Chapter 15

Oscillatory Motion III

Chapter 9

Gravitation

S H O - Velocity and Acceleration

It is useful to consider the velocity and acceleration as it relates to the displacement. For this we will use the solution that includes the phase.

$$x(t) = A\cos(\omega t + \phi)$$

The velocity is the first derivative:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

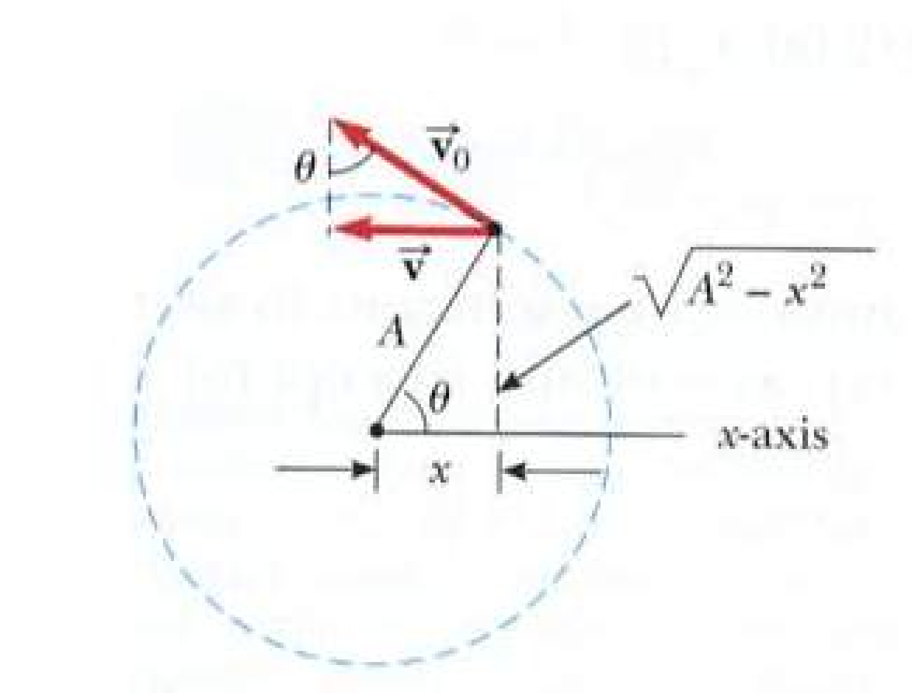
From this we see that the velocity is out of phase with the displacement. When the displacement is maximum, the velocity is zero. Similarly when the velocity is maximum the displacement is zero.

The acceleration is the second derivative:

$$a(t) = \frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

The acceleration **always** has the opposite sign of the displacement, i.e. the object is under the influence of a restoring force!

Uniform Circular Motion and SHO

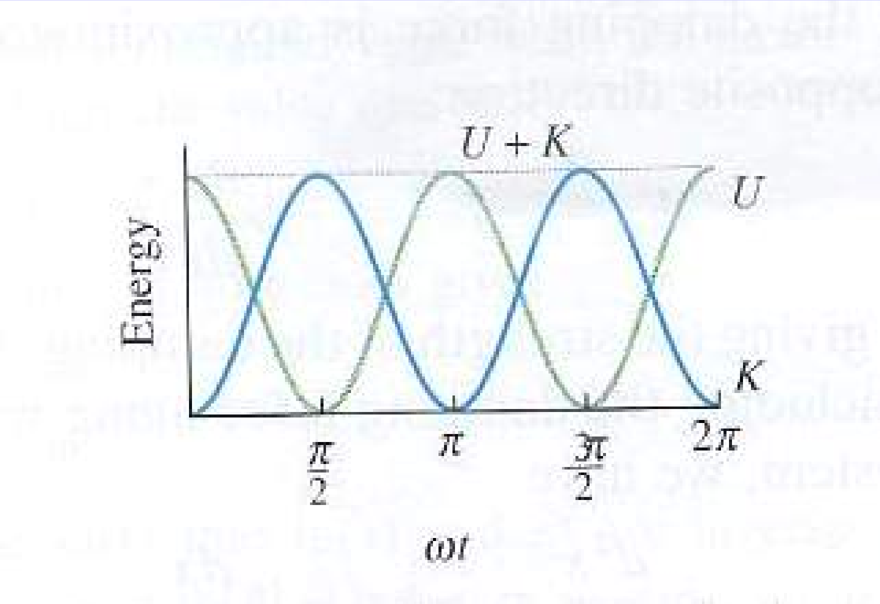


We can think of SHO as the ***x*** component of an object undergoing circular motion with a uniform angular velocity ***ω***. In the figure **$\theta = \omega(t)$** and **$x = A \cos\theta$**.

The tangential velocity is ***ωA*** and the ***x*** component of this velocity is proportional to **$\sin\theta$**. Also from the figure we see that the ***x*** component of the velocity is pointing in negative direction when **$\sin\theta$** is positive. Hence, **$v = -\omega A \sin\theta$**.

This should help you to understand why we used ***ωt*** as the argument for the solution to the displacement of an object under a linear restoring force.

Energy in Simple Harmonic Oscillations



For the mass spring system the potential energy is $U(x) = \frac{1}{2} k x^2$, where x is the displacement from equilibrium.

The kinetic energy is $K = \frac{1}{2} m v^2$.

Assuming that $x = A \cos(\omega t)$ we find:

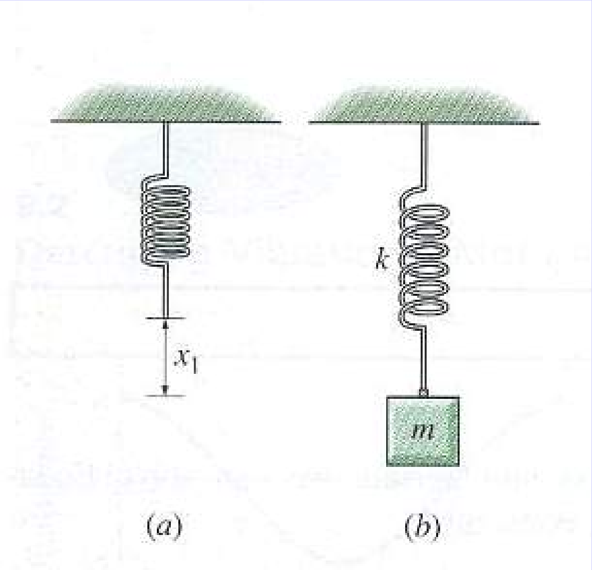
$$U = \frac{1}{2} k A^2 \cos^2 \omega t$$
$$K = \frac{1}{2} m \omega^2 A^2 \sin^2 \omega t$$

However $\omega^2 = k/m$. Hence the total energy is:

$$E = U + K = \frac{1}{2} k A^2 \cos^2 \omega t + \frac{1}{2} k A^2 \sin^2 \omega t = \frac{1}{2} k A^2$$

The kinetic and potential energy are out of phase so that when one is a minimum the other is a maximum and vice versa. Their total is a constant!

Example: Vertical Mass Spring



There are now two forces acting on the mass ***m***. The force of gravity and that due to the spring. The resulting differential equation from Newton's 2nd is:

$$mg - kx = m\frac{d^2x}{dt^2} \rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = g$$

The solution $\boldsymbol{x_h = A\cos(\omega t+\phi)}$ satisfies:

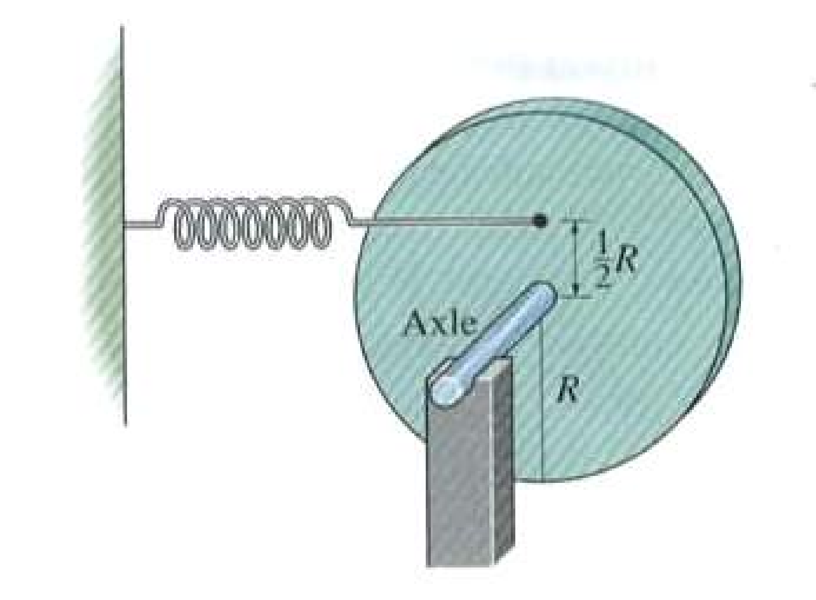
$$\frac{d^2x_h}{dt^2} + \frac{k}{m}x_h = 0, \quad \omega^2 = k/m$$

Simply adding $\boldsymbol{x_1 = mg/k}$ to the *homogeneous solution* yields the full solution:

$$x = x_h + x_1 = A\cos(\omega t + \phi) + mg/k$$

The mass continues to oscillate at the same frequency as before. It simply oscillates about a new equilibrium position, $\boldsymbol{x_1 = mg/k}$.

Example: Spring and Rotating Disk



A uniform disk of mass ***M*** and radius ***R*** is mounted on a horizontal axle. A horizontal spring of spring constant ***k*** at equilibrium is connected to the disk at a distance ***R/2*** above the axle. What is the angular frequency for small amplitude oscillations?

If the disk is rotated through an angle ***θ*,** the torque and angular acceleration are related:

$$\tau = -kx\frac{R\cos\theta}{2} = -k\frac{R\cos\theta}{2}\frac{R\sin\theta}{2} = I\alpha$$

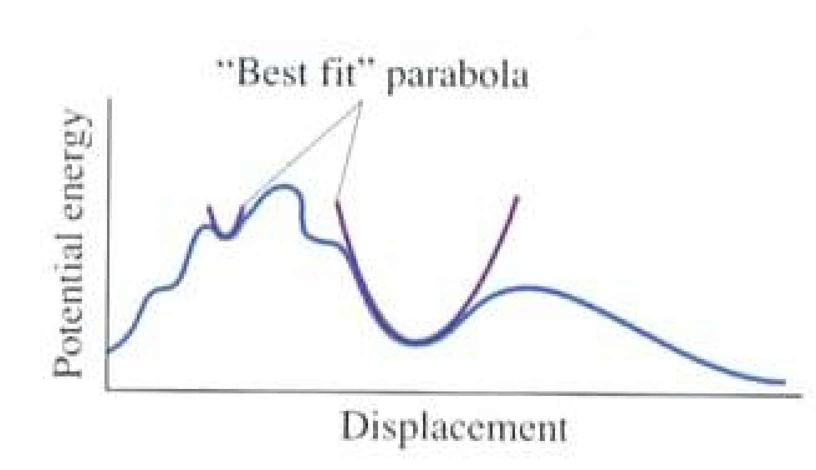
The moment of inertia is ***I* = ½ *MR*²**. Hence the EOM for small oscillations is:

$$\frac{d^2\theta}{dt^2} + \frac{k}{2M}\theta = 0$$

Hence the angular frequency is found from:

$$\omega^2 = k/2M$$

Oscillatory Motion and Potential Energy Functions



For small displacements from a position of stable equilibrium, $\boldsymbol{x_o}$, the potential can usually be approximated by an upright parabola. Consider a Taylor series about $\boldsymbol{x_o}$:

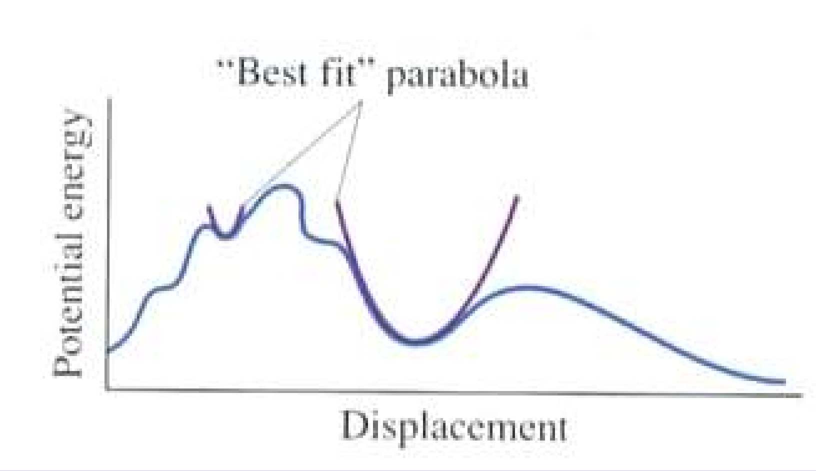
$$U(x) \simeq U(x_o) + \frac{dU(x_o)}{dx}(x - x_o) + \frac{1}{2}\frac{d^2U(x_o)}{dx^2}(x - x_o)^2$$

At the minimum the first derivative of the potential vanishes. Any potential energy has an arbitrary constant as it is the change in potential energy that is important. As long as the displacement from equilibrium is small we have:

$$\Delta U(x) \simeq \frac{1}{2}\frac{d^2U(x_o)}{dx^2}(x - x_o)^2 \text{ and } F = -\frac{dU}{dx} = -\frac{d^2U(x_o)}{dx^2}(x - x_o)$$

Oscillatory Motion and Potential Energy Functions

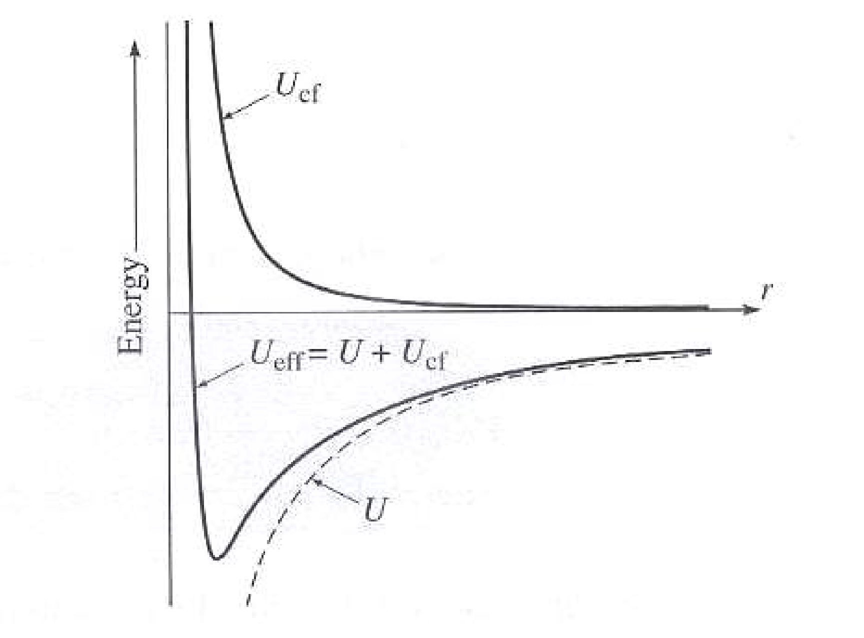
For small displacements from a position of stable equilibrium, $\boldsymbol{x_o}$, the potential can usually be approximated by an upright parabola.



$$F = -\frac{dU}{dx} = -\frac{d^2U(x_o)}{dx^2}(x - x_o) = -k(x - x_o) \text{ with } k = \frac{d^2U(x_o)}{dx^2}$$

This is Hook's law all over again with an effective spring constant being given by the second derivative of the potential at the point of equilibrium. So SHO is a very general phenomena. As it turns out it is also very general phenomena even in quantum mechanics. Only when the second derivative vanishes or when the displacements from equilibrium are large is this not a good approximation!

Effective Gravitational Potential and Oscillatory Motion



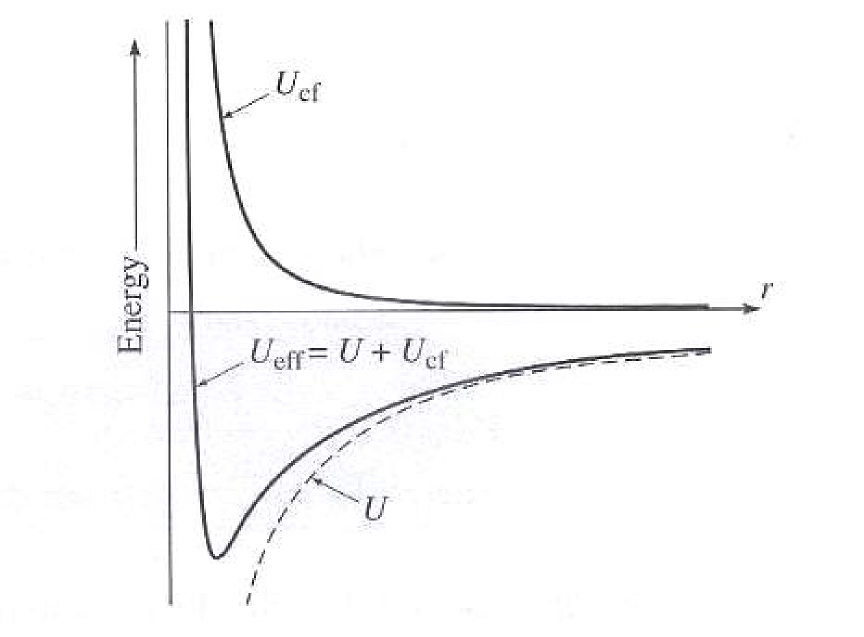
The effective gravitational potential for an planet or comet orbiting a star (or moon orbiting a planet) is a sum of (i) a term due to the centrifugal force term and (ii) a term due to the gravitational potential itself.

$$U_{eff}(r) = -\frac{GMm}{r} + \frac{1}{2}\frac{L^2}{mr^2}$$

A plot of this effective potential is shown in the figure. ***L*** is the angular momentum of the orbiting object.

A object in a circular orbit has a fixed radius. This radius occurs at the minimum in the potential. Since its radius will not change at this point, the object has no kinetic energy related to a changing distance from the star, ***dr/dt***. How about orbits that are slightly displaced from this radius?

Effective Gravitational Potential and Oscillatory Motion



The effective gravitational potential for an orbiting object is:

$$U_{eff}(r) = -\frac{GMm}{r} + \frac{1}{2}\frac{L^2}{mr^2}$$

First we have to determine $\boldsymbol{r_o}$.

$$\frac{dU(r_o)}{dr} = \frac{GMm}{r_o^2} - \frac{L^2}{mr_o^3} = 0$$

$$\frac{GMm}{r_o^2} = \frac{m^2 r_o^4 \varpi^2}{mr_o^3} \rightarrow r_o^3 = \frac{GM}{\varpi^2}$$

Here ϖ is the angular velocity of the orbiting object. The effective spring constant is:

$$\frac{d^2U(r_o)}{dr^2} = -2\frac{GMm}{r_o^3} + 3\frac{L^2}{mr_o^4} = m\varpi^2 = k \rightarrow \boxed{\omega^2 = k/m = \varpi^2}$$

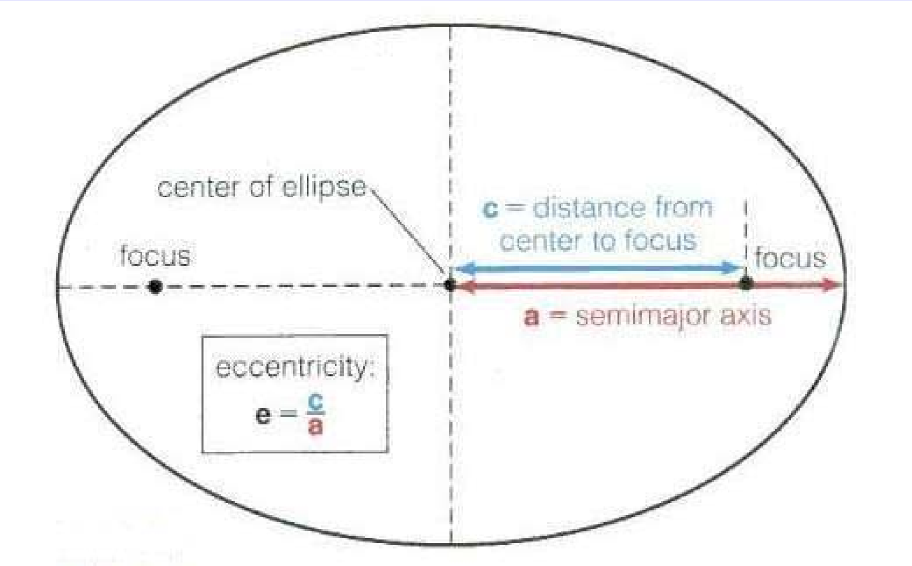
If the angular velocity equals the angular oscillating frequency, **closed orbits!**

Chapter 9

Gravitation

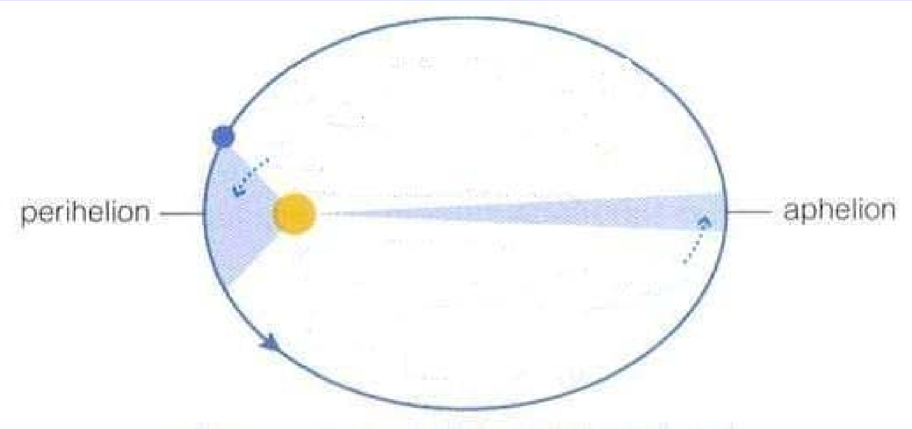
Kepler's Laws of Planetary Motion

Kepler's three laws make no effort to explain the planetary motion. Instead, they are mathematical descriptions of the planet's motion.



1. The planets orbit the Sun in ellipses with the Sun at one focus.

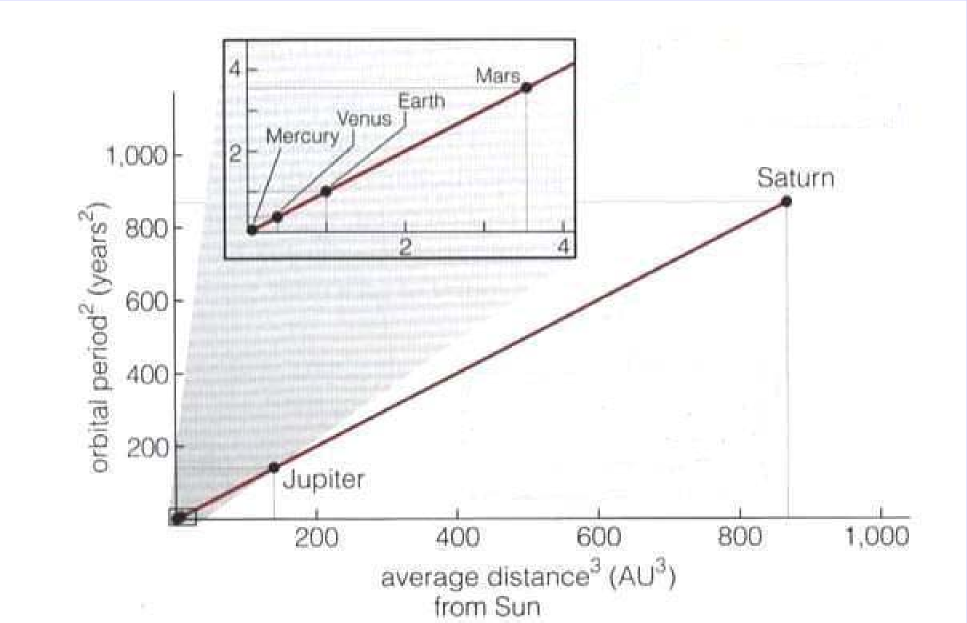
Discuss Keper's breakthrough with Brahe's data. "Those 8 arcminutes of Mars orbit could not be ignored."



2. A line joining the Sun and a planet sweeps out equal areas in equal times.

This is a straightforward result of the conservation of angular momentum.

Kepler's Laws of Planetary Motion



3. The square of the planet's orbital period is proportional to the cube of the semimajor axis of its orbit.

In units of Earth years and Astronomical Units, the average distance from the Earth to the Sun, this law is expressed as $\mathbf{T^2 = \mathit{a}^3}$.

This final observation occurred several years after the first two.

It was Newton's prediction of these observations using his law of gravity that resulted in a basic understanding of orbital motion and (weak) gravity in general. In fact Kepler's third law (in SI units) is a straightforward extension of our knowledge of the angular velocity of an orbiting object.

$$\varpi^2 = \frac{4\pi^2}{T^2} = \frac{GM}{r_o^3} \rightarrow T^2 = \frac{4\pi^2}{GM} r_o^3$$

Newton's Law of Gravity

Newton realized that the motion of the falling apple and the motion of the moon around the Earth were due to the same force. They were both falling toward the Earth due to the force of gravity.

$$F_g = -\frac{GMm}{r^2}$$ Universal Gravitation

This force obeys the inverse square law. Also the minus sign indicates that this force is attractive.

G is the universal constant of gravitational attraction and is given by

$$G = 6.673 \times 10^{-11} Nm^2/kg^2$$

Strictly speaking it only applies to point objects. However, for spherically symmetrical objects ***r*** is the distance between their centers. As long as the size of the object is small compared to ***r***, then it is simply the distance between them.

Acceleration Due to Gravity on the Earth

The force of gravity everyone feels on the surface of the Earth is given by Newton's law where $\boldsymbol{r = R_E}$. Given that the mass of the Earth is $\boldsymbol{M_E = 5.974 \mathrm{x} 10^{24} kg}$ and the radius of the Earth is $\boldsymbol{R_E = 6.378 \mathrm{x} 10^{6} m}$ the force divided its mass of an object on the surface of the Earth is:

$$F/m = -\frac{6.673 \times 10^{-11} \times 5.974 \times 10^{24}}{(6.378 \times 10^{6})^{2}} = -9.7998 m/\sec^{2}$$

The first thing to note is that this acceleration due to the pull of gravity from the Earth is **INDEPENDENT** of the mass of the object. Next, it is not appropriate to use 5 significant figures in the result as we only used 4 significant figures in all of the quantities used to find this acceleration. Rounding off this result we find $\boldsymbol{g = 9.800 m/s^2}$.

Gravity is Weak!

When calculating the electron energy levels in the hydrogen atom, the gravitational attraction between the proton and the electron is neglected. It is of interest to compare the difference between the electrical force and the gravitational force at the same separation. The force between two static charges is:

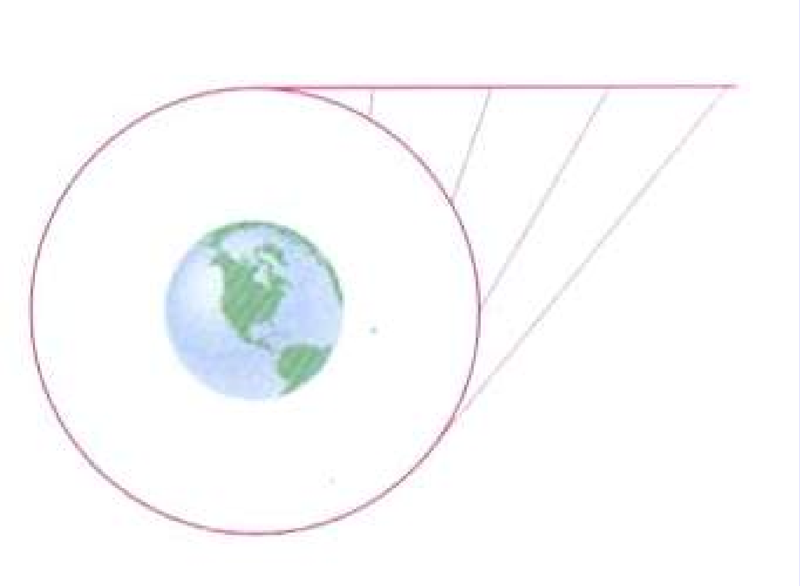
$$F_C = \frac{q_1 q_2}{4\pi\epsilon_o r^2} \rightarrow \text{Coulomb's Law}$$

This law also satisfies an inverse square law. Hence the ratio of the electrical attraction and the gravitational attraction will be independent of the distance between the two particles. Using the masses of the proton and electron, the electron charge, and the Coulomb interaction constant we find:

$$F_g/F_C = \frac{G m_p m_e}{q_p q_e/(4\pi\epsilon_o)} = 4.4 \times 10^{-40}!$$

Gravity may be MUCH weaker than the electrical force, but it is the dominant long range force in the universe. It is ALWAYS attractive while electrical charges appear to occur in equal numbers, hence they balance each other out over long ranges.

Orbital Motion



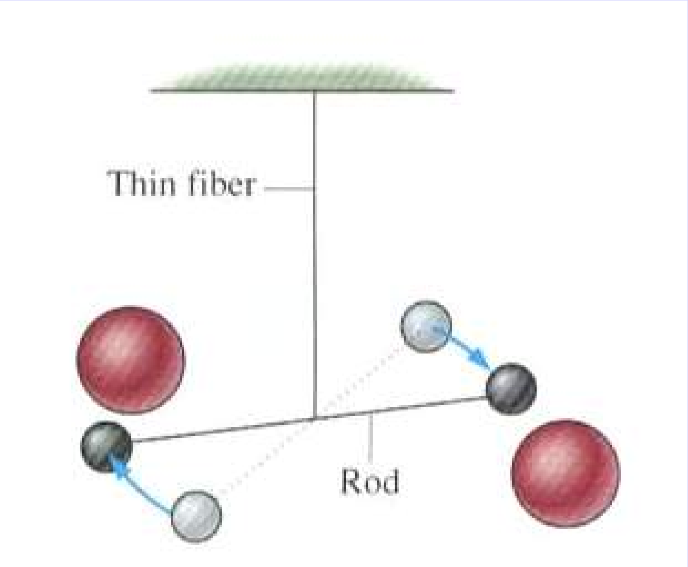
An object orbiting the Earth (or any other object orbiting a large massive object) is accelerating toward the center of the Earth. The blue lines indicate the path of an object in the absence of gravity. From our study of circular motion we know that gravity must provide the force for radial acceleration. This leads to the period for a circular orbit:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} = m\omega^2 r \rightarrow \omega^2 = \frac{GM}{r^3}$$

$$\frac{4\pi^2}{T^2} = \frac{GM}{r^3} \rightarrow T^2 = \frac{4\pi^2}{GM} r^3$$

We have proved Kepler's 3rd law for circular orbits. Note that this expression is independent of the object's **mass**. This law is the primary way astronomers measure the product ***GM*** of objects throughout our galaxy.

Cavendish Experiment

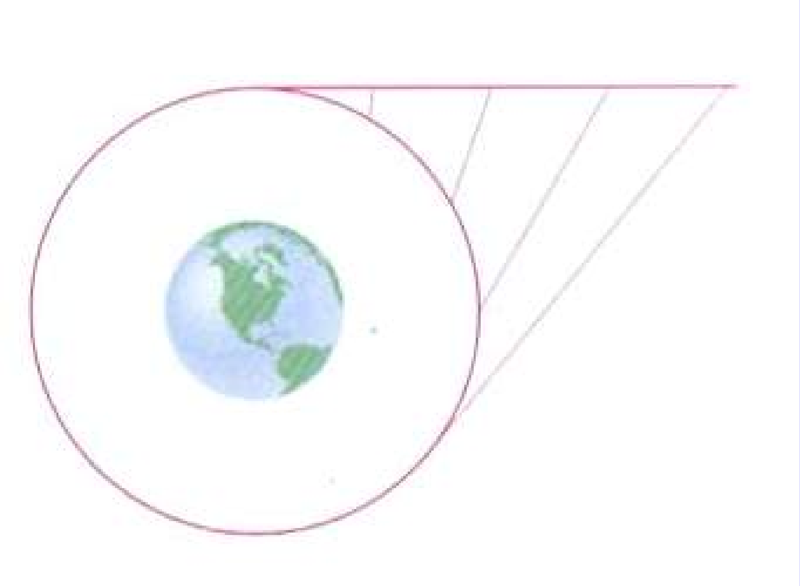


Astronomers measure the product ***GM*** for orbiting objects, but what about ***G*** itself? It is extremely weak but Cavendish found a way. He suspended two ***5cm*** diameter lead spheres connected by a thin rod on the end of a thin fiber. He then brought two ***30cm*** lead spheres close to the suspended lead spheres and measured the small rotation of the rod. Knowing the torsional property, κ, of the fiber, he was able to calculate ***G***.

Actually Cavendish's purpose was to measure the mass of the Earth. Knowing the gravitational acceleration at the surface of the Earth as well as the rotational period of the moon, he knew the product ***GM***. His final result for ***G*** in SI units was $G = 6.74 \times 10^{-11} Nm^2/kg^2$.

It is the product ***GM*** that determines the dynamical properties of objects in a gravitational field. Astronomers routinely measure this product to ***5*** parts in 10^8 while ***G*** itself is only known to ***5*** parts in 10^4.

Example: Geosynchronous Orbit



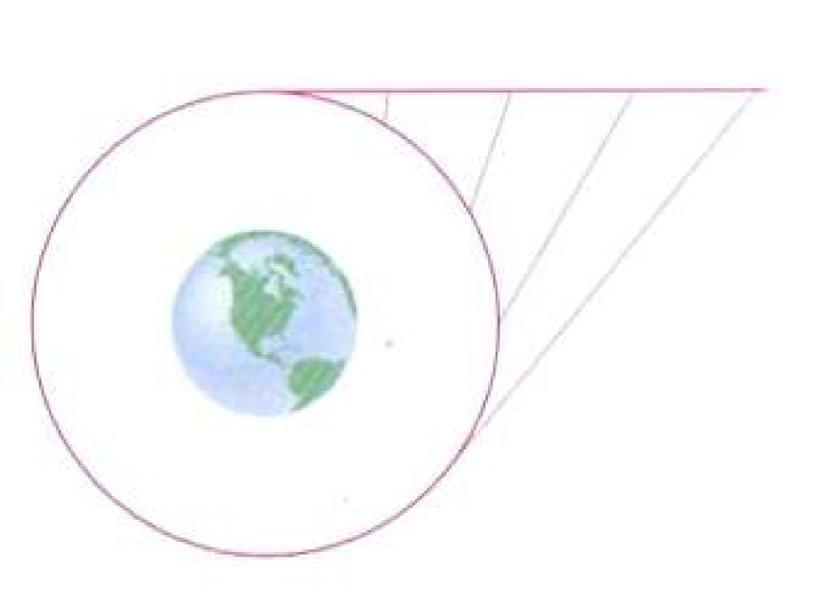
Geosynchronous orbits are important for communications satellites. For example, Direct Dish TV. We are now in a position to determine the altitude for a geosynchronous orbit. The period must be one day! From Kepler's 3rd law:

$$r^3 = \frac{GM_E T^2}{4\pi^2} = 7.538 \times 10^{22} m^3$$

$$r = 4.224 \times 10^7 m$$

However this quantity is the distance from the center of the Earth, not its altitude. The altitude is $\boldsymbol{r - R_E = 3.59 \times 10^7 m}$.

Orbital Motion



From our study of circular motion we know that gravity must provide the force for radial acceleration. This leads to the orbital velocity for a circular orbit:

$$\frac{GMm}{r^2} = \frac{mv_{orb}^2}{r} \rightarrow v_{orb}^2 = \frac{GM}{r}$$

Since the orbital velocity is independent of the object's mass, all objects in orbit are traveling at the same speed. For example, an astronaut on a space walk is not left behind by the space shuttle.