$$

Example: Daredevil Motorcyclist



(b) If he exceeds this velocity by 50% how far from the rim, *d*, will he land?

Again, this problem is easily solved by making use of the parabolic trajectory equation.

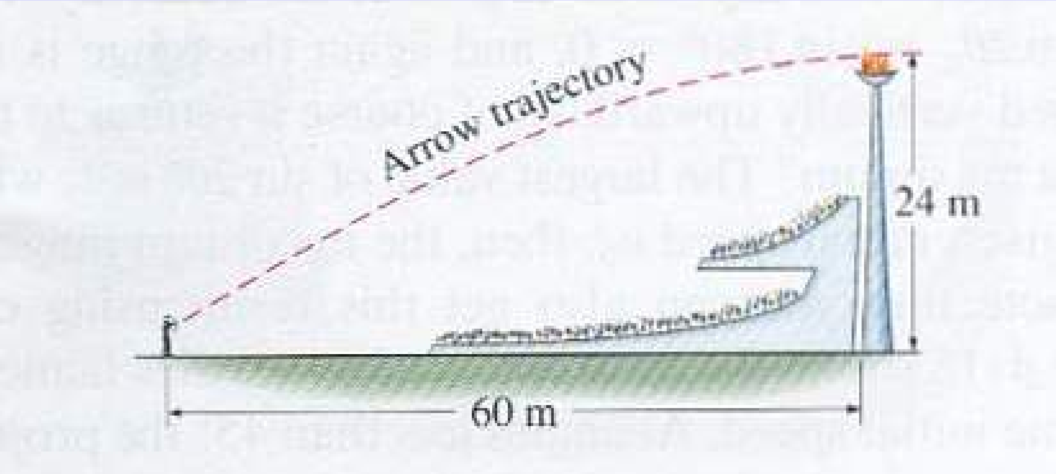
$$y = \tan(\theta)x - \frac{g}{2V_0^2\cos^2(\theta)}x^2 \quad \text{with} \quad y = -5.9m,\ x = 48m,\ \theta = 15^\circ$$

The resulting quadratic equation for *x* with $V_0 = 38.1m/s$ is

$$y - \tan(\theta)x + \frac{g}{2V_0^2\cos^2(\theta)}x^2 = 0 \rightarrow -5.9 - .268x + .00364x^2 = 0$$

Since $x = 91.4m$ the distance *d* is: $d = x - 48 = 91.4 - 48 = 43.4m$

Example: Olympic Flame



In the ’92 Olympics the Olympic Flame was lit by a flaming arrow. Given the geometry shown, find the initial velocity to reach flame at the peak of the trajectory.

The peak of the trajectory equation occurs when *dy/dx* = 0, or:

$$y = \tan(\theta)x - \frac{1}{2}\frac{g}{V_0^2\cos^2\theta}x^2$$

$$\left.\frac{dy}{dx}\right|_{x=\ell} = \tan\theta - \frac{g\ell}{V_0^2\cos^2\theta} = 0$$

$$V_0^2\cos^2\theta = \frac{g\ell}{\tan\theta} \rightarrow V_0^2 = \frac{g\ell}{\sin\theta\cos\theta}$$

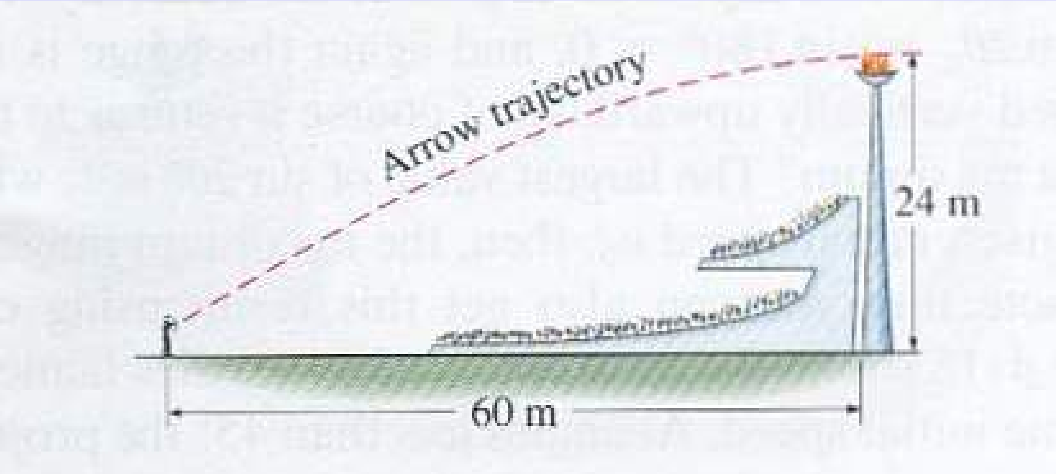
Finding the height when *dy/dx* = 0:

$$y = \tan(\theta)x - \frac{1}{2}\frac{g}{V_0^2\cos^2\theta}x^2$$

$$y(\ell) = h = \tan(\theta)\ell - \frac{1}{2}\frac{g\ell^2}{g\ell/\tan\theta}$$

$$h = \frac{1}{2}\tan(\theta)\ell$$

Example: Olympic Flame



In the ’92 Olympics the Olympic Flame was lit by a flaming arrow. Given the geometry shown, find the initial velocity to reach flame at the peak of the trajectory.

Since we know both h and ℓ the initial angle of the trajectory is:

$$\theta = \tan^{-1}(2h/\ell) = \tan^{-1}(2h/\ell) = \tan^{-1}.8 = .675 rad = 38.7^\circ$$

Knowing the initial angle of the trajectory the initial speed is:

$$V_0 = \sqrt{2g\ell/\sin 2\theta} = 34.7 m/s$$

The initial velocity is:

$$\vec{V}_0 = V_0 \cos\theta \hat{i} + V_0 \sin\theta \hat{j}$$

$$\vec{V}_0 = 27.1 \hat{i} + 21.7 \hat{j} \, m/s$$

Example: Ballistic Cart

A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

Where will it land?

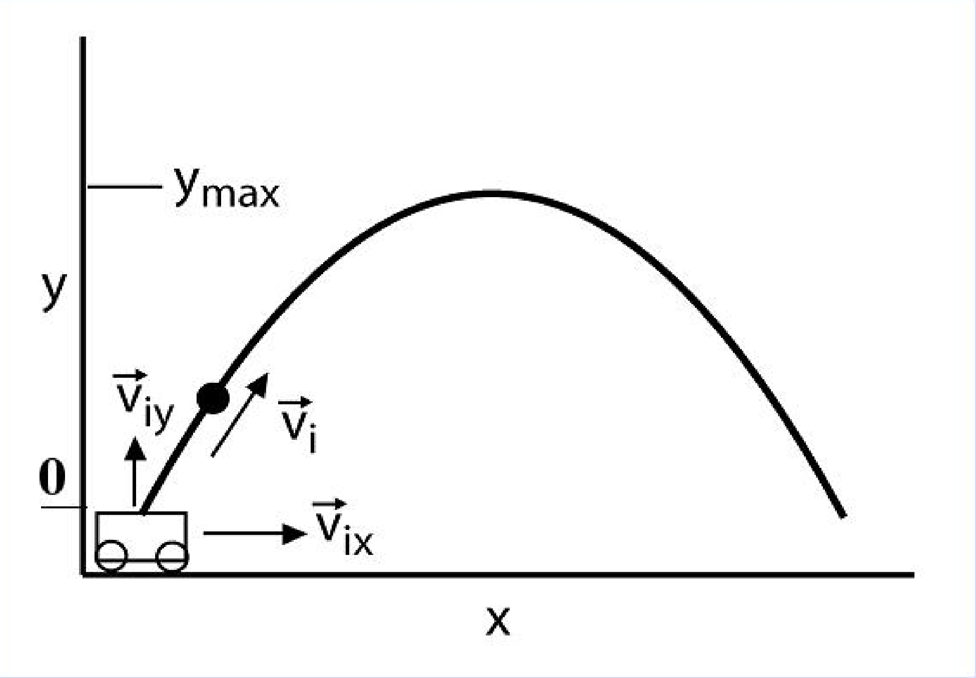
The projectile has reached its range at time t_f when y=0

$$y_f = V_{iy} t_f - \frac{1}{2} g t_f^2 = 0$$

$$t_f = \frac{2V_{iy}}{g}$$

At this time the x location is

$$x_f = V_{ix} \frac{2V_{iy}}{g}$$



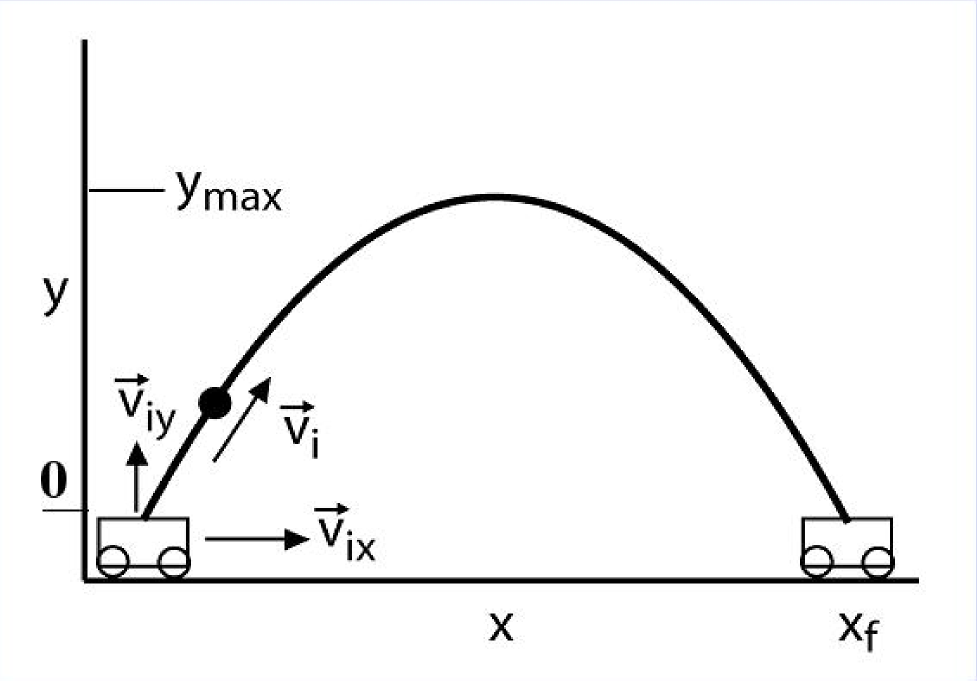
Example: Ballistic Cart

A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

Where will it land?

The cart will have also had the same velocity in the x direction and the same x position at all times

$$x_{f-cart} = V_{ix}\frac{2V_{iy}}{g}$$



The ball will land back in the cart!

In fact the ball is always directly above the cart!

Example: Ballistic Cart

A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

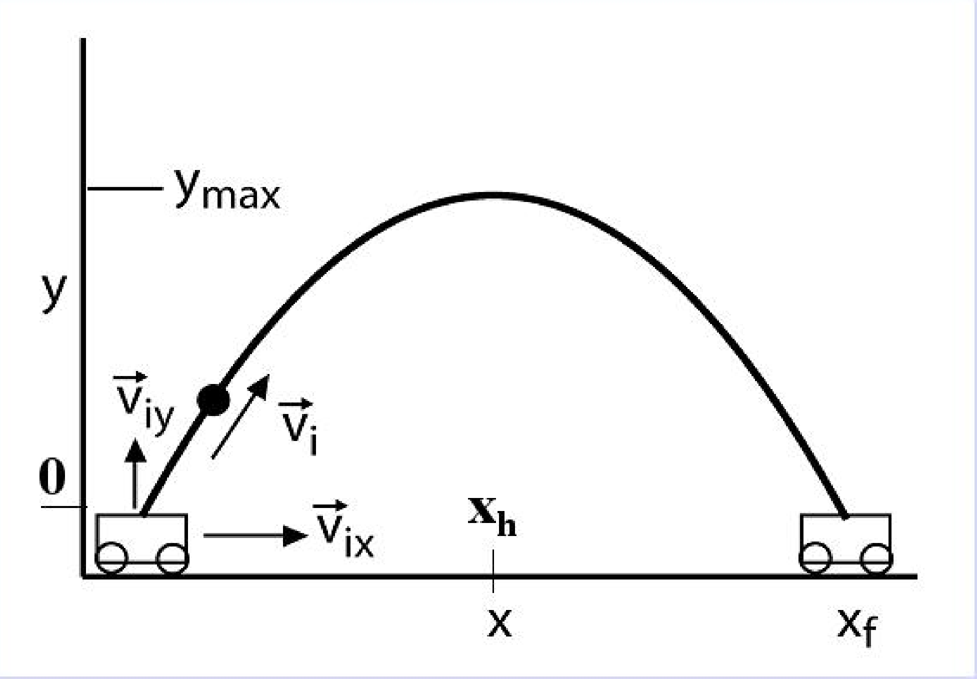
What is $y_{max}(t_f)$

The height of the trajectory is defined by v_y=0 when t=t_h

$$V_y = V_{iy} - g t_h = 0$$

At this time the x location is

$$x_h = V_{ix} \frac{V_{iy}}{g}$$



Example: Ballistic Cart

A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

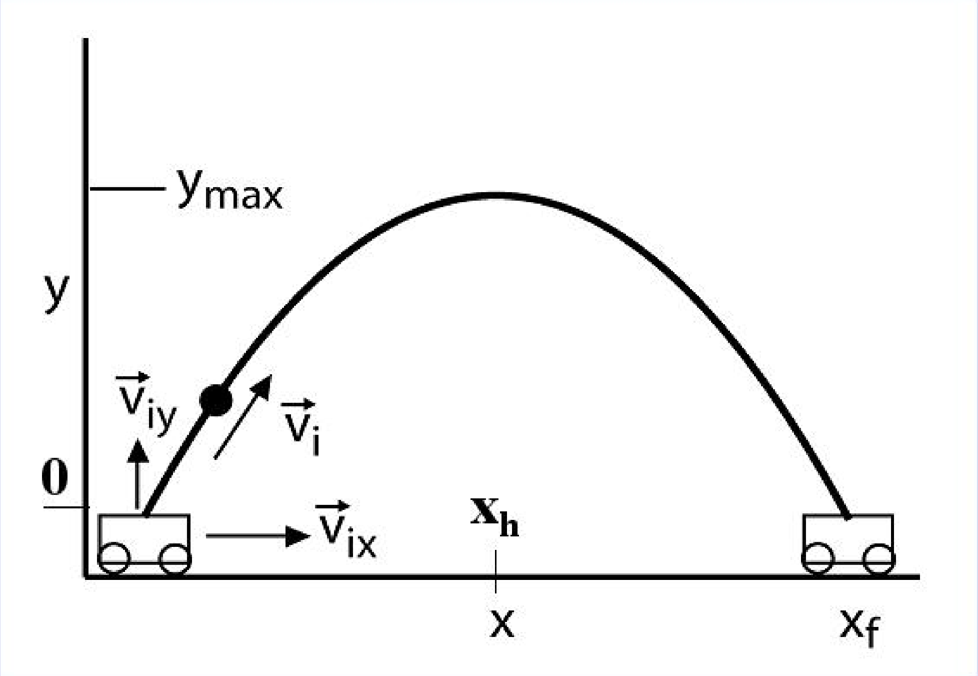
What is $y_{max}(t_f)$

So, the time to the top is

$$t_f = \frac{2v_{iy}}{g} = 2t_h$$

$$t_h = \frac{t_f}{2}$$

This occurs exactly midway through the flight



Example: Ballistic Cart

A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

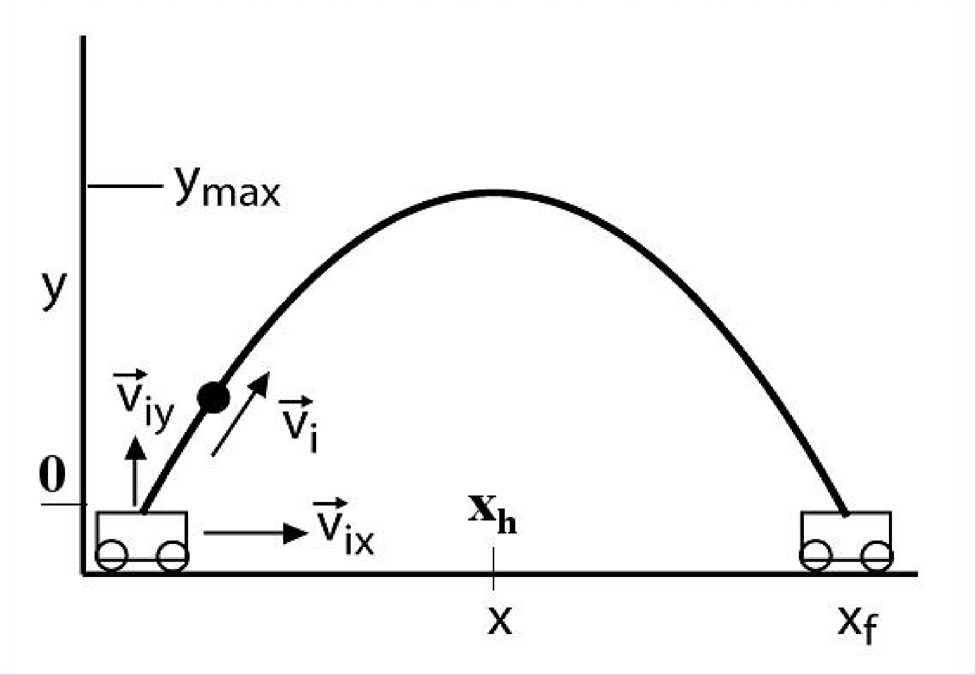
What is $y_{max}(t_f)$

Substituting in for t_h in the equation for y_{max}:

$$y_{max} = V_{iy} t_h - \frac{1}{2} g t_h^2$$

$$y_{max} = V_{iy} \frac{t_f}{2} - \frac{g t_f^2}{8}$$

We want a function of time of flight, so substitute in for v_{iy}.



Example: Ballistic Cart

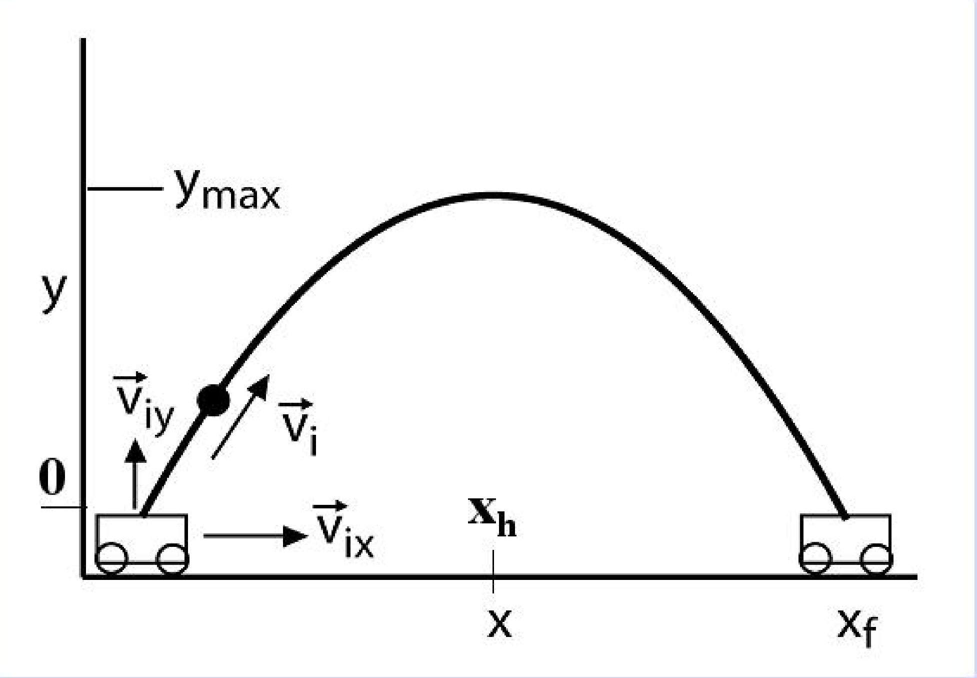
A cart is traveling at constant speed v_i. If a ball is shot straight up from the cart, where will it land, what is the maximum height the ball traveled as a function of time of flight?

What is $y_{max}(t_f)$

We only want a function of time of flight, so substitute in for viy.

$$V_{iy} = gt_h = \frac{gt_f}{2}$$

$$y_{max} = \frac{gt_f^2}{4} - \frac{gt_f^2}{8} = \boxed{\frac{gt_f^2}{8}}$$



Test this for a measured time t_f: does it make sense?

For 2s trajectory y_{max}=4.9m=16ft

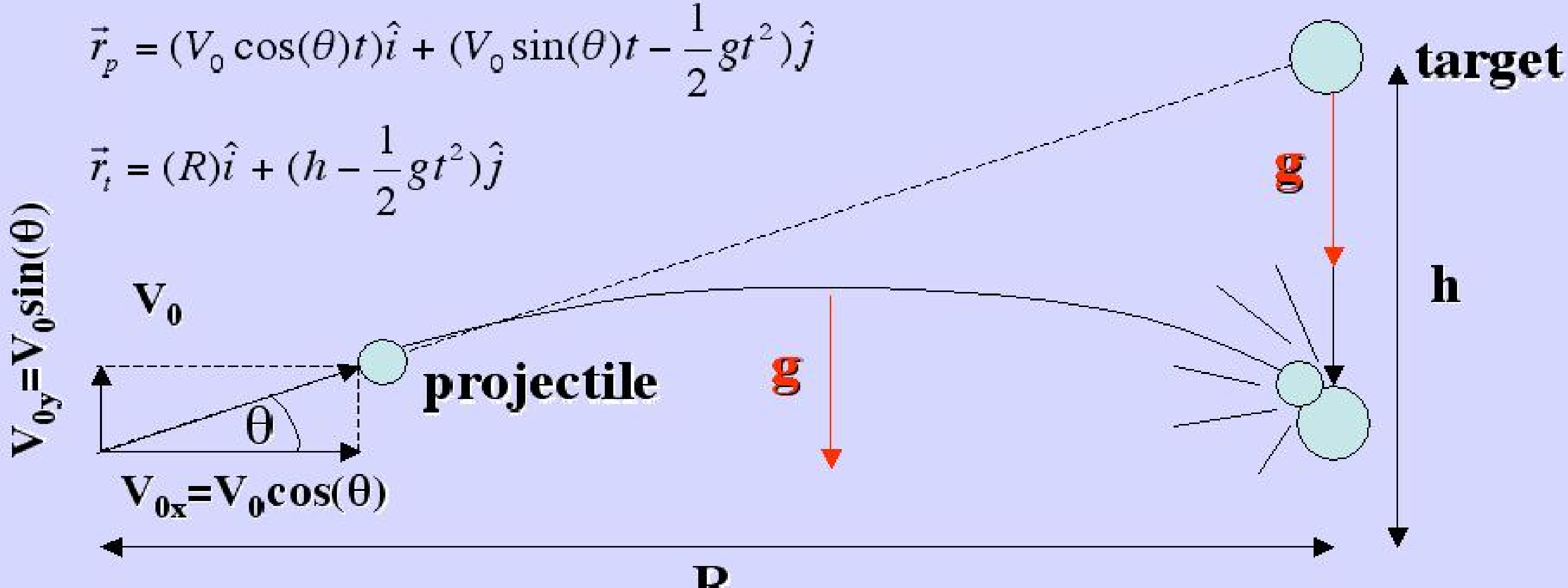
Example: Shoot the Monkey!

We first construct the position vectors

where $h = R\tan(\theta)$

$$\vec{r}_p = (V_0\cos(\theta)t)\hat{i} + (V_0\sin(\theta)t - \frac{1}{2}gt^2)\hat{j}$$

$$\vec{r}_t = (R)\hat{i} + (h - \frac{1}{2}gt^2)\hat{j}$$



When projectile reaches R,

$$R = V_0\cos(\theta)t$$

$$\boxed{t = \frac{R}{V_0\cos(\theta)}}$$

At what time are the y positions equal?

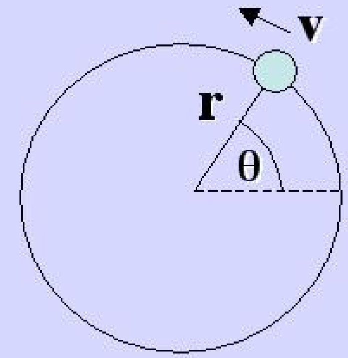
$$h - \frac{1}{2}gt^2 = V_{0y}t - \frac{1}{2}gt^2$$

$$t = \frac{h}{V_{0y}} = \frac{R\tan(\theta)}{V_0\sin(\theta)} = \boxed{\frac{R}{V_0\cos(\theta)}}$$

The Same Time! They Collide!

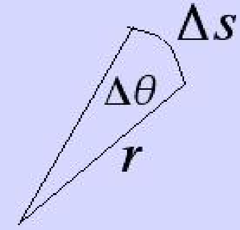
Circular Motion 1

Uniform Circular Motion



A body moving in a circle of radius r with uniform speed v

In a time interval Δt **the arc length traversed is**



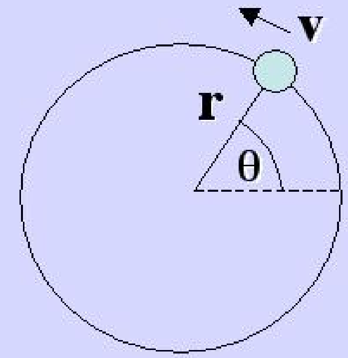
$\Delta s = r\Delta\theta$ $\qquad$ θ **in radians** $0 \rightarrow 2\pi$

The limit as that time interval goes to zero is v

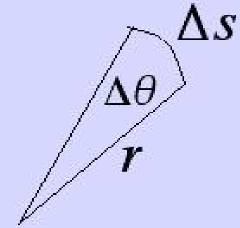
$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = r\frac{d\theta}{dt}$$

$\frac{d\theta}{dt}$ **is called the angular velocity** ω **where** $v = r\omega$

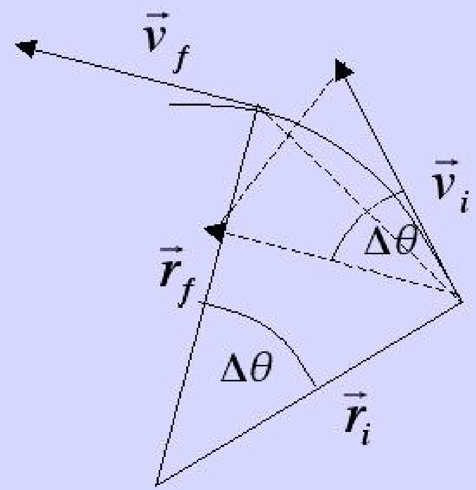
Angular Acceleration



The direction of v is always tangent to the circle. So, in time Δt the vector v changes direction by the angle between $\mathbf{v}_i$ and $\mathbf{v}_f$.



The angle made by $\mathbf{r}_i$ and $\mathbf{r}_f$ is equal to the angle made by $\mathbf{v}_i$ and $\mathbf{v}_f$ because the two triangles defined by the vectors are formally similar.



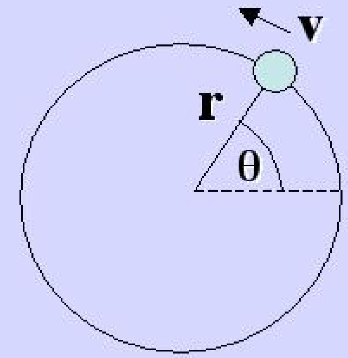
Thus the ratio of corresponding sides are equal:

$$\frac{\Delta \vec{v}}{\left|v_i\right|} = \frac{\Delta s}{r}$$

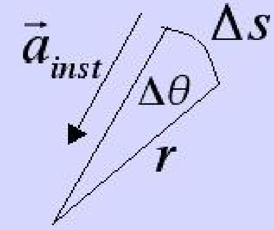
Giving an Average Angular Acceleration:

$$a_{ave} = \frac{\Delta v}{\Delta t} = \frac{\left|v_i\right|}{r}\frac{\Delta s}{\Delta t}$$

Instantaneous Acceleration



In the same way as the limit as the time interval vanished defined the linear velocity and acceleration, and the angular velocity, the instantaneous acceleration in circular motion is defined by this limit.

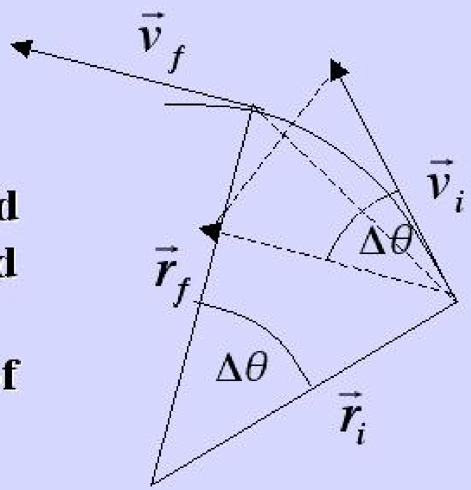


$$a_{ave} = \frac{|v_i|}{r}\frac{\Delta s}{\Delta t}$$

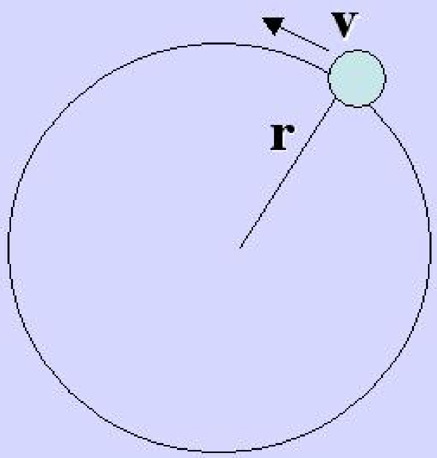
$$a_{inst} = \lim_{\Delta t \to 0} a_{ave}$$

$$a_{inst} = \frac{|v_i|}{r}\lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \boxed{\frac{v^2}{r}}$$

a_{inst} is always pointed along the radius, and points toward the center of the circle of motion.



Example: Circular Motion



A motorized rotator with radius r has a frequency of rotation of N times in 10s. What are the angular velocity and instantaneous acceleration?

First, find the **angular velocity** from the period of rotation and the circumference of the orbit:

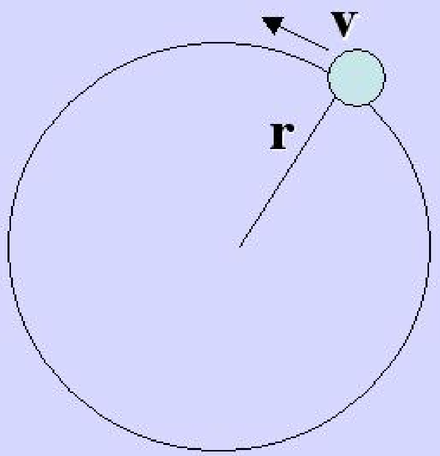
radians/s

$$v = r\omega = r\left(\frac{2\pi}{\tau_p}\right) \quad \text{where} \quad \tau_p = \frac{10}{N} s$$

Is the period of one orbit.

Example: Circular Motion

A motorized rotator with radius r has a frequency of rotation of N times in 10s. What are the angular velocity and instantaneous acceleration?



Then, the **instantaneous acceleration** is

$$a_{inst} = \frac{v^2}{r} = \frac{\left(\frac{2\pi r}{\tau_p}\right)^2}{r}$$

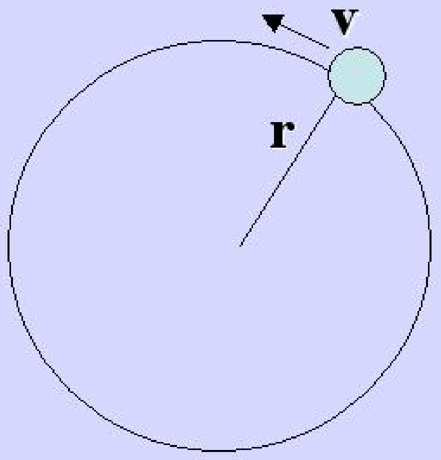
$$\boxed{a_{inst} = \frac{4\pi^2 r}{\tau_p^2}}$$

where $\tau_p = \frac{10}{N} s$

Does this make sense?
NEXT TIME

Example: Circular Motion

A motorized rotator with radius r has a frequency of rotation of N times in 10s. What are the angular velocity and instantaneous acceleration?



In class we calculated 18 turns in 10s for the rotator with a radius r~0.2m.

Therefore $$\tau_p = \frac{10}{18}s = \boxed{0.556s}$$

$$v = r\omega = r(\frac{2\pi}{\tau_p}) = 0.2m(\frac{2\pi}{0.556s}) = \boxed{2.26m/s}$$

$$a_{inst} = \frac{4\pi^2 r}{\tau_p^2} = \frac{4\pi^2(0.2m)}{(0.556s)^2} = \boxed{25.5m/s^2}$$