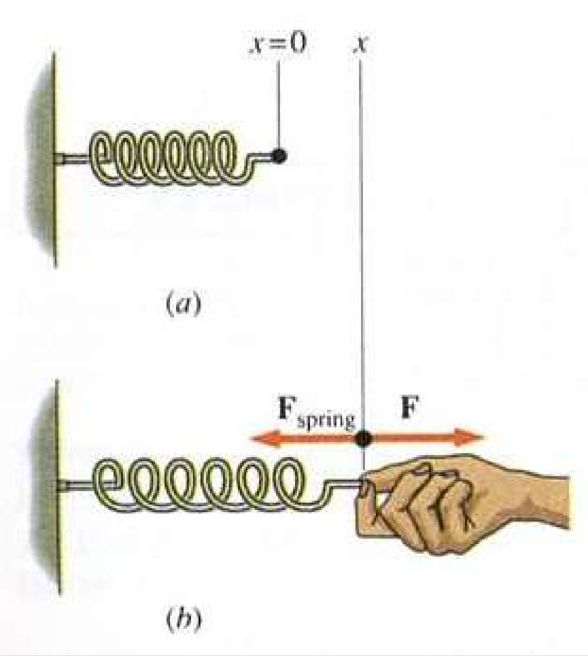
Today's Lecture

Lecture 17: Course Review Continued;
Hook's Law,
Nonconservative Forces,
System of Particles,

Chapter 11,
Impulse, Conservation of Momentum, Collisions

Springs
and
Hook’s Law

Example: Work to Stretch a Spring

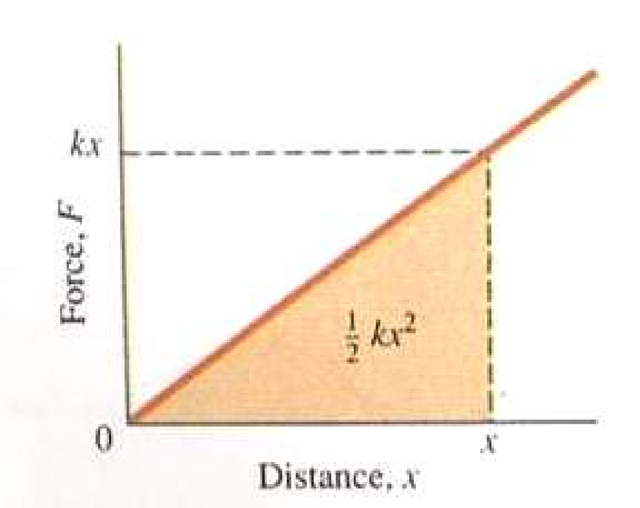


From Hook's Law a spring exerts a force proportional to its displacement from equilibrium:

$$F = -kx$$

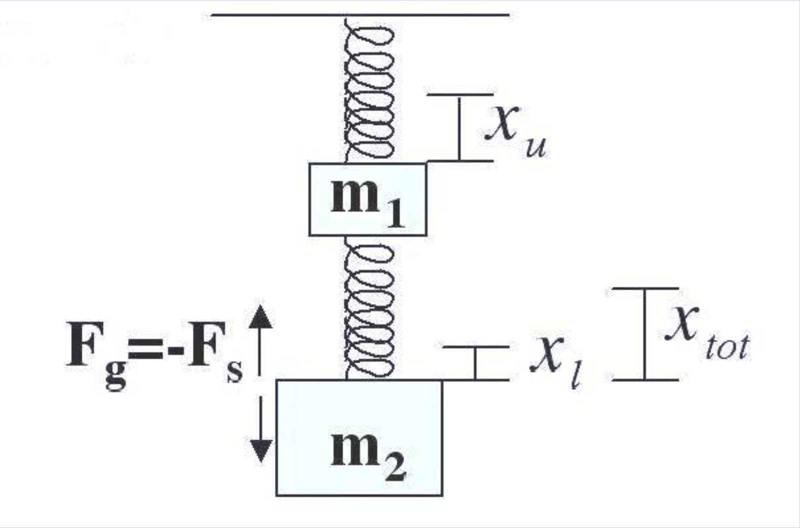
This is the force by the spring on the hand stretching it. From Newton's 3rd, the force exerted by the hand is ***kx***. The work done by the hand is the integral:

$$W = \int_0^x kx'\,dx' = \frac{1}{2}kx^2$$



What would the work be if the hand compressed the spring?

Example: Springs in Series with Two Masses



Two springs are in series, both with a spring constant of ***k = 20N/m.*** Mass ***m_1 = .2kg*** and mass ***m_2 = .4kg***. Find the displacement of the lower mass.

Newton's 2nd for the lower mass is:

$$T_l - m_2 g = kx_l - m_2 g = 0 \rightarrow x_l = m_2 g/k$$

Newton's 2nd for the upper mass is:

$$T_u - T_l - m_1 g = kx_u - kx_l - m_1 g = 0$$
$$T_u - m_2 g - m_1 g = 0 \rightarrow x_u = (m_1 + m_2)g/k$$

The total displacement of the lower mass is:

$$x_{tot} = (2m_2 - m_1)g/k = 9.8/20 = .49m = 49cm$$

Work – Kinetic Energy Theorem

Conservation of Energy

From the Work-Energy Theorem the work done on an object is equal to the change in its kinetic energy:

$$W = m\int \frac{d\vec{v}}{dt} \bullet d\vec{r} = \int \vec{F}_{net} \bullet d\vec{r} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

If we consider separately the work done by conservative forces, W_c, and non-conservative forces, $\boldsymbol{W_{nc}}$:

$$\Delta K = W_c + W_{nc}$$

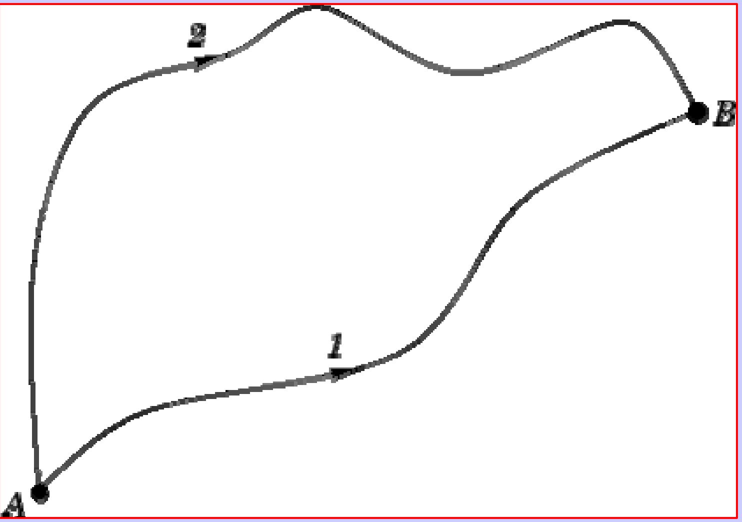
Potential energy was defined as the negative of the work done by conservative forces: $\boldsymbol{\Delta U = - W_c}$. Hence:

$$\Delta K + \Delta U = W_{nc}$$

In the absence of non-conservative forces, the total mechanical energy is conserved!

$$\boxed{\Delta K + \Delta U = 0 \rightarrow \frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f = E}$$

Conservative and Nonconservative Forces



If the work done between points **A** and **B** is path independent then we can state:

$$W_{AB} = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_2$$

The work done by the force ***F*** going from ***A*** to ***B*** and back to ***A***, $\boldsymbol{W_{ABA}}$, is:

$$W_{ABA} = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 - \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_2 = \int_A^B \left[\vec{F} \cdot d\vec{r}\right]_1 + \int_B^A \left[\vec{F} \cdot d\vec{r}\right]_2 = 0$$

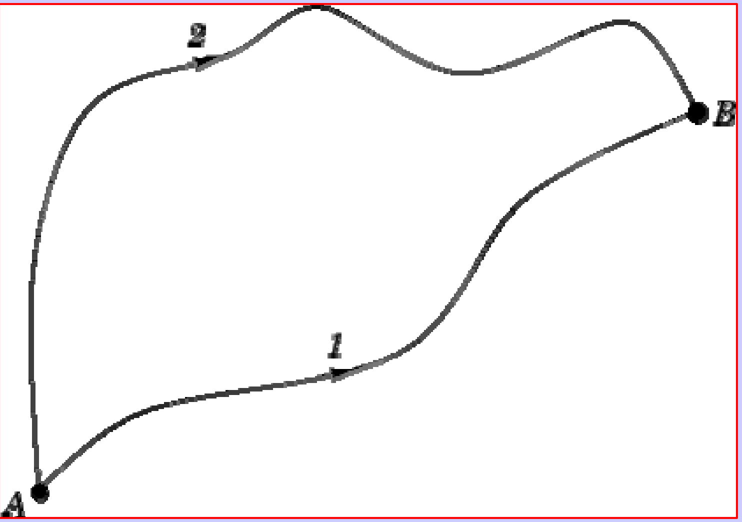
If the total work done by a force over a closed path vanishes,

$$\boxed{W_{ABA} = \oint \vec{F} \cdot d\vec{r} = 0!}$$

the force is said to be **conservative!**

Work done by frictional forces is proportional to ***–dr*** in both directions, ergo – **nonconservative!**

Potential Energy



The negative of the work done by a conservative force along any arbitrary path between two points is defined to be the change in the **potential energy** (associated with that force) between those two points.

$$\Delta U = -\int_A^B \vec{F} \cdot d\vec{r}$$

The difference in potential energy depends only on the location of the endpoints! Also the zero of **U** is arbitrary. It is usually chosen for convenience.

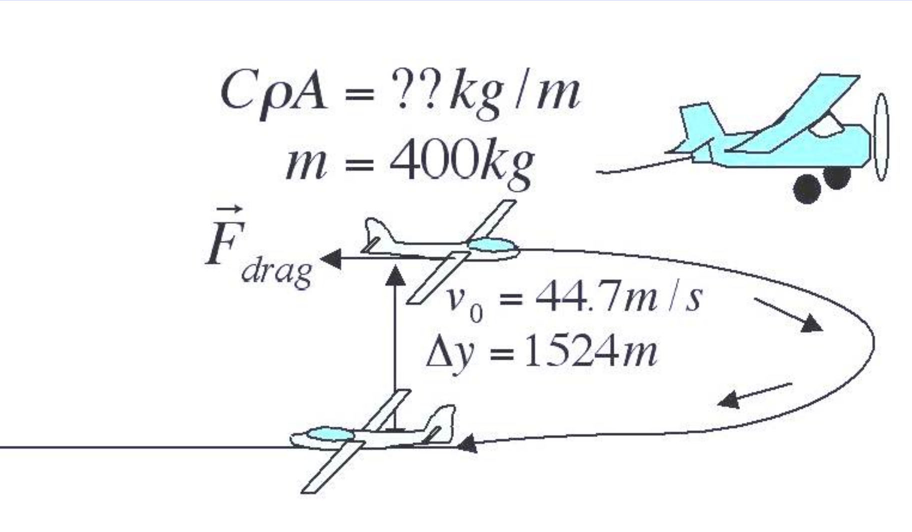
The change in gravitational potential energy:

$$\Delta U_g = -\int_0^y \left(-mg\hat{j}\right) \cdot d\vec{r} = mg\int_0^y dy = mgy$$

The change in a spring's potential energy:

$$\Delta U_k = -\int_0^x \left(-kx\hat{i}\right) \cdot d\vec{r} = k\int_0^x xdx = \frac{1}{2}kx^2$$

Example: Releasing the Glider



A glider with a mass of ***400kg*** is released at an altitude of ***1524m*** and an initial velocity of ***100mph = 44.7m/s***. If it lands on the runway at ***10m/s***, how much work was done by the drag force on the way down?

From the work energy theorem the change in kinetic energy is found from the net work on the glider. The work done by gravity is ***mgh***. Hence the nonconservative work is:

$$W_{nc} = \Delta K - mgh = \frac{1}{2}400(10^2 - 44.7^2) - 400(9.8)1524 = -6.35 \times 10^6 J$$

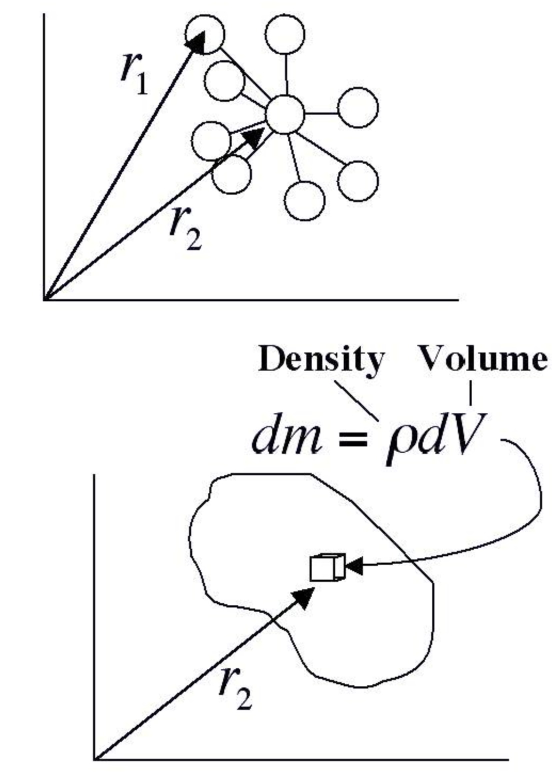
An interesting point to note is that by using the work-energy theorem we did not need to know the drag coefficient, the average velocity, or the distance traveled prior to landing!

Systems of Particles

Center of Mass

Mathematically we define the center of mass as the average of the mass weighted vector displacement of the individual particles. Defining the total mass as ***M***.

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{N} m_i$$



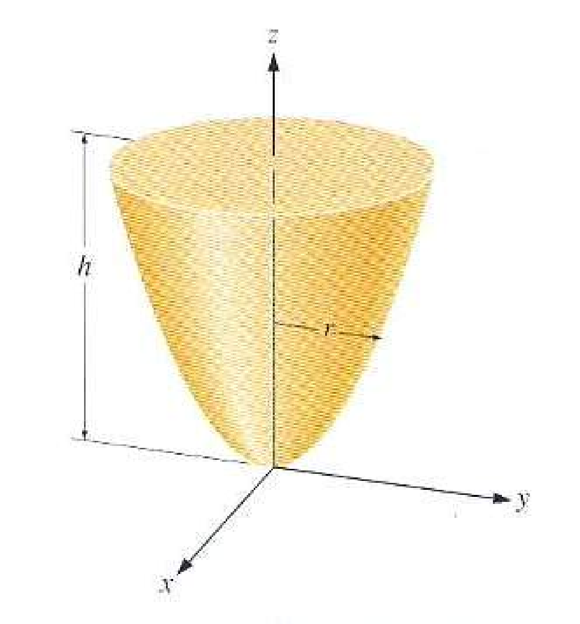
This allows us to define the center of mass as:

$$\vec{R}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots) = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

For continuous media both sums become integrals:

$$M = \int dm, \quad \text{and} \quad \vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Example: COM for Paraboloidal Solid



Consider a parabolidal solid of height ***h*** and uniform density $\boldsymbol{\rho}$. The height of the solid is given by $\boldsymbol{z = a\, r^2}$. Find (a) the mass, $\boldsymbol{M}$, of the solid and (b) the $\boldsymbol{z}$ coordinate, $\mathbf{Z_{cm}}$, for the center of mass.

(a) The total mass is found from the integral

$$M = \int \rho dV = \int \rho A(z) dz$$

where

$$A(z) = \pi r^2 = \frac{\pi}{a} z$$

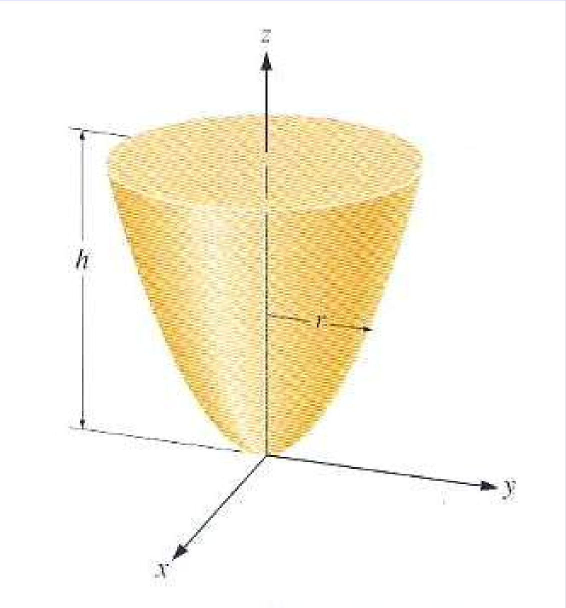
Performing this integral we find:

$$M = \int_0^h \rho \frac{\pi}{a} z dz = \rho \frac{\pi}{a} \frac{h^2}{2} = \rho V$$

(b) The center of mass $\mathbf{Z_{cm}}$ is found from the integral

$$Z_{cm} = \frac{1}{M} \int z \rho dV = \frac{1}{V} \int_0^h z A(z) dz = \frac{1}{V} \int_0^h \frac{\pi}{a} z^2 dz$$

Example: COM for Paraboloidal Solid



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(b) The center of mass $\boldsymbol{Z_{cm}}$ is found from the integral

$$Z_{cm} = \frac{1}{M}\int z\rho dV = \frac{1}{V}\int_0^h zA(z)dz = \frac{1}{V}\int_0^h \frac{\pi}{a}z^2 dz$$

Evaluating this integral we find

$$Z_{cm} = \frac{1}{V}\frac{\pi}{3a}h^3 = \frac{\pi}{3a}h^3/\left(\frac{\pi}{2a}h^2\right) = \frac{2}{3}h$$

For the cone $\boldsymbol{Z_{cm} = 3h/4}$. In light of this result, does the height for the paraboloidal solid make sense?

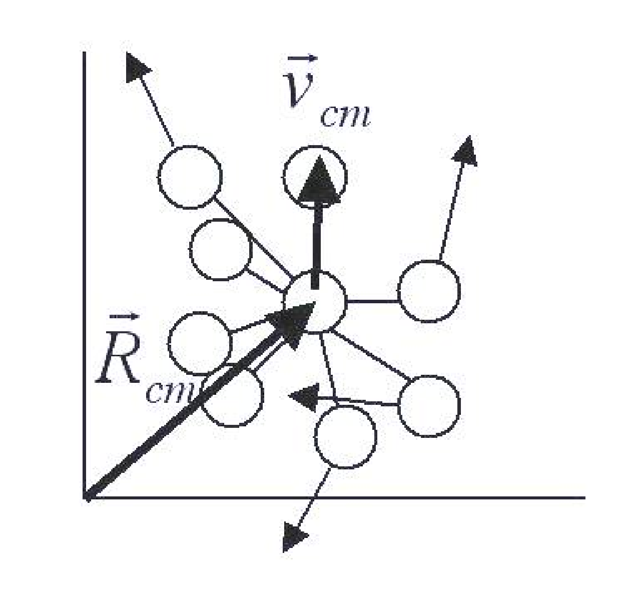
Motion of the Center of Mass

The total momentum of a system of particles is equal to the momentum of the center of mass. In the absence of any **net external force** this momentum is **conserved**.

To see this consider the time derivative of the center of mass:

$$\frac{d}{dt}\vec{R}_{cm} = \vec{v}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

$$\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i$$



Even though individual particles may be moving relative to the center of mass, the center of mass maintains a uniform velocity.

External Forces and the Center of Mass

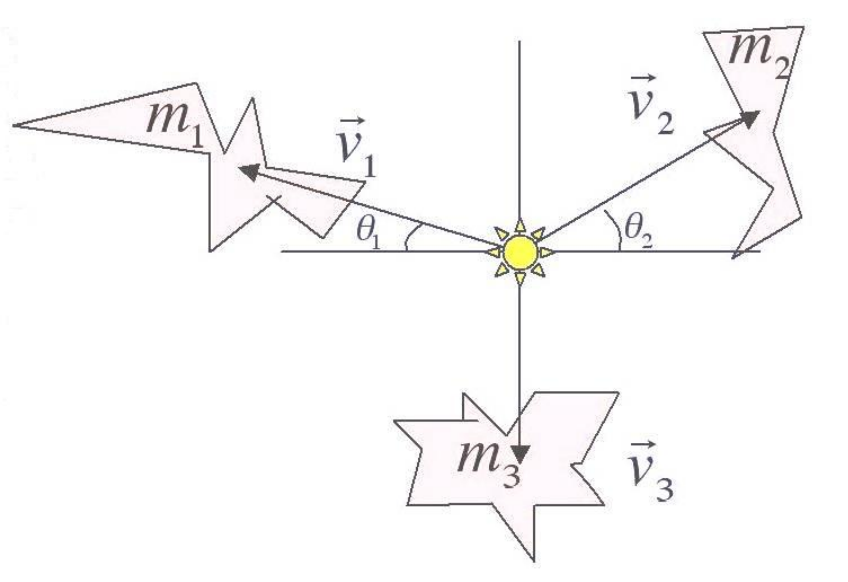
The sum of all the net forces on each of the particles determines the acceleration of the center of mass.

$$\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{v}_i = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{i-net}$$

However, we need to consider the sum of the forces on each of the particles. Some of the forces on the ***i****th* particle are due to external forces (e.g. external gravitational field). There are also forces between the particles themselves (at least a gravitational attraction). This could make the problem virtually intractable, but Newton's 3rd comes to the rescue. It is the basis for recognizing that **the sum of the internal forces over all of the particles cancel!** It is only the sum of all the external forces that induce an acceleration of the center of mass.

$$\boxed{\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} \vec{F}_{i-ext}}$$

Example: Asteroid Explodes



An explosion breaks up an asteroid at rest into three pieces whose centers of mass travel away in a plane as shown. Find the expressions for the speeds of masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$ as a function of mass $\boldsymbol{m_3}$ and the angles θ_1 and θ_2.

Recognizing that the initial momentum is zero and there are no external forces:

$$\vec{P}_1 + \vec{P}_2 + \vec{P}_3 = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = 0$$

In component form we have 2 algebraic equations and 2 unknowns:

$$x: \quad -m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 = 0$$

$$y: \quad m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 = m_3 v_3$$

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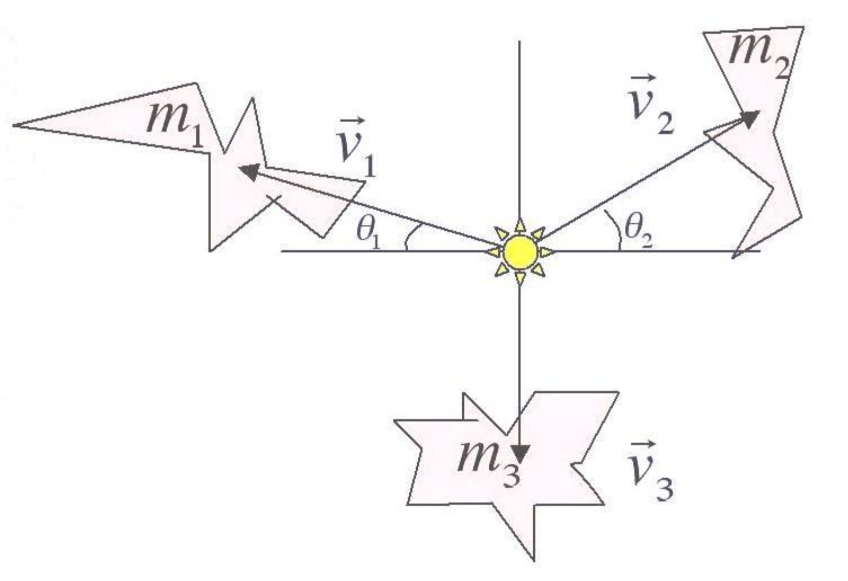

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$$x: \quad -m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 = 0$$

$$y: \quad m_1 v_1 \sin\theta_1 + m_2 v_2 \sin\theta_2 = m_3 v_3$$

Solving these algebraic equations with the determinate approach:

$$\det = -m_1 m_2 \sin(\theta_1 + \theta_2)$$

and:

$$v_1 = -m_3 m_2 v_3 \cos\theta_2 / \det = \frac{m_3}{m_1} \frac{v_3 \cos\theta_2}{\sin(\theta_1 + \theta_2)}$$

$$v_2 = -m_1 m_3 v_3 \cos\theta_1 / \det = \frac{m_3}{m_2} \frac{v_3 \cos\theta_1}{\sin(\theta_1 + \theta_2)}$$

Example: Rockets

A spacecraft designed to probe interstellar medium is moving away from the Sun with a velocity of $\boldsymbol{v_i = 35\ km/s}$ (at the orbit of Pluto). At that point an advanced rocket exhaust fuel at a velocity of $\boldsymbol{v_{ex} = 47\ km/s}$. If the mass of the rocket (sans fuel) is ***750kg***, how much fuel must be used to accelerate the rocket to ***150 km/s*** ?

From the rocket equation:

$$\Delta v = v_{ex} \ln \frac{M_i}{M_f} \rightarrow v_f - v_i = v_{ex} \ln \frac{M_i}{M_f}$$

$$\ln \frac{M_i}{M_f} = \frac{150 - 35}{47} = 2.45$$

Hence the fuel mass is:

$$M_f = M_i e^{-2.447} \rightarrow 750 = (750 + M_{fuel}) e^{-2.447}$$

$$M_{fuel} = 750 e^{2.447} - 750 = 7915 kg$$

Chapter 11:

Impluse, Collisions, and Conservations Laws

Impulse

Given a collision between two objects, we know that the force on the object is it's change in momentum with time:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

The impulse is the integrated change in momentum over the time during the collision:

Impulse: $I = \int \vec{F} dt = \int d\vec{p}$

$$\vec{I} = \Delta\vec{P} = \vec{P}_f - \vec{P}_i$$

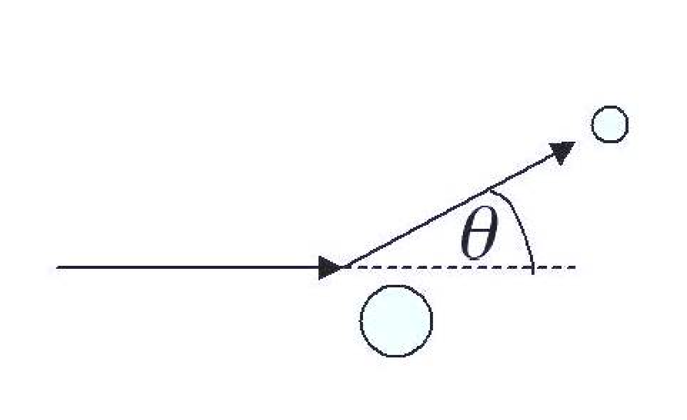
The average impulse force is this integrated change in momentum divided by the total time:

$$\vec{F}_{ave} = \frac{\vec{I}}{\Delta t}$$

Note: compare impulse to work: Impulse is analog in time instead of space. Momentum instead of Energy

Example: Impulse

6. A proton moving the positive x direction at 4.3Mm/s collides with a nucleus. The collision lasts 0.12fs. And the average impulse force is $\vec{F}_{av} = (42\hat{i} + 17\hat{j})\mu N$ **(a) Find the velocity of the proton after the collision. (b) Through what angle has the proton's motion been deflected?**



In this case the average force and collision time is given. The impulse is then:

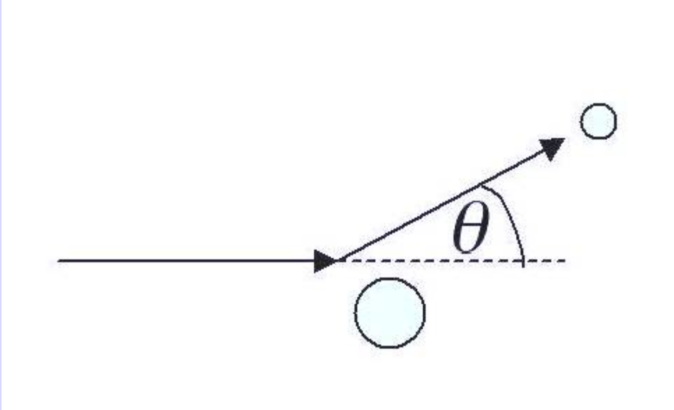
$$\vec{I} = \Delta\vec{p} = m\vec{v}_f - m\vec{v}_i = \vec{F}_{av}\Delta t$$

Solving for the final velocity:

$$\vec{v}_f = \vec{v}_i + \frac{\vec{F}_{av}\Delta t}{m}$$

$$\vec{v}_f = (4.3x10^6 m/s)\hat{i} + \frac{(42\hat{i} + 17\hat{j})(10^{-6}N)(0.12x10^{-15}s)}{1.67x10^{-27}kg}$$

Example: Impulse



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Thus the final velocity is:

$$\boxed{\vec{v}_f = (7.32\hat{i} + 1.22\hat{j})(10^6)m/s}$$

The deflection angle can be gotten from the velocity vector:

$$\boxed{\theta_f = \tan^{-1}(\frac{1.22}{7.32}) = 9.48^o}$$

The impulse itself is not asked for, but is equal to:

$$\vec{I} = \Delta\vec{p} = \vec{F}_{av}\Delta t = (5\hat{i} + 2\hat{j})(10^{-9})Ns$$

Inelastic vs. Elastic Collisions

Inelastic collisions: the momentum is conserved, not the energy

Elastic collisions: the momentum and energy are conserved

For a totally inelastic collision between two objects, the objects stick together and become one mass. Only momentum conserved:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}_f$$

For a totally elastic collision between two objects, the objects are unchanged in form, and rebound perfectly while conserving kinetic energy as well as momentum:

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

$$\frac{1}{2}m_1v_{1i}^2 + \frac{1}{2}m_2v_{2i}^2 = \frac{1}{2}m_1v_{1f}^2 + \frac{1}{2}m_2v_{2f}^2$$

Inelastic collisions are rarely totally inelastic, elastic collisions are rarely totally elastic. Reality is somewhere between and these are the limits.

Example: Inelastic Collision

20) In an ice show stunt, a 70kg skater catches a 150g baseball moving at 23m/s. (a) If the skater was initially at rest, what is his final speed? (b) If the catch takes 36ms, what is the average impulsive force exerted by the ball?

Assume that the skating is frictionless and aligned with the direction of the ball. The skater catches the ball so the collision is totally elastic.

(a) $$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

Initially at rest

$$\vec{v}_f = \frac{m_1\vec{v}_{1i} + \cancel{m_2\vec{v}_{2i}}^{\,0}}{(m_1 + m_2)} = \frac{m_1}{(m_1 + m_2)}\vec{v}_{2i}$$

$$v_f = \frac{0.15}{70.15}(23)m/s = \boxed{0.05m/s}$$

(b) $$F_{av} = \frac{\Delta p}{\Delta t} = \frac{(70kg)(0.05m/s)}{(0.036s)} = \boxed{95.6N}$$

Note: in the case of inelastic collisions in one dimension, typically there is one equation for one unknown

Elastic Collisions in One-Dimension

For an elastic collisions in one-dimension both energy and momentum are conserved:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$
$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Note that for elastic collisions in one dimension there are two equations for two unknowns.

Rearranging and simplifying:

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$
$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right)$$

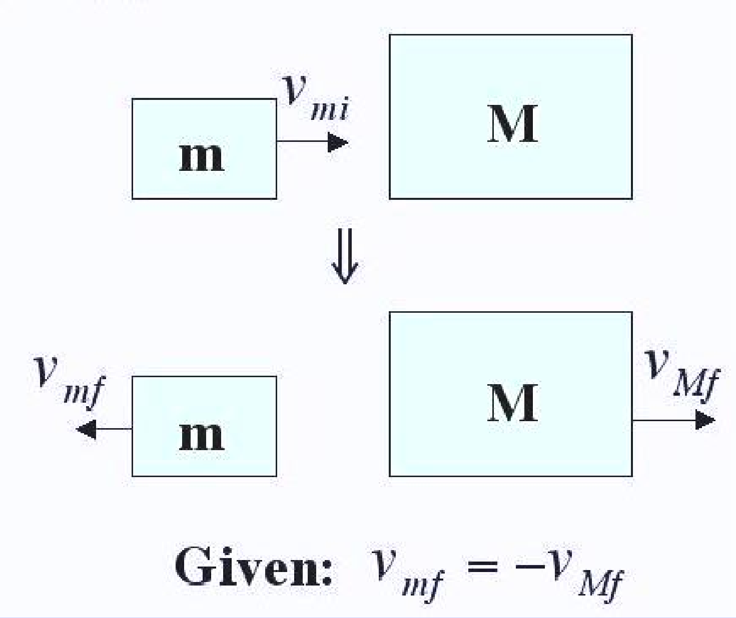
Now we can divide the second equation by the first and then rearrange:

$$m_1 v_{1i} + m_2 v_{2i} = m_2 v_{2f} + m_1 v_{1f}$$
$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

This pair of equations (linear) is much easier to work with than the original. Also note that the relative velocities become reversed!

Elastic Collisions in One-Dimension

Example:



In a one dimensional elastic collision ***M*** is initially at rest. If both masses end up with the same speed, $\boldsymbol{v_{mf} = - v_{Mf}}$, how are ***m*** and ***M*** related?

With $\boldsymbol{v_{Mi} = 0}$ and $\boldsymbol{v_{mf} = - v_{Mf}}$, our equations become:

$$mv_{mi} = mv_{mf} + Mv_{Mf} \rightarrow mv_{mi} = mv_{mf} - Mv_{mf}$$

$$v_{mi} = v_{Mf} - v_{mf} \rightarrow v_{mi} = -2v_{mf}$$

Substituting for $\boldsymbol{v_{mi}}$ we find:

$$-2mv_{mf} = mv_{mf} - Mv_{mf}$$

with the result

$$M = 3m$$

As we already noted, this pair of equations is much easier to work with than the original conservation of energy and conservation of momentum!