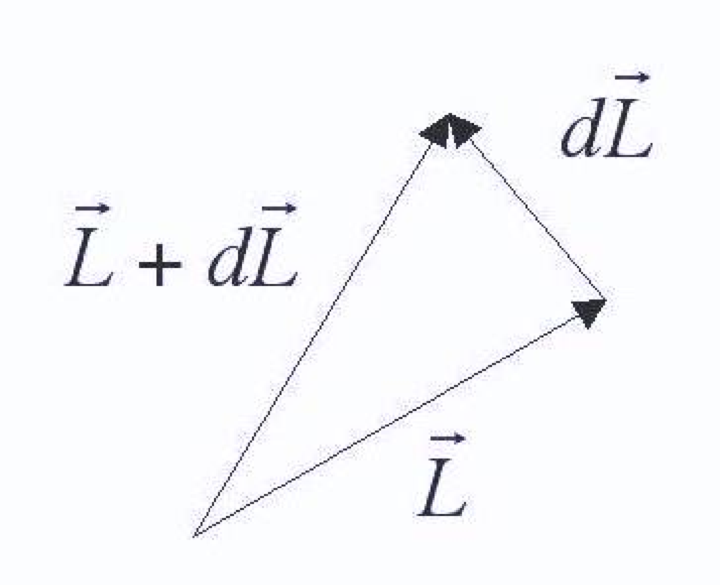
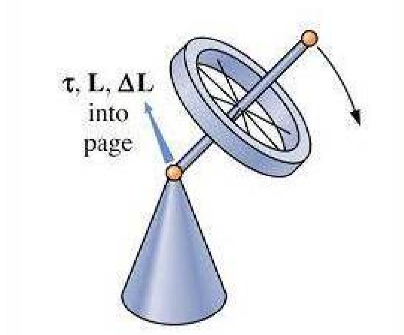
Chapter 13
Rotational Motion

Chapter 14
Static Equilibrium

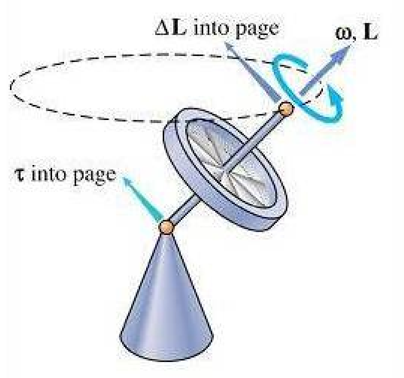
3D Angular Momentum: Precession

If an applied torque imparts angular momentum perpendicular to a spinning object's angular momentum, and the rotational axis is fixed at a point, the magnitude of the angular momentum won't change, only the direction. This is called precession.



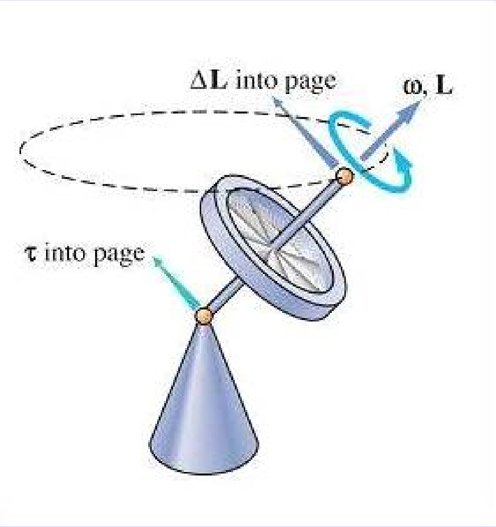


Without the spin, the gyroscope falls over and the torque is perpendicular to the rotation of the fall.

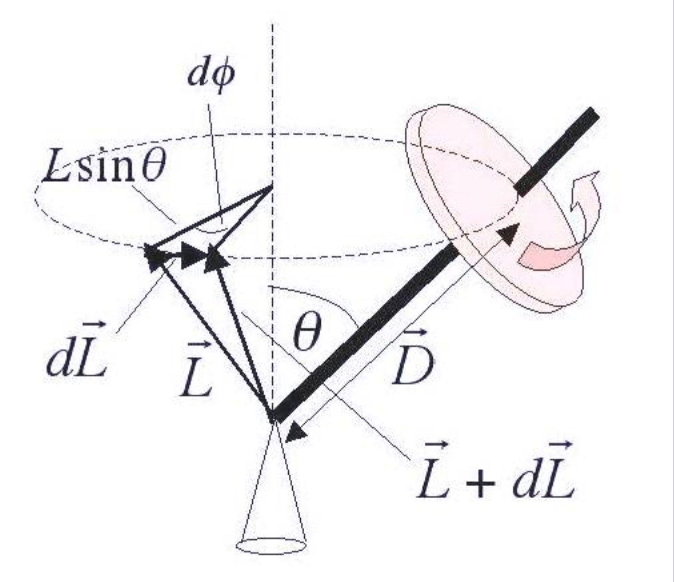


With the spin, the torque from the pull of gravity is perpendicular to the angular momentum of the spin. This causes L to change direction around the pivot point.

3D Angular Momentum: Precession



What is the rate of precession for the gyro? First we determine the torques and the angular momentum, then apply ***τ* = *dL/dt***. Consider the figure below:



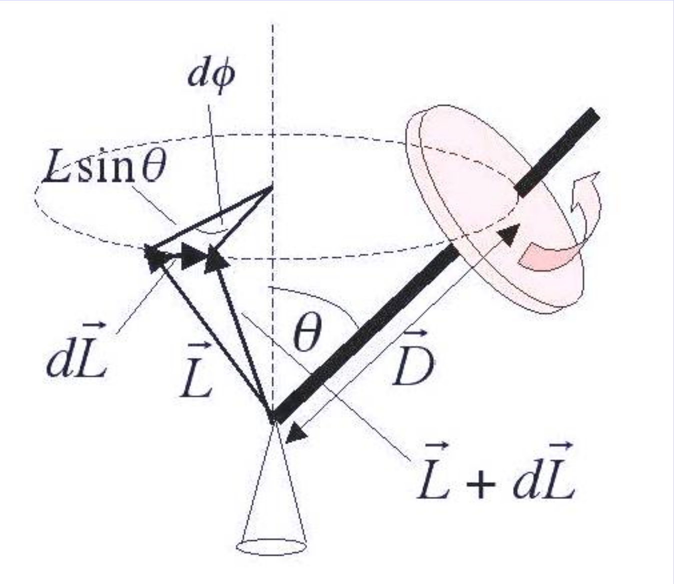
Assume that the rotational axis is massless. Then the torque due to gravity is:

$$\vec{\tau} = \vec{D} \times m\vec{g} \rightarrow \tau = mgD\sin\theta$$

A small change in the angular momentum, ***dL***, is:

$$dL = L\sin\theta d\phi$$

3D Angular Momentum: Precession



What is the rate of precession for the gyro? First we determine the torques and the angular momentum, then apply $\boldsymbol{\tau} = \boldsymbol{dL/dt}$.

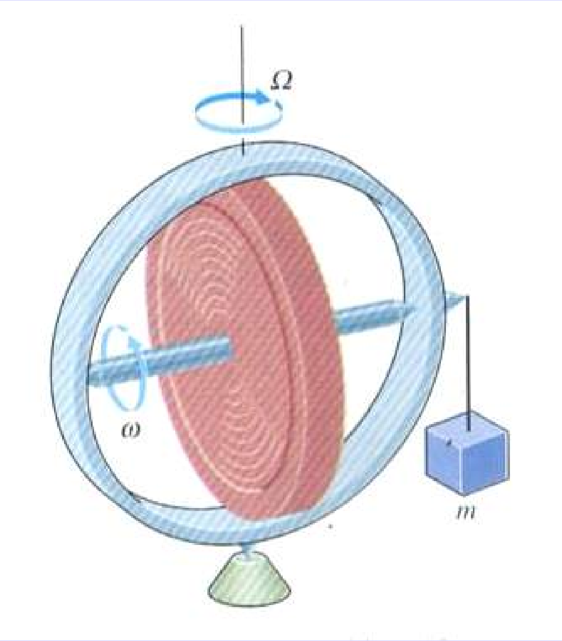
$$dL = \tau dt = mgD\sin\theta dt = L\sin\theta d\phi$$

Solving for the precession rate, $\boldsymbol{\Omega = d\phi/dt}$:

$$\boxed{\Omega = \frac{d\phi}{dt} = \frac{mgD}{L}}$$

The precession rate is independent of $\boldsymbol{\theta}$!

Example: Precession



Initially a gyroscope is spinning with angular speed ω and is perfectly balanced so that it is not precessing. When a mass m is hung from the frame the gyro precesses about the vertical axis at a rate $\mathbf{\Omega}$. Find the rotational inertia of the gyro.

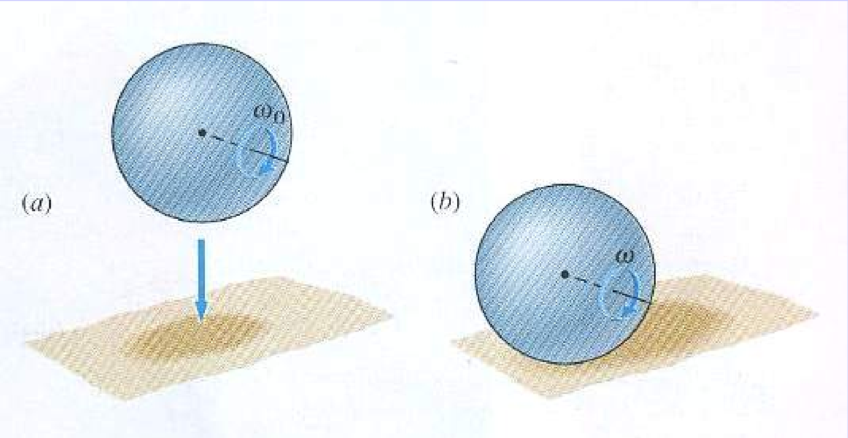
The torque due to gravity is: $\tau = mgR$

The torque is the rate of change of the angular momentum. Hence:

$$\tau = mgR = \frac{dL}{dt} = L\Omega = I\omega\Omega$$

Solving for ***I***: $I = \dfrac{mgR}{\omega\Omega}$

Example: Rolling with Friction



A solid sphere of mass ***M*** and radius ***R*** is spinning with angular velocity $\boldsymbol{\omega_o}$ about a horizontal axis. It is dropped vertically onto a surface with a coefficient of kinetic friction $\boldsymbol{\mu_k}$. (a) Find the expression for the final angular velocity once its achieved pure rolling motion.

There are two things happening in this scenario. First the sphere is accelerating in the ***x*** direction. It is the frictional force inducing this linear acceleration:

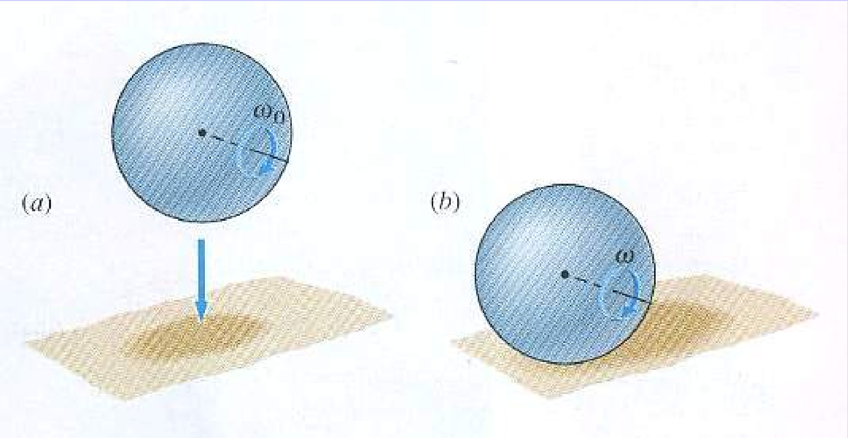
$$F_f = \mu_k Mg = Ma \rightarrow a = \mu_k g \rightarrow v = \mu_k gt$$

The frictional force (via torque) is also responsible for slowing the angular rate:

$$\tau = \mu_k MgR = I\alpha \rightarrow \alpha = \mu_k MgR/(2MR^2/5) = 5\mu_k g/2R$$

$$\omega = \omega_o - \alpha t = \omega_o - 5\mu_k gt/2R$$

Example: Rolling with Friction



A solid sphere of mass ***M*** and radius ***R*** is spinning with angular velocity $\boldsymbol{\omega_o}$ about a horizontal axis. It is dropped vertically onto a surface with a coefficient of kinetic friction $\boldsymbol{\mu_k}$. (b) Find the time this takes.

(a) The no slip condition means that $\boldsymbol{v = R\omega}$. Substituting for ***t*** now yields:

$$\omega = \omega_o - 5\mu_k gt/2R = \omega_o - 5v/2R = \omega_o - 5\omega/2$$

$$7\omega/2 = \omega_o \rightarrow \omega = 2\omega_o/7$$

(b) Since $\boldsymbol{v = \mu_k\, gt = \omega R}$:

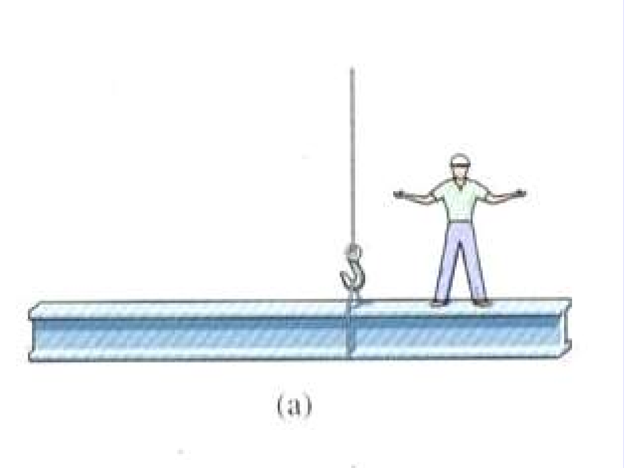
$$t = \omega R/\mu_k g = 2\omega_o R/7\mu_k g$$

Chapter 14

Static Equilibrium

Conditions for Equilibrium

A body is in **static** equilibrium when it is stationary and when both "the net external force and the net external torque are zero!"



$$\sum_i \vec{F}_i = 0$$

and

$$\sum_i \vec{\tau}_i = 0$$

If the tension in the support cable does not act at the center of "gravity" for the beam and the worker then the beam will rotate! Not good!

For static equilibrium we also have

$$\sum_i \vec{p}_i = 0$$

Center of Gravity

The center of gravity is the point at which the gravitational force acts in balance.

Consider the calculation of the torque acting on a group of point masses due to gravitational force.

$$\vec{\tau}_{net} = \sum_i \vec{\tau}_i$$

For each point mass if the acceleration of gravity is g (uniform) the force is $m_i g$

$$\vec{\tau}_{net} = \sum_i \vec{r}_i \times m_i \vec{g}$$

The mass is scalar and can be moved, and we can multiply and divide by the total mass M without changing τ.

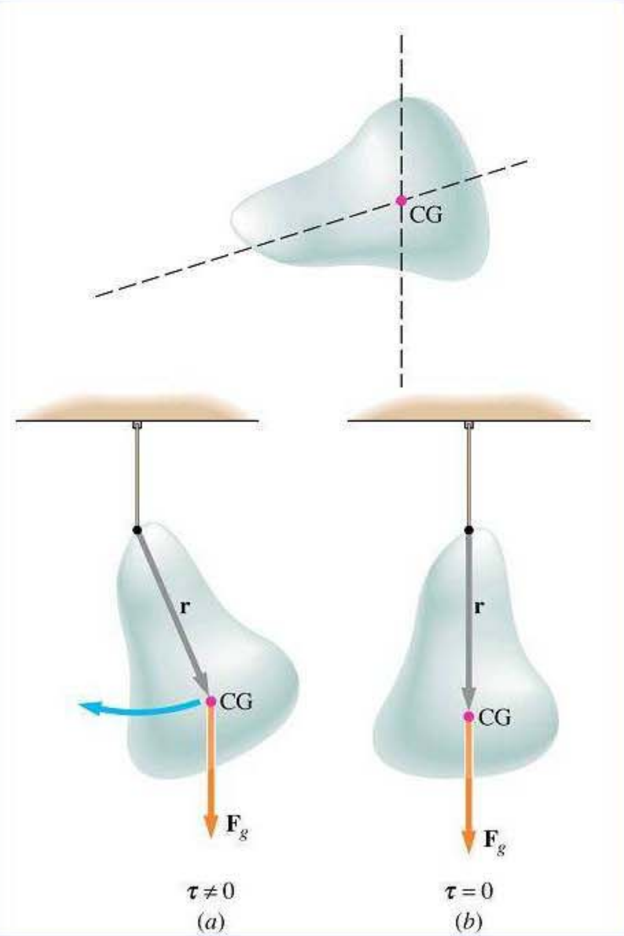
$$\vec{\tau}_{net} = \sum_i \frac{m_i \vec{r}_i}{M} \times M\vec{g}$$

The first term is the center of mass. The second is the net gravitational force on all the objects.

$$\boxed{\vec{\tau}_{net} = \vec{R}_{cm} \times M\vec{g}}$$

Thus if g is uniform, R_{cm} is the center of gravity.

Example: Hanging Equilibrium



For a **hanging** solid **object** the **center of gravity** will lie **directly under** the **hanging point** when in equilibrium.

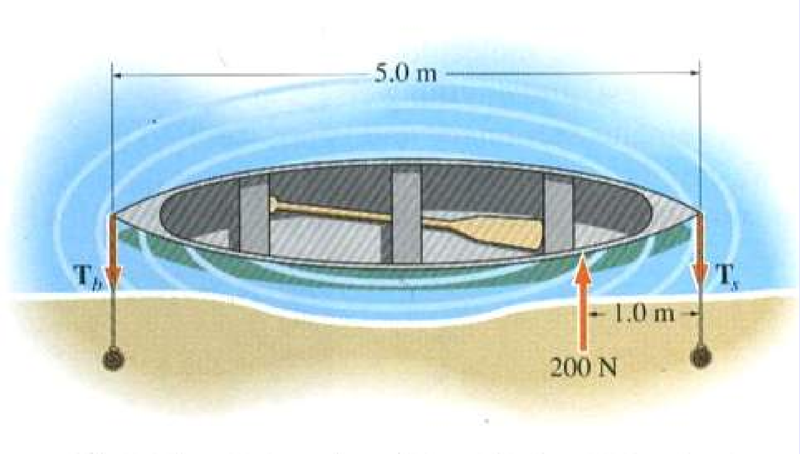
If the object is hung from **two pivot points**, the vertical **lines beneath** them will **intersect** at the **center of gravity**.

The center of gravity will behave like a point mass on a pendulum for a hanging body. When the center of gravity is not beneath the pivot, a torque acts on the body, and it is not in equilibrium

Later we will define the **equilibrium condition mathematically** as
For the pendulum, x is the arclength along the swing.
The derivative of gravitational potential at the bottom is zero.

$$\frac{dU}{dx} = 0$$

Example: Static Equilibrium



A canoe is tied to the shore with two ropes one at the bow and the other at the stern. As a rower boards the canoe he exerts a force of **200*N*** on the canoe (as shown). Find the tensions in the ropes.

Since the sum of the forces is zero we have:

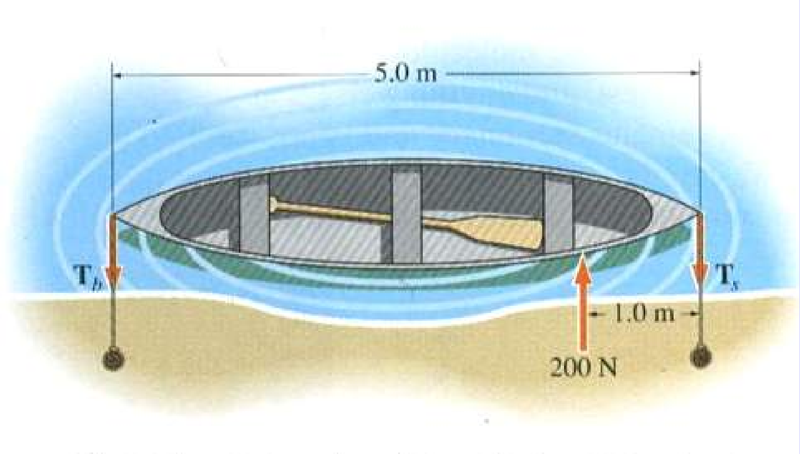
$$T_b + T_s = 200$$

To find $\boldsymbol{T_b}$ we will take the origin to be the bow of the boat. Summing the torques:

$$5T_s = 200(4) \rightarrow T_s = 160N$$

Since the tensions sum to **200*N*** the tension in the bow rope is $\boldsymbol{T_b = 40N}$

Example: Static Equilibrium



A canoe is tied to the shore with two ropes one at the bow and the other at the stern. As a rower boards the canoe he exerts a force of **200*N*** on the canoe (as shown). Find the tensions in the ropes.

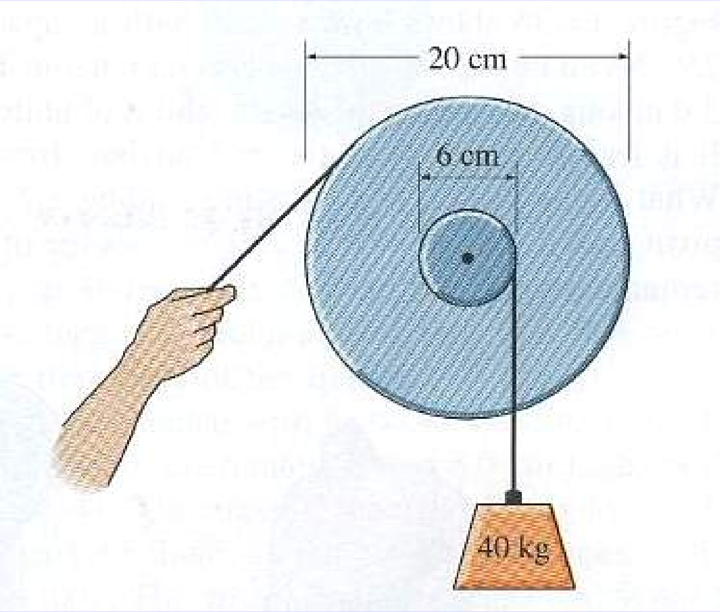
What if we decided to take the origin as the stern? Then the sum of the torques is:

$$5T_b = 200 \rightarrow T_b = 40N$$

Since the tensions sum to **200*N*** the tension in the stern rope is $\boldsymbol{T_s = 160N}$

The answers are the same as they had to be. The important point here is the choice of the origin is arbitrary and is often chosen for convenience.

Example: Static Equilibrium



Two pulleys are mounted on a horizontal axis as shown. The inner pulley has a diameter of ***6cm*** and the outer pulley a diameter of ***20cm***. Find the force on the outer rope required to support the ***40kg*** mass.

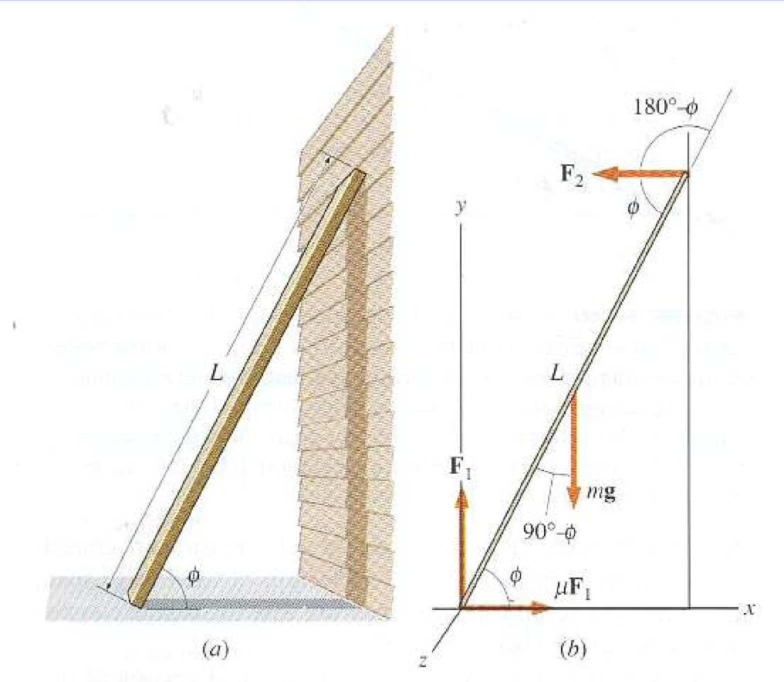
The forces are balanced by normal forces between the axel and the pulleys.

Balancing torques yields:

$$FR = mgr \rightarrow 10F = 40(9.8)3$$
$$F = 12(9.8) = 118N$$

Since both ropes are always tangent to the outer rim of their respective pulleys, ***sinθ* = 1**. The angle that the hand is pulling the rope determines the normal forces of the axel on the pulleys for the forces to sum to zero.

Example: Leaning Board Against a Wall



A board of mass m and length L is leaning against a wall. The wall is frictionless and the coefficient of static friction between the floor and the board is $\boldsymbol{\mu}$. Find the minimum angle $\boldsymbol{\phi}$ at which the board can be leaned without slipping.

The component force equations yield:

$$F_1 = mg \text{ and } \mu F_1 = F_2$$

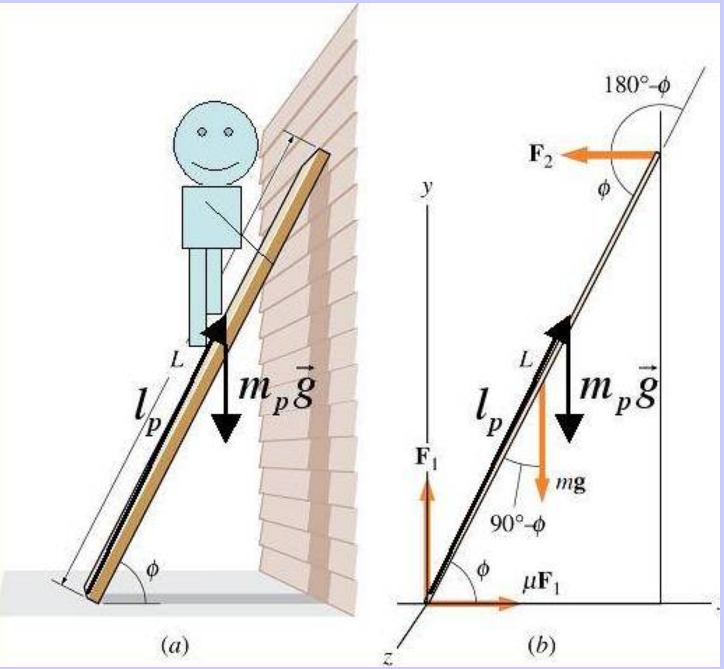
The most convenient origin for the torque equation is the bottom of the board. Balancing torques about that point:

$$mg\frac{L}{2}\sin(\pi/2-\phi) = F_2 L\sin\phi = \mu mgL\sin\phi$$

$$2\mu = \sin(\pi/2-\phi)/2\sin\phi = \cot\phi$$

Solving for $\boldsymbol{\phi}$: $\tan\phi = 1/2\mu \rightarrow \phi = \tan^{-1}(1/2\mu)$ Does this make sense?

Example: Ladder with Person



This is the same problem as the board leaning up against the wall, except now we will consider the additional effects of a person on a ladder. The person has a mass $\boldsymbol{m_p}$ and is a distance $\boldsymbol{l_p}$ up the ladder.

The component force equations yield:

$$F_1 = mg + m_p g \quad \text{and} \quad \mu F_1 = F_2$$

Again the most convenient origin for the torque equation is the bottom of the ladder. Balancing torques at that point:

$$mg\frac{L}{2}\cos\phi + m_p g l_p \cos\phi = F_2 L\sin\phi = \mu(mg + m_p g)L\sin\phi$$

$$mL\cos\phi + 2m_p l_p\cos\phi = 2\mu(m + m_p)L\sin\phi$$

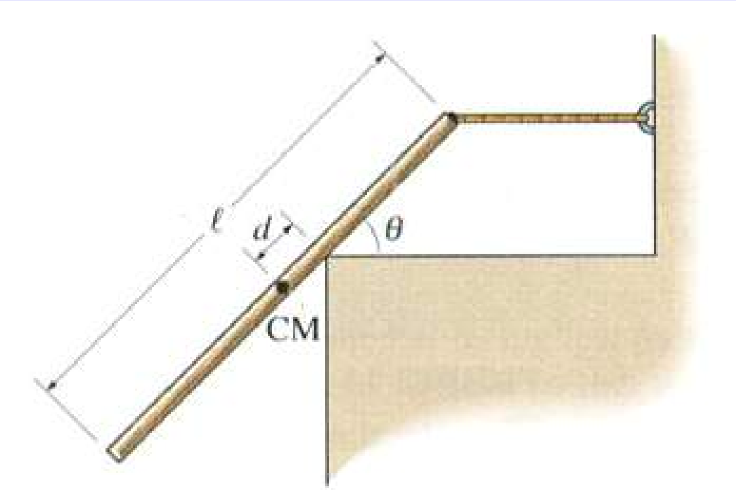
$$\tan\phi = (mL + 2m_p l_p)/(2\mu(m + m_p)L)$$

Does this answer make sense?

What if $\boldsymbol{m = 0}$?

What if $\boldsymbol{l_p = L}$?

Example: Balancing a Board Over The Edge



A uniform board of length ℓ is balanced over a frictionless edge as shown secured by a horizontal rope. The center of mass is a distance ***d*** from the edge. Find the angle, $\boldsymbol{\theta}$, that the board makes with the horizontal.

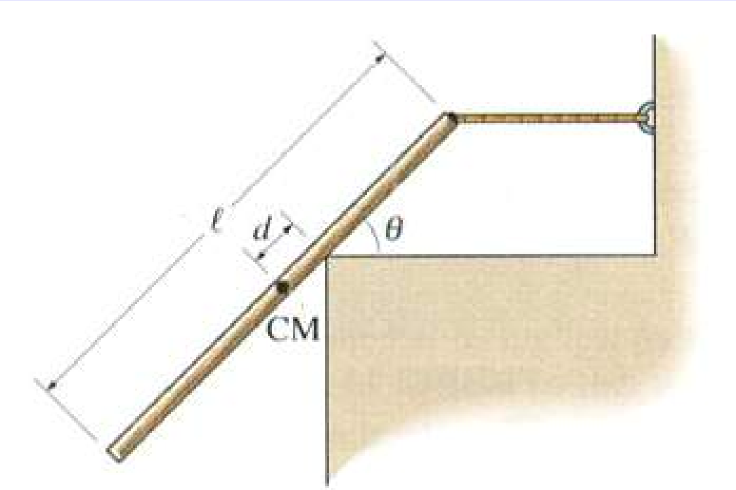
Again we have to balance the forces and torques. The torque equation is straightforward. Using the point of contact between the board and the edge:

$$T(\ell/2 - d)\sin\theta = mgd\sin(\pi/2 + \theta) = mgd\cos\theta$$

To solve for ***T*** we have to balance forces. At the contact point there is only a normal force, ***N***, (frictionless). Since this force is normal to the board balancing forces for both components yields:

$$mg = N\cos\theta \quad \text{and} \quad T = N\sin\theta$$

Example: Balancing a Board Over The Edge



A uniform board of length ℓ is balanced over a frictionless edge as shown secured by a horizontal rope. The center of mass is a distance ***d*** from the edge. Find the angle, $\boldsymbol{\theta}$, that the board makes with the horizontal.

There are three unknowns, $\boldsymbol{\theta}$, ***T***, and ***N***. However we have three equations. Solving for ***T*** from the force equations:

$$T = mg\frac{\sin\theta}{\cos\theta}$$

Substituting this result into the torque equation yields:

$$T(\ell/2 - d)\sin\theta = mgd\cos\theta$$

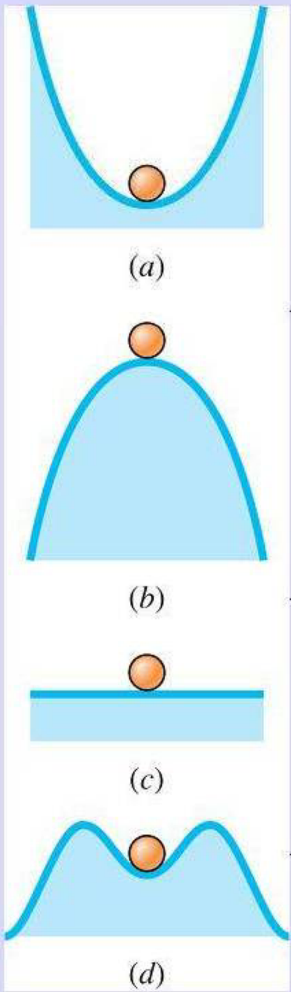
$$mg(\ell/2 - d)\sin^2\theta = mgd\cos^2\theta = mgd(1 - \sin^2\theta)$$

$$(\ell/2)\sin^2\theta = d \rightarrow \sin\theta = \sqrt{2d/\ell}$$

If this condition is not satisfied then the rope won’t stay horizontal!

Stability of Equilibria

An unperturbed body in equilibrium will have no translational or rotational acceleration. A perturbation adds a small amount displacement to the system, and the response of the system to that displacement defines the stability.



$$\frac{d^2U}{dx^2} > 0$$

stable

A perturbed body in a stable equilibrium will return to that equilibrium state as the energy of the perturbation dissipates or damps.

$$\frac{d^2U}{dx^2} < 0$$

unstable

A perturbed body in unstable equilibrium will cause the perturbation to grow.

$$\frac{d^2U}{dx^2} = 0$$

neutral

Neutral equilibrium will remain as perturbed.

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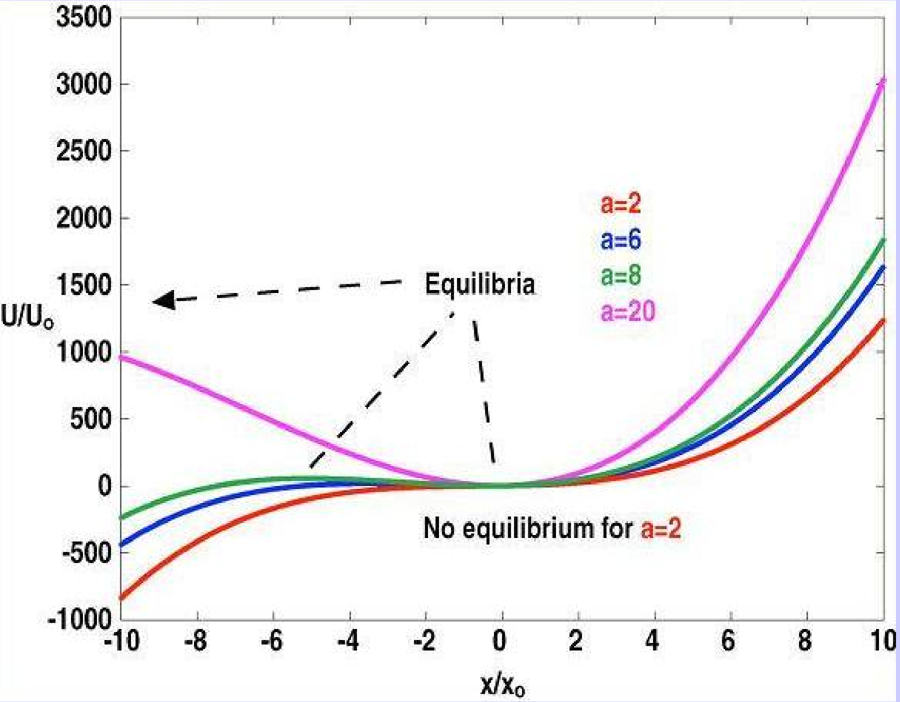
metastable

Metastable equilibrium has a threshold for instability. (locally stable)

Example: Equilibrium Conditions



The potential energy as a function of ***x*** is:

$$U(x) = U_o\left(\frac{x^3}{x_o^3} + a\frac{x^2}{x_o^2} + 4\frac{x}{x_o}\right)$$

For what values of a will there be two static equilibria? Comment on the stability of these equilibria.

Equilibrium is determined by ***dU/dx* = 0**. Taking the derivative of ***U***:

$$\frac{dU}{dx} = \frac{U_o}{x_o}\left(3\frac{x^2}{x_o^2} + 2a\frac{x}{x_o} + 4\right) = 0 \rightarrow \frac{x_{1,2}}{x_o} = \frac{1}{3}\left(-a \pm \sqrt{a^2 - 12}\right)$$

For two real roots $\boldsymbol{a^2 > 12}$, $\boldsymbol{a \sim 3.5}$. Taking the second derivative yields:

$$\frac{d^2U(x_{1,2})}{dx^2} = \frac{U_o}{x_o^2}\left(6\frac{x_{1,2}}{x_o} + 2a\right) = \pm 2\frac{U_o}{x_o^2}\sqrt{a^2 - 12}$$

Right point is metastable.
Left point is unstable.