Chapter 12

Rotational Dynamics

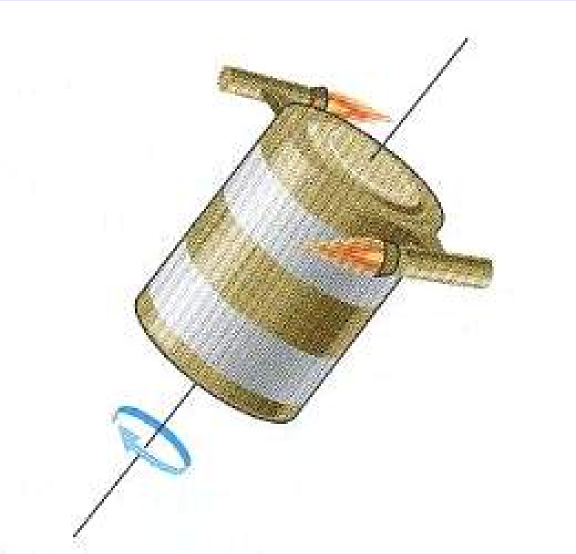
Chapter 13

Rotational Vectors

Angular Momentum

Rotational Dynamics

The knowledge of a body's moment of inertia allows us to use the rotational analog of Newton's second law to determine the object's behavior.



A cylindrical satellite of radius ***R*** is spinning at a frequency ***f***. It must be stopped so that a space shuttle crew can make repairs. Two small jets each with a thrust ***F*** are mounted tangent to the satellite's surface as shown. How long must they fire to bring the satellite to rest?

$$\Delta\omega = \alpha t \text{ and } \tau = 2RF$$

$$\Delta\omega = 2\pi f = \frac{\tau}{I}t = \frac{2RF}{I}t$$

$$t = \frac{2\pi f MR^2/2}{2RF} = \frac{\pi f MR}{2F}$$

Note that the moment of inertia for a cylinder is the same as that for a disk!

Does this result make sense?

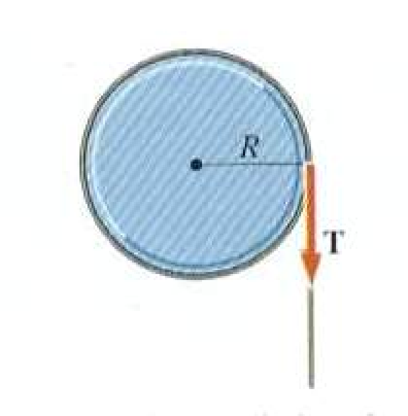
Example: Rotational Dynamics

A solid cylinder of mass ***M*** and radius ***R*** is used to support a massless rope and a bucket of mass ***m***. Find the rate of acceleration of the bucket into the well.

First consider the free body diagram for the bucket:

$$mg - T = ma$$

Next the free body diagram for the cylinder. The tangential acceleration of the cylinder must equal ***a***.



$$a = \alpha R = \frac{\tau}{I} R = \frac{TR^2}{MR^2/2} = \frac{2T}{M}$$

Example: Rotational Dynamics

A solid cylinder of mass ***M*** and radius ***R*** is used to support a massless rope and a bucket of mass ***m***. Find the rate of acceleration of the bucket into the well.

Substituting for ***T*** to solve for ***a***:

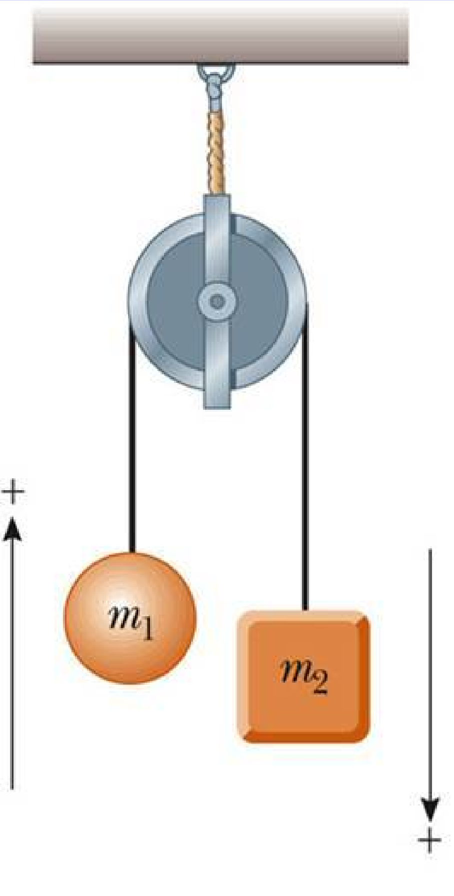
$$a = 2T/M = 2m(g - a)/M \rightarrow a\left(1 + \frac{2m}{M}\right) = 2\frac{m}{M}g$$

$$a = \frac{2m}{M + 2m}g$$

Note that the acceleration is reduced due to the rotational inertia of the cylinder.

Does this result make sense?

Example: Atwood Machine (again)



Now find the acceleration for the masses in an Atwood machine with a cylindrical pulley of radius ***R*** and a moment of inertia ***I***.

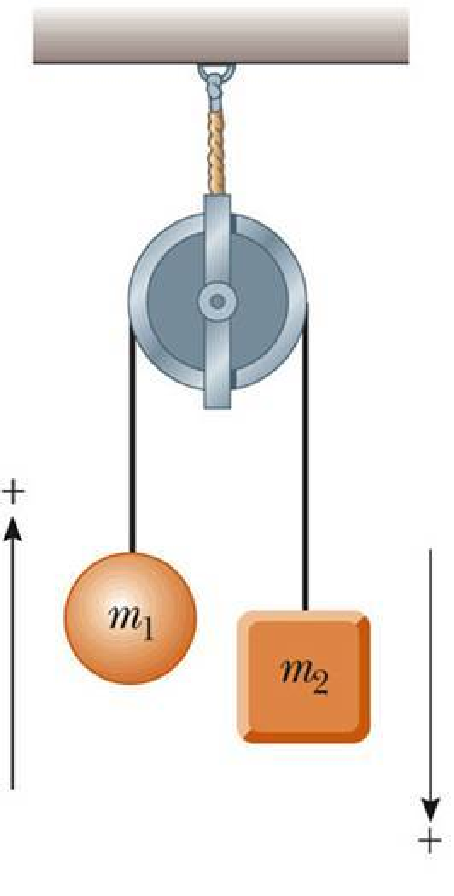
For this case the tension in the rope is different on each side. There are now three free body diagrams to consider. The two for the suspended masses leads to the equations:

$$m_2 g - T_2 = m_2 a \quad \text{and} \quad T_1 - m_1 g = m_1 a$$

There is now a net torque on the pulley and after applying the no slip condition its equation of motion is:

$$\tau = (T_2 - T_1)R = I\alpha = Ia/R$$

Example: Atwood Machine (again)



Now consider an Atwood machine with a cylindrical pulley of radius ***R*** and a moment of inertia ***I***.

Dividing the torque equation by ***R*** and adding the two equations yields:

$$(m_2 - m_1)g = (m_1 + m_2 + I/R^2)a$$

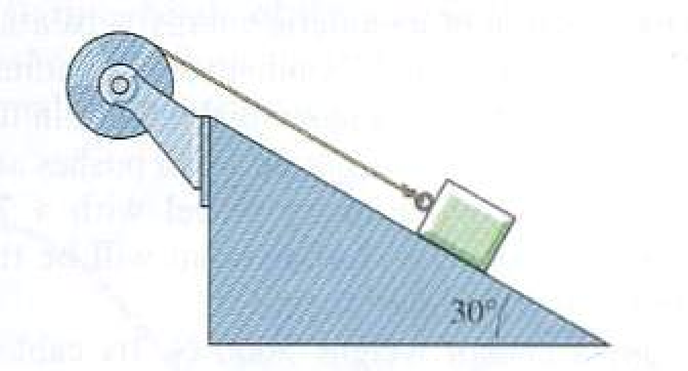
$$a = \frac{(m_2 - m_1)g}{m_1 + m_2 + I/R^2}$$

Compare this to the solution for a massless pulley.

$$a = \frac{m_2 - m_1}{m_1 + m_2} g$$

Again the acceleration is reduced due to the rotational inertia of the pulley.

Example: Incline Plane w Pulley & Friction



A block of mass ***m*** is attached by a massless string to a solid cylinder pulley of mass ***M*** and radius ***R***. When the mass is released it accelerates down the plane with an acceleration ***a***. Find the coefficient of kinetic friction, μ_k.

Newton's 2[nd] for the mass ***m*** yields:

$$mg\sin\theta - T - \mu_k mg\cos\theta = ma$$

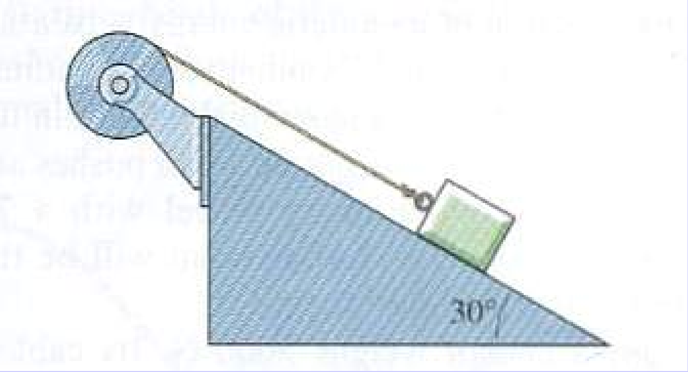
Again (the rope does not slip) the tangential acceleration of the pulley is equal to this acceleration:

$$a = \alpha R = \frac{\tau}{I}R = \frac{R^2}{I}T$$

For a solid cylinder ***I = ½ MR²*** and after combining the equations:

$$(m + M/2)a = mg\sin\theta - \mu_k mg\cos\theta$$

Example: Incline Plane w Pulley & Friction



A block of mass ***m*** is attached by a massless string to a solid cylinder pulley of mass ***M*** and radius ***R***. When the mass is released it accelerates down the plane with an acceleration ***a***. Find the coefficient of kinetic friction, $\boldsymbol{\mu_k}$.

$$(m + M/2)a = mg\sin\theta - \mu_k mg\cos\theta$$

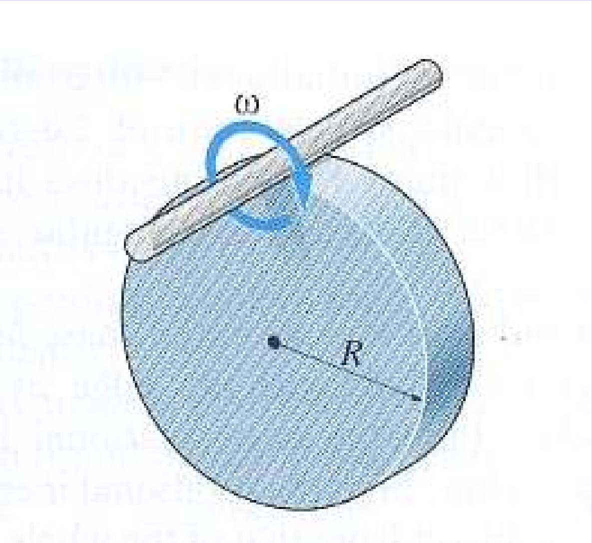
Solving for $\boldsymbol{\mu_k}$:

$$\mu_k = \frac{mg\sin\theta - (m + M/2)a}{mg\cos\theta} = \tan\theta - \frac{(m + M/2)}{m\cos\theta}\frac{a}{g}$$

If $\boldsymbol{\mu_k = \tan\theta}$ then there is no acceleration and the moment of inertia of the pulley is irrelevant. However for steeper angles the moment of inertia again reduces the acceleration of the mass sliding down the incline.

Could we have solved this problem with the work-energy theorem? You betcha! Now $\boldsymbol{\Delta K}$ includes rotational kinetic energy as well. Give it a try!

Example: Disk Rotating About Axis on Edge



A uniform disk or radius ***R*** is rotating about a horizontal axis at its edge. What is its minimum angular speed, ω, required for it to make a complete circle?

From the conservation of energy:

$$E_{bot} = \frac{1}{2} I\omega^2 = E_{top} = 2mgR$$

From the parallel axis theorem the momentum of inertia about an axis on its edge is:

$$I = \frac{1}{2} mR^2 + mR^2 = \frac{3}{2} mR^2$$

Solving for ω:

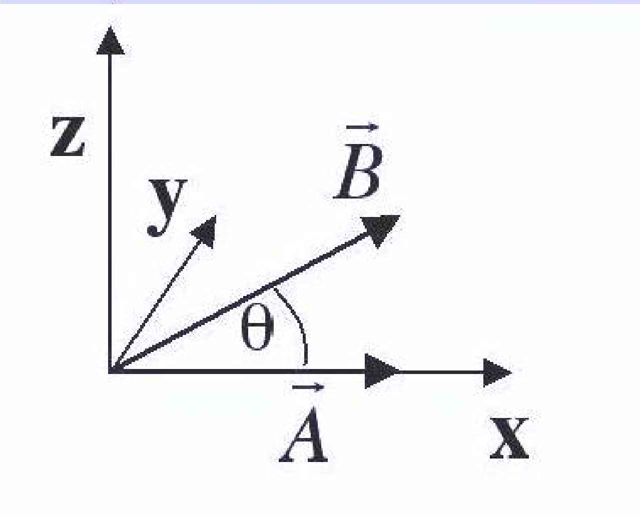
$$\omega^2 = \frac{4mgR}{I} = \frac{4mgR}{3mR^2/2} = \frac{8}{3}\frac{g}{R}$$

$$\omega = 2\sqrt{\frac{2g}{3R}}$$

Chapter 13

Rotational Vectors
Angular Momentum

The Vector Product (Cross Product) Vs. Scalar Product



In terms of the angle between the two vectors, the vector and scalar products are written:

$$\vec{A} \bullet \vec{B} = AB\cos\theta$$

$$\vec{A} \times \vec{B} = AB\sin\theta\,\hat{k}$$

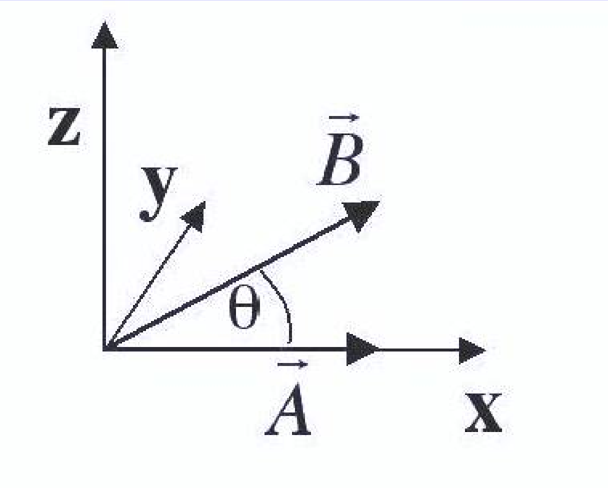
Where the unit vector **k** points **perpendicular to both A and B.**

What is the vector product in unit vector notation?

We've discussed scalar products in previous lectures. An example of a scalar product is Work.

$$W = \vec{F} \cdot \Delta\vec{r} = (F_x\Delta x) + (F_y\Delta y) + (F_z\Delta z) = F\Delta r\cos\theta$$

The Vector Product (Cross Product)



The vector product can be formed by expanding the vectors in their basis and applying the sinθ between the individual basis vectors.

$$\hat{i}\times\hat{i}=0,\quad \hat{j}\times\hat{j}=0,\quad \hat{k}\times\hat{k}=0$$
$$\hat{i}\times\hat{j}=\hat{k},\quad \hat{j}\times\hat{k}=\hat{i},\quad \hat{k}\times\hat{i}=\hat{j}$$
$$\hat{j}\times\hat{i}=-\hat{k},\quad \hat{k}\times\hat{j}=-\hat{i},\quad \hat{i}\times\hat{k}=-\hat{j}$$

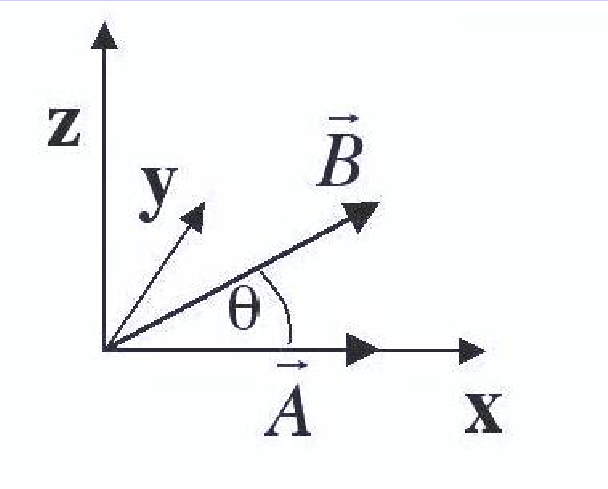
The net result is:

$$\vec{A}\times\vec{B}=\left(A_x\hat{i}+A_y\hat{j}+A_z\hat{k}\right)\times\left(B_x\hat{i}+B_y\hat{j}+B_z\hat{k}\right)$$
$$\vec{A}\times\vec{B}=(A_yB_z-A_zB_y)\hat{i}+(A_zB_x-A_xB_z)\hat{j}+(A_xB_y-A_yB_x)\hat{k}$$

For **A** and **B** in the x – y plane as shown:

$$\vec{A}\times\vec{B}=AB\sin\theta\,\hat{k}$$

The Vector Product (Cross Product)



The net result of this is we can express the vector product of **A** and **B** as the determinant:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & A_x & B_x \\ \hat{j} & A_y & B_y \\ \hat{k} & A_z & B_z \end{vmatrix}$$

The magnitude of the vector product of **A** and **B**:

$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta$$

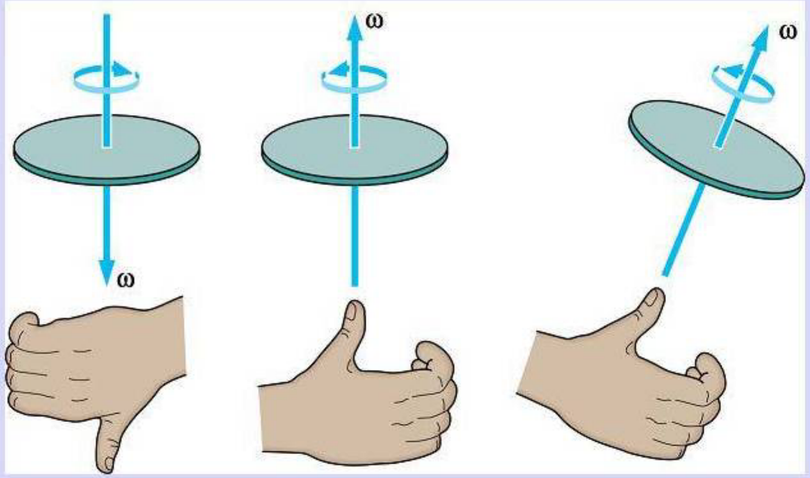
The vector **A** x **B** is normal to the plane formed by the two vectors **A** and **B** with a direction determined by the "**right hand rule**".

The "right hand rule" and the angular velocity vector

The "right hand rule" refers to the direction of a vector product result being in the **direction of your thumb** as you wrap your fingers around from the **first vector to the second**. This applies to **ALL vector products**.

The angular velocity vector actually comes from the vector product

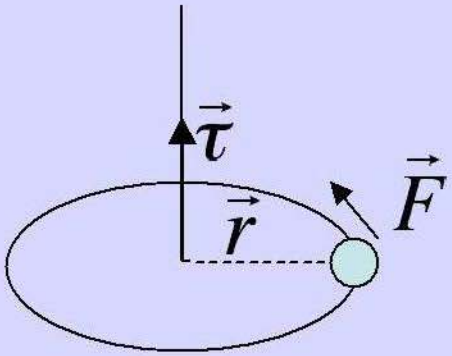
$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2}$$



and points **along the axis of rotation** according to the right hand rule. Sign of ω obviously relates to direction of rotation.

Torque as a Vector

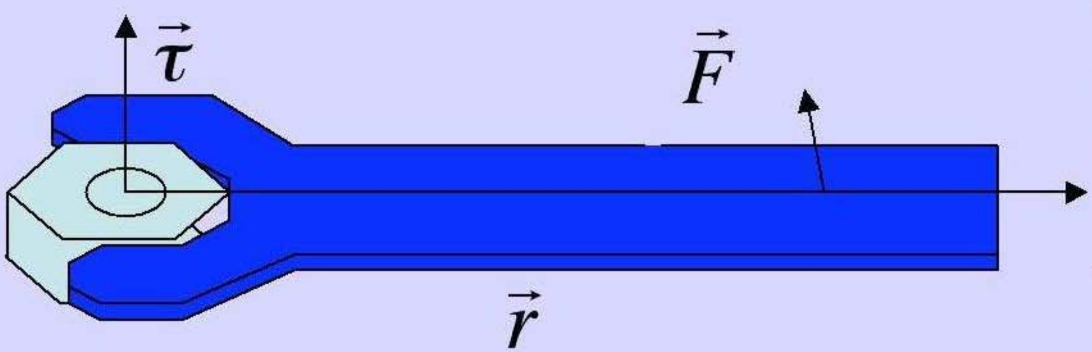
Torque also formally comes from the vector product



$\vec{\tau} = \vec{r} \times \vec{F}$ depends on the choice of origin

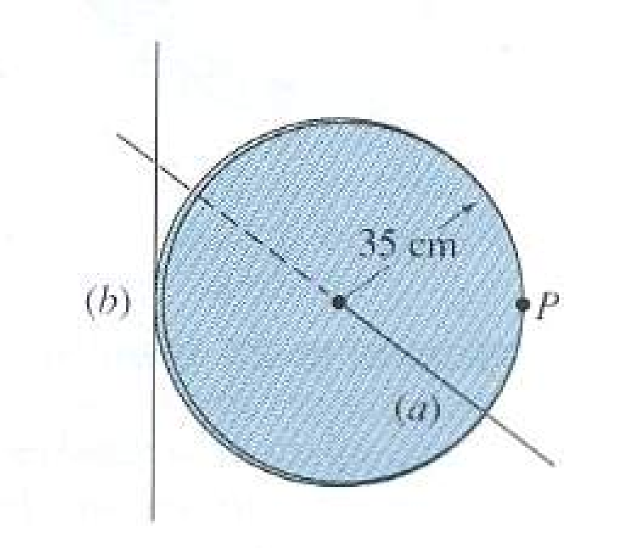
So, as in the figure shown, the torque is always perpendicular to both r and F.

Screws and nuts are typically threaded to tighten in the direction of torque.



For a given amount of force the torque increases with the radius at which it is applied. This is the reason why door knobs are at the outer end of doors. And why car steering wheels are so big.

Example: Torque



Give the magnitude and direction of the smallest force you could apply at **P** to produce a torque of ***1.2 N-m***

(a) about an axis through the center of the disk

(b) about a vertical axis tangent to the edge.

(a) For an axis through the center, the lever arm is ***r = 35cm***. The maximum for **sin*θ*** is when ***θ* = *π*/2**.

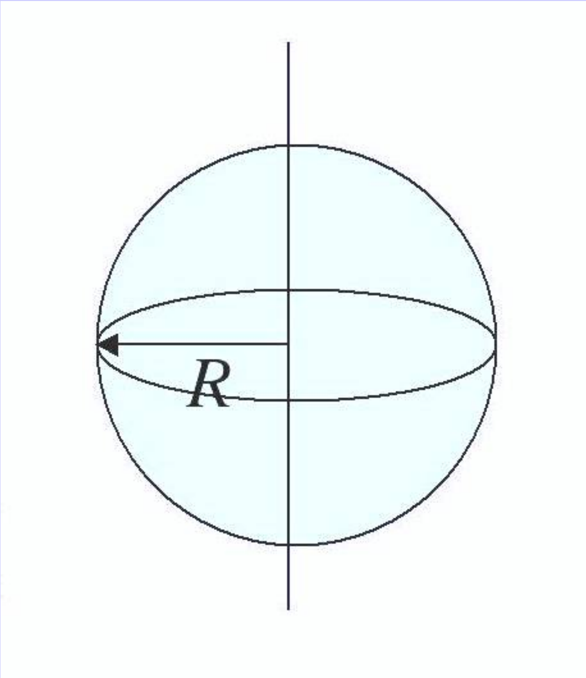
$$\tau = rF \rightarrow F = \tau/r = 1.2/.35 = 3.4N$$

(b) For an axis through the edge, the force must be perpendicular to the disk or else some component of the force will be parallel to the moment arm.

$$\tau = rF \rightarrow F = \tau/r = 1.2/.7 = 1.7N$$

To minimize the force there must be no component of the force parallel to the moment arm.

Example: Torque



A sphere of radius ***R = 1m*** is centered at the origin. At what coordinates on the surface of the sphere would you apply a force ***F = 5N*** in the ***+y*** direction to produce (about the origin)

(a) ***5Nm*** torque pointing in the ***-z*** direction

(b) ***3.4Nm*** torque pointing in the ***+z*** direction

(c) ***3.4Nm*** torque pointing in the ***–x*** direction?

The torque is given by ***τ = r x F*** or:

$$\vec{\tau} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right) \times 5\hat{j} = 5x\hat{k} - 5z\hat{i} \quad \text{and}$$

$$R^2 = x^2 + y^2 + z^2 = 1$$

Hence the answers are:

(a) $x = -1,\ y = z = 0$

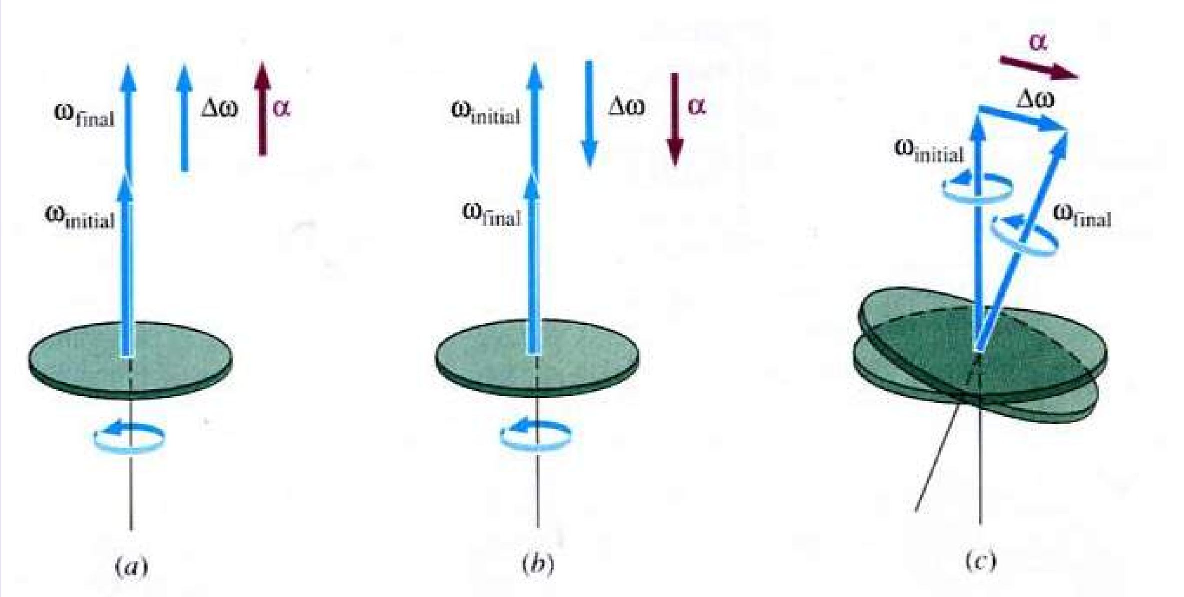
(b) $x = .68,\ y = \pm\sqrt{1-(.68)^2} = \pm.73,\ z = 0$

(c) $x = 0,\ y = \pm.73,\ z = .68$

The finite values for ***y*** allow for force to lie on the surface of the sphere.

Angular Velocity and Acceleration as Vectors

(a) An increase in angular speed alone means that $\boldsymbol{\Delta\omega}$ is parallel to ω. (b) A decrease in angular speed alone means that $\boldsymbol{\Delta\omega}$ is antiparallel to w. (c) If ω only changes direction, then for small $\boldsymbol{\Delta\omega}$ the angular acceleration is perpendicular to ω.



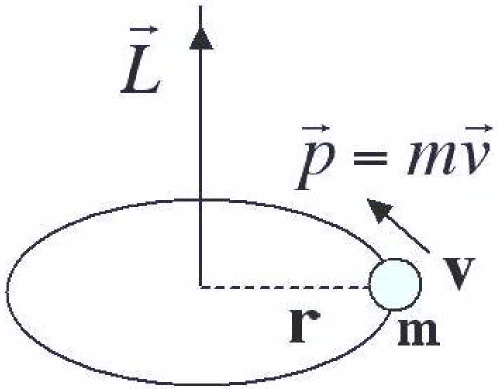
A change in the direction of the angular velocity without a change in angular speed is analogous to the radial acceleration for circular motion. Remember angular velocity and angular acceleration are vectors!

$$\boxed{\vec{\alpha} = \frac{d\vec{\omega}}{dt}}$$

Angular Momentum

The angular momentum L is defined with the vectors of linear motion:

$$\vec{L} = \vec{r} \times \vec{p}$$



Here the object may or may not be in circular motion. The angular momentum is simply defined instantaneously at a location and about an axis.

For a system of particles $\vec{L} = \sum_i \vec{L}_i$

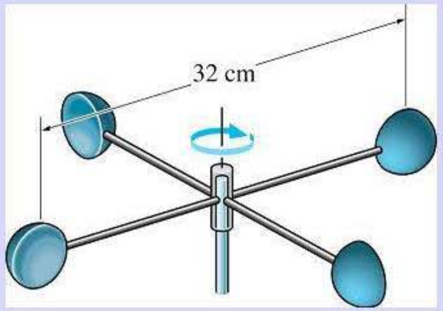
Typically, but not always, we can write this in the analogous rotational mechanics form:

$$\vec{L} = I\vec{\omega}$$

The torque can be written as the change in the angular momentum, just as the linear force was written as the change in the linear momentum.

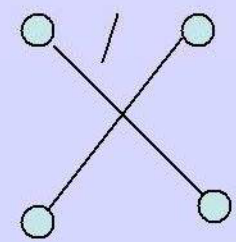
$$\vec{\tau} = I\vec{\alpha} = \frac{d\vec{L}}{dt}$$ **analogous to** $\vec{F} = \frac{d\vec{p}}{dt}$ **from linear motion**

Example: Angular Momentum



13-20 An anemometer for measuring wind speed consists of four small metal cups, each of mass 120g, mounted on the ends of essentially massless rods 32cm long. Find (a) the anemometers rotational inertia about its central axis and (b) its angular momentum when it's spinning at 12 rev/s in the direction indicated.

$$R = 16cm$$



Looking down

(a) The moment of inertia for the four cup anemometer is

$$I = \sum r^2 m = 4R^2 M$$

$$I = 4(0.12kg)(0.16m)^2 = \boxed{1.23 \times 10^{-2}\, kgm^2}$$

(a) Using this moment of inertia,

$$\omega = (12)(2\pi)rad/s$$

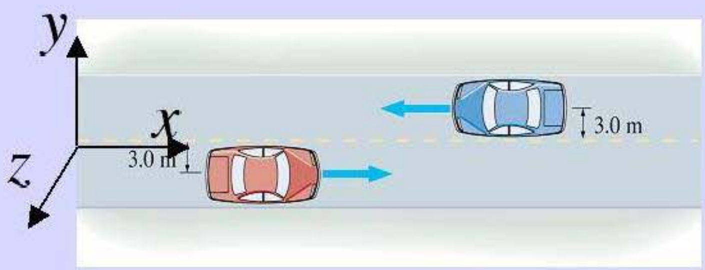
$$L = I\omega$$

$$L = (1.23 \times 10^{-2})(12)(2\pi)Js = \boxed{0.926Js}$$

Directed downward by right hand rule

Would we get the same from $\vec{L} = \vec{r} \times \vec{p}$? Yes.

Example: Angular Momentum



13-27 Two identical 1800kg cars are traveling in opposite directions at 90km/h. Each car's center is 3.0m from the center of the highway. What are the magnitude and direction of the angular momentum of the system consisting of the two cars, about a point on the center line of the highway?

The angular momentum can be written $\vec{L} = \vec{r} \times \vec{p}$

$$\vec{L} = (r_x\hat{i} + r_y\hat{j}) \times (p_x\hat{i} + p_y\hat{j})$$

$\vec{L} = -r_y p_x \hat{k}$ **For one car**

For both cars

$$\vec{L} = \sum_i \vec{L}_i = -(3.0)(1800)(-900/36) - (-3.0)(1800)(900/36) Js\hat{k}$$

$\boxed{\vec{L} = 2.7 \times 10^5 Js\hat{k}}$ **This is out of the board.**