Today's Lecture

Lecture 18: Chapter 11,
Impulse,
Conservation of Momentum,
Inelastic and Elastic Collisions

Impulse

Given a collision between two objects, we know that the force on the object is it's change in momentum with time:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The impulse is the integrated change in momentum over the time during the collision:

Impulse: $I = \int \vec{F} dt = \int d\vec{p}$

$$\vec{I} = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

The average impulse force is this integrated change in momentum divided by the total time:

$$\vec{F}_{ave} = \frac{\vec{I}}{\Delta t}$$

Note: compare impulse to work: Impulse is analog in time instead of space. Momentum instead of Energy

Example: Impluse

A ***59g*** tennis ball is thrown up and at the peak of its trajectory it is hit by a racket with a horizontal force given by $\boldsymbol{F = at - bt^2}$ where $\boldsymbol{t}$ is measured in milliseconds, $\boldsymbol{a = 1200N/ms}$, and $\boldsymbol{b = 400N/ms^2}$. The ball and racket separate after ***3ms***. Find (a) the impulse and (b) the average impulsive force, and (c) the speed of the ball as it leaves the racket.

(a) The impulse is the integral:

$$I = \int F dt = \int_0^{3ms} (at - bt^2) dt$$

$$I = 600N/ms \times (3ms)^2 - \frac{400}{3} N/ms^2 \times (3ms)^3 = (5.4 - 3.6) = 1.8kgm/s$$

(b) The average impulsive force: $\langle F \rangle = I/\Delta t = 1.8/(3 \times 10^{-3}) = 600N$

(c) The speed of the ball: $v = P/m = I/m = 1.8/.059 = 30.5m/s$

Inelastic vs. Elastic Collisions

Inelastic collisions: the **momentum** is conserved, **not** the **energy**

Elastic collisions: the **momentum and energy** are conserved

For a **totally inelastic** collision between two objects, the objects stick together and become one mass. Only momentum conserved:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_f$$

For a **totally elastic** collision between two objects, the objects are unchanged in form, and rebound perfectly while conserving kinetic energy as well as momentum:

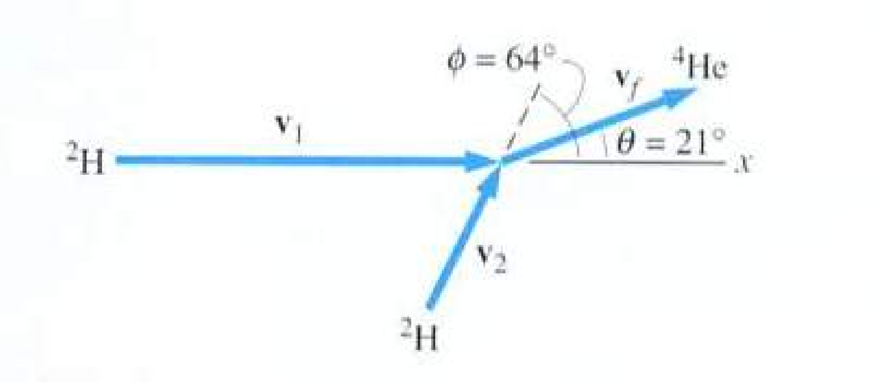
$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Inelastic collisions are rarely totally inelastic, elastic collisions are rarely totally elastic. Reality is somewhere between and these are the limits.

Inelastic Collisions in Two-Dimensions Example: Elementary Particles

Two deuterium nuclei combine to form an α particle. One of the incident nuclei is detected moving at ***3.5 Mm/s*** and the α particle moving at ***2.3 Mm/s*** at an angle $\boldsymbol{\theta = 21^o}$ to the ***x*** axis. Find the initial velocity of the second deuteron.



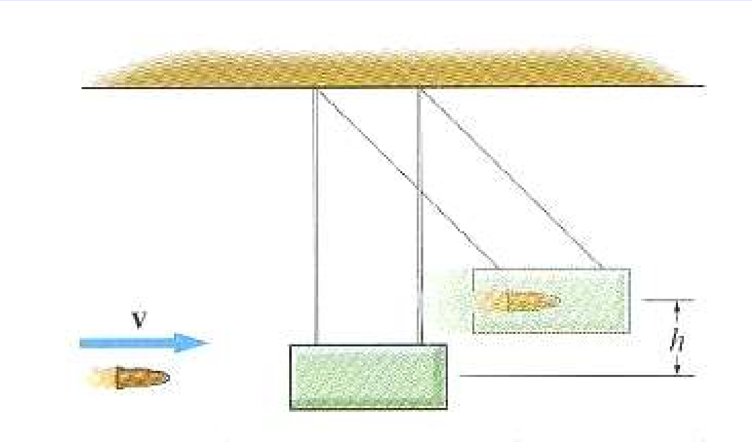
This is a perfectly inelastic collision so only momentum is conserved.
Solving for $\vec{v}_2$:

$$m_d \vec{v}_2 + m_d \vec{v}_1 = m_\alpha \vec{v}_f \rightarrow \vec{v}_2 = \frac{m_\alpha}{m_d}\vec{v}_f - \vec{v}_1$$

Considering each component we find:

$$v_{2x} = \frac{m_\alpha}{m_d} v_{fx} - v_{1x} = 2 \times 2.3\cos 21^\circ - 3.5 = .79 Mm/s$$
$$v_{2y} = \frac{m_\alpha}{m_d} v_{fy} = 2 \times 2.3\sin 21^\circ = 1.65 Mm/s$$

Inelastic Collisions in Two-Dimensions
Example: Ballistic Pendulum



If a bullet with mass ***m*** strikes the wooden block with mass ***M***, find the initial velocity of the bullet if the block rises to a height ***h*** after impact.

Initially momentum is conserved so that:

$$mv = (M+m)V$$

Conserving energy as the block (and bullet) rise in the gravitational field we have:

$$\frac{1}{2}(M+m)V^2 = (M+m)gh \rightarrow V = \sqrt{2gh}$$

Solving for ***v*** :

$$v = \frac{M+m}{m}\sqrt{2gh}$$

Elastic Collisions in One-Dimension

For an elastic collisions in one-dimension both energy and momentum are conserved:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Note that for elastic collisions in one dimension there are two equations for two unknowns.

Rearranging and simplifying:

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$

$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right)$$

Now we can divide the second equation by the first and then rearrange:

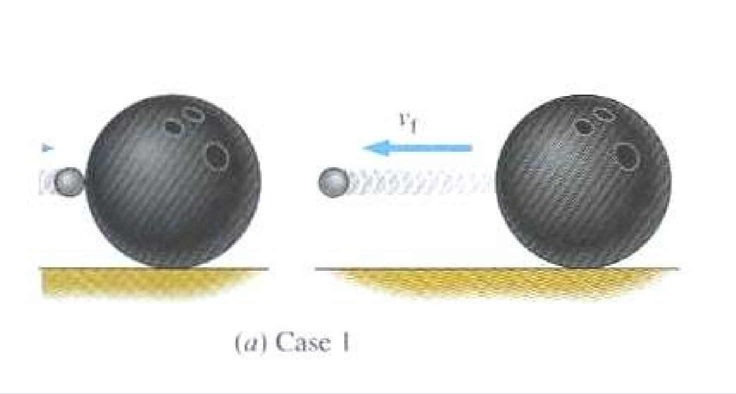
$$m_1 v_{1i} + m_2 v_{2i} = m_2 v_{2f} + m_1 v_{1f}$$

$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

This pair of equations (linear) is much easier to work with than the original. Also note that the relative velocities become reversed!

Elastic Collisions in One-Dimension

Case 1 $m_1 << m_2$ and $v_{2i} = 0$



The conservation equations are:

$$m_1 v_{1i} = m_2 v_{2f} + m_1 v_{1f}$$
$$v_{1i} = v_{2f} - v_{1f}$$

Eliminating v_{2f} we find:

$$(m_1 - m_2) v_{1i} = (m_1 + m_2) v_{1f}$$
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$
$$v_{1f} \simeq -v_{1i}$$

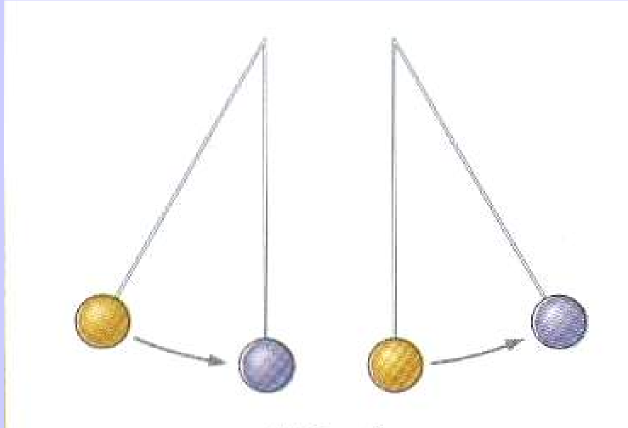
A ping pong ball would bounce off of a bowling ball with essentially its initial speed.

The solution for v_{2f} is:

$$v_{2f} = v_{1f} + v_{1i} = \frac{2m_1}{m_1 + m_2} v_{1i} << v_{1i}$$

Elastic Collisions in One-Dimension

Case 2 $m_1 = m_2$ and $v_{2i} = 0$


(b) Case 2

The conservation equations are:

$$m_1 v_{1i} = m_2 v_{2f} + m_1 v_{1f}$$
$$v_{1i} = v_{2f} - v_{1f}$$

Eliminating $\boldsymbol{v_{2f}}$ we find:

$$(m_1 - m_2) v_{1i} = (m_1 + m_2) v_{1f}$$
$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$
$$v_{1f} = 0$$

Clearly the solution for $\boldsymbol{v_{2f}}$ is:

$$v_{2f} = v_{1f} + v_{1i} = v_{1i}$$

The first object stops, transferring all of its energy and momentum to the second object.

Elastic Collisions in One-Dimension

Case 3 $m_1 >> m_2$ and $v_{2i} = 0$


(c) Case 3

The conservation equations are:

$$m_1 v_{1i} = m_2 v_{2f} + m_1 v_{1f}$$
$$v_{1i} = v_{2f} - v_{1f}$$

Eliminating $\boldsymbol{v_{1f}}$ we find:

$$2m_1 v_{1i} = (m_1 + m_2) v_{2f}$$
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} \simeq 2v_{1i}$$

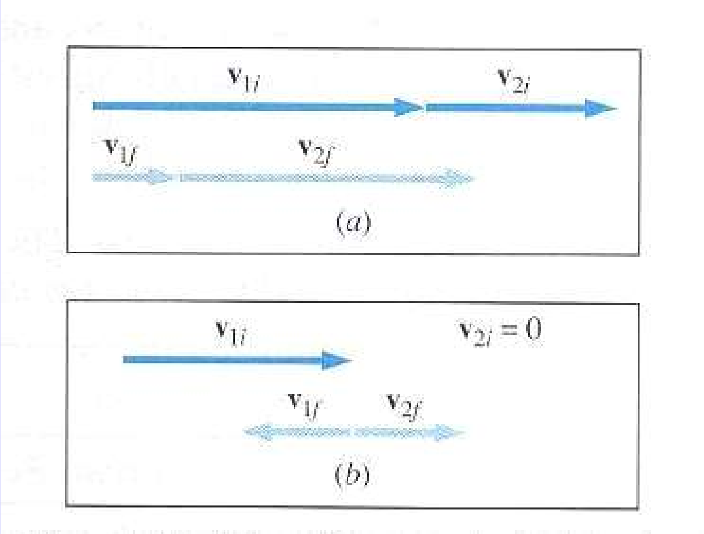
The solution for $\boldsymbol{v_{1f}}$ is:

$$v_{1f} = v_{2f} - v_{1i} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} \simeq v_{1i}$$

How are energy and momentum conserved? Essentially all of the energy and momentum remain with the more massive object.

Elastic Collisions in One-Dimension
Mass of an Elementary Particle

A particle of unknown mass moving at ***460 km/s*** collides head-on with a carbon nucleus moving at ***220 km/s*** in the same direction. After the collision the carbon continues on at ***340 km/s***
Find the mass of the unknown particle.



The conservation equations are:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

If carbon is particle 2 then eliminate $\boldsymbol{v_{1f}}$ and find:

$$2m_1 v_{1i} + (m_2 - m_1) v_{2i} = (m_2 + m_1) v_{2f}$$

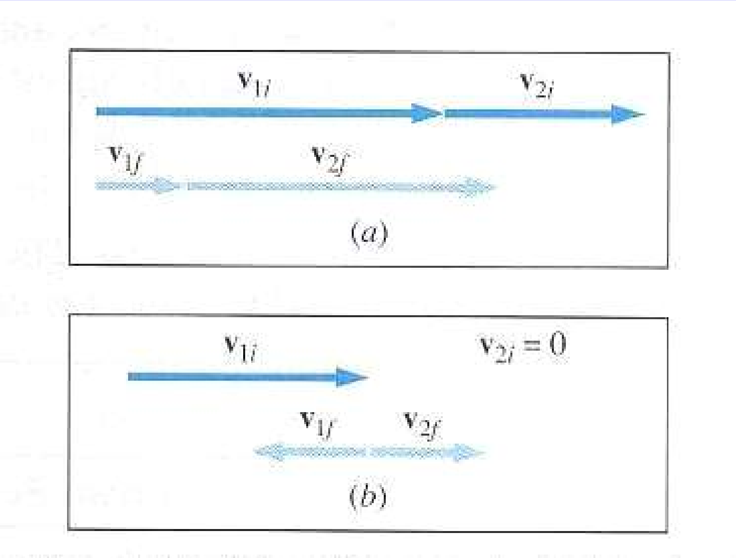
Solving for $\boldsymbol{m_1}$:

$$m_1 = m_2(v_{2f} - v_{2i})/(2v_{1i} - v_{2i} - v_{2f})$$
$$m_1 = 12(340 - 220)/(920 - 220 - 340) = 4.0$$

An alpha particle.

Elastic Collisions in One-Dimension
Mass of an Elementary Particle

A particle of unknown mass moving at ***460 km/s*** collides head-on with a carbon nucleus moving at ***220 km/s*** in the same direction. After the collision the carbon continues on at ***340 km/s*** Find the final velocity of the unknown particle.



The conservation equations are:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

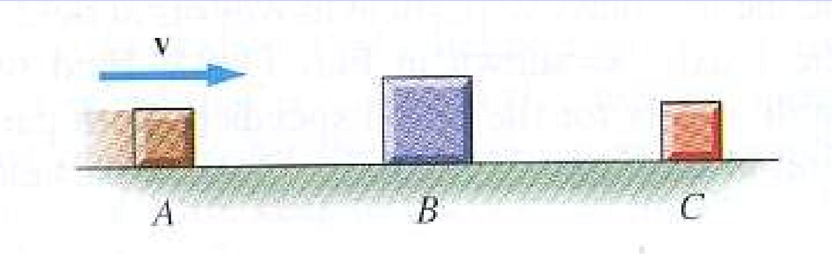
$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

From the second conservation equation $\boldsymbol{v_{1f}}$ is given by:

$$v_{1f} = v_{2f} - v_{1i} + v_{2i} = 340 - 460 + 220 = 100 km/s$$

Elastic Collisions in One-Dimension
Three Blocks

Blocks B and C have masses ***2m*** and ***m*** respectively and are at rest on a frictionless surface. Block A also of mass ***m*** is heading at block B with a velocity ***v***. Assuming that all collisions are elastic what is the final velocity of each block



The conservation equations for each collision are:

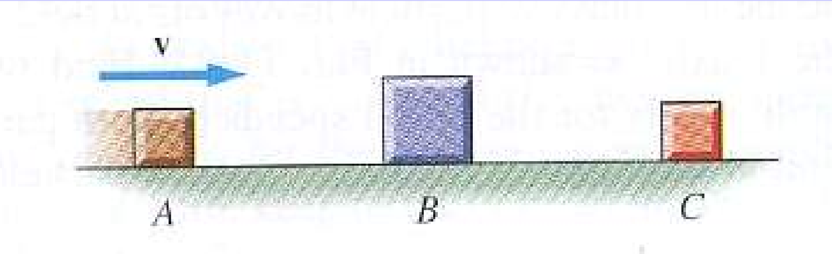
$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \text{ and } v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

For the first collision these become: $v = v_{Af} + 2v_{Bf}$ and $v = v_{Bf} - v_{Af}$

With solutions: $v_{Bf} = 2v/3$ and $\boxed{v_{Af} = -v/3}$

Elastic Collisions in One-Dimension
Three Blocks

Blocks B and C have masses ***2m*** and ***m*** respectively and are at rest on a frictionless surface. Block A also of mass ***m*** is heading at block B with a velocity ***v***. Assuming that all collisions are elastic what is the final velocity of each block



The conservation equations for each collision are:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \text{ and } v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

For the second collision these become: $4v/3 = 2v_{Bf} + v_{Cf}$ and $2v/3 = v_{Cf} - v_{Bf}$

With solutions: $\boxed{v_{Cf} = 8v/9}$ and $\boxed{v_{Bf} = 2v/9}$

Note that the total final momentum is equal to *mv*.

Elastic Collisions in Two-Dimensions

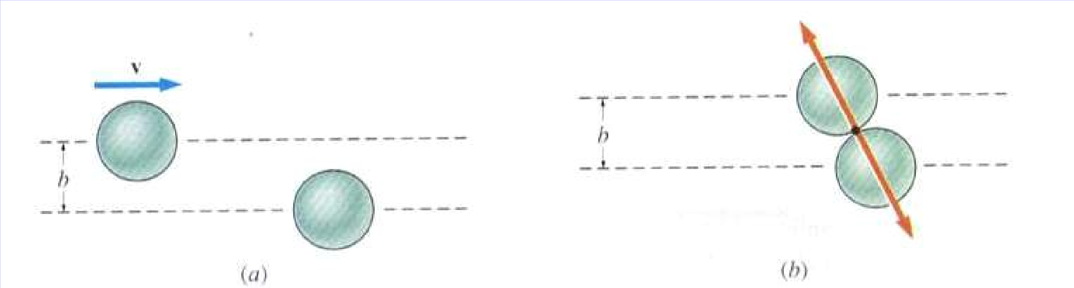
For an elastic collisions in two-dimensions both energy and momentum are conserved, but now the conservation of momentum is a two-dimensional vector equation.

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1 v_{1i}^2 + \frac{1}{2}m_2 v_{2i}^2 = \frac{1}{2}m_1 v_{1f}^2 + \frac{1}{2}m_2 v_{2f}^2$$

Unlike elastic collisions in one dimension there are now three equations for four unknowns.

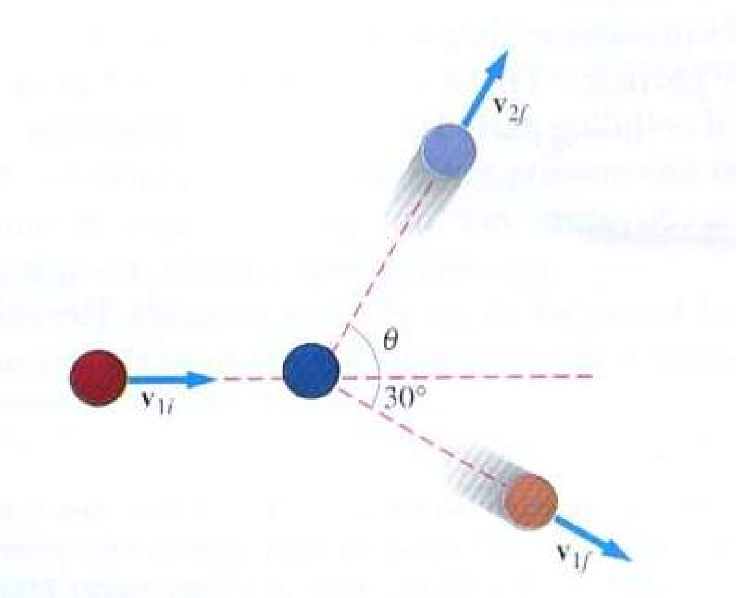
Now there is an impact parameter ***b*** which is the distance between the centers of the spheres.



Knowing the direction of one of the particles after the collision provides enough information to analyze the collision if the masses and initial conditions are known.

Elastic Collisions in Two-Dimensions Example: Billards

The cue ball strikes the five ball (of equal mass) and the cue ball continues on 30° to its original direction. In what direction does the five ball move?



This is an elastic collision and we could grind through the conservations equations. But let's be a little clever. The equations are:

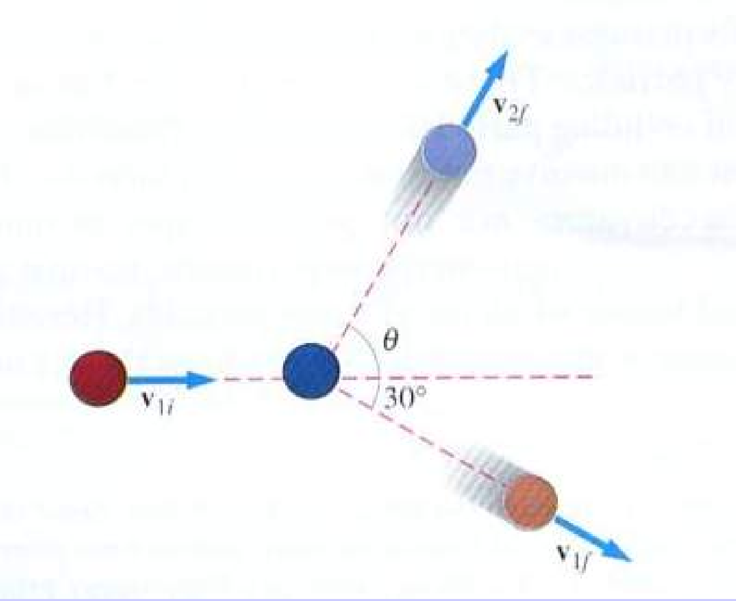
$$\vec{v}_{1i} = \vec{v}_{1f} + \vec{v}_{2f} \quad \text{and} \quad v_{1i}^2 = v_{1f}^2 + v_{2f}^2$$

Taking the scalar product of the momentum equation with itself yields

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2\vec{v}_{1f} \cdot \vec{v}_{2f}$$

Elastic Collisions in Two-Dimensions Example: Billards

The cue ball strikes the five ball (of equal mass) and the cue ball continues on 30° to its original direction. In what direction does the five ball move?



Taking the scalar product of the momentum equation with itself yields

$$v_{1i}^2 = v_{1f}^2 + v_{2f}^2 + 2\vec{v}_{1f} \cdot \vec{v}_{2f}$$

Comparing this relation with the conservation of energy yields:

$$\vec{v}_{1f} \cdot \vec{v}_{2f} = v_{1f} v_{2f} \cos(\theta + 30^\circ) = 0$$

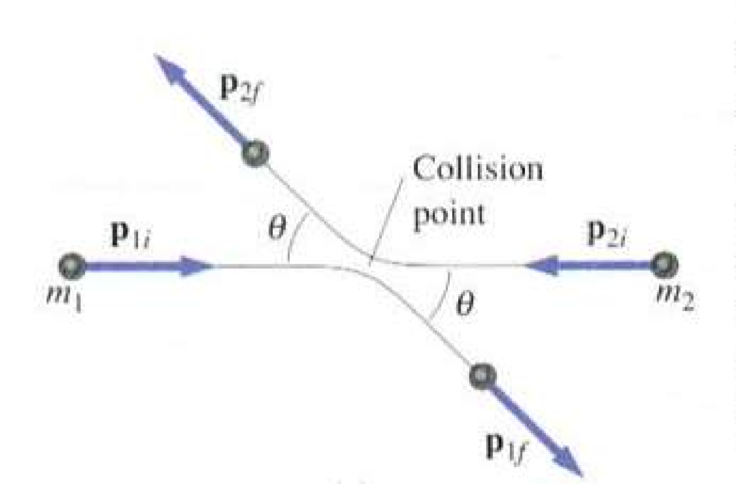
Hence: $$\theta + 30^\circ = 90^\circ \rightarrow \theta = 60^\circ$$

The unknown speeds can now be determined from the original conservation of momentum equations.

Couldn't v_{1f}=0? Yes, but that solution was covered in the one-dimension case.

Center of Mass Frame

Two dimensional collisions take a particularly simple form in the center-of-mass frame. In that frame the total momentum is zero. The conservation of momentum guarantees that the total momentum both before and after collision remains zero. Therefore with two particles they always have equal and opposite momentum.

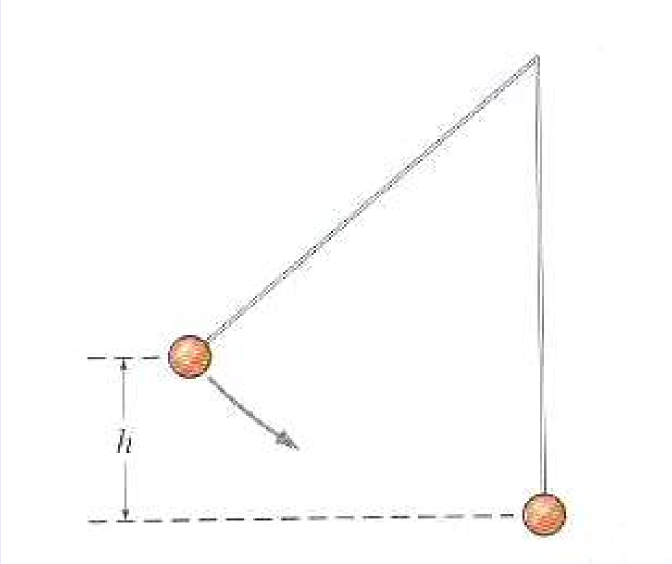


This means that it is almost always easier to analyze a collision by transforming to center-of-mass frame, do the analysis and then transform back to the original or "lab" frame.

As an example consider an incident particle with mass $2m$ and velocity v. It makes a 1D collision with a stationary particle of mass m. The center of mass velocity is $v_{cm} = 2v/3$. In the cm frame $v_{1i} = v/3$ and $v_{2i} = -2v/3$. After the collision the velocities in the cm frame are $v_{1f} = -v/3$ and $v_{2f} = 2v/3$. In the "lab" frame the velocities are $v_{1f} = v/3$ and $v_{2f} = 4v/3$.

One Final Example

Two identical pendulum bobs are suspended from strings of equal length. One is released from a height ***h***. When it hits the second bob they stick together. What is the maximum height that the pair rise to on the opposite side?



Clearly the answer is ***h*/4**, WHY?