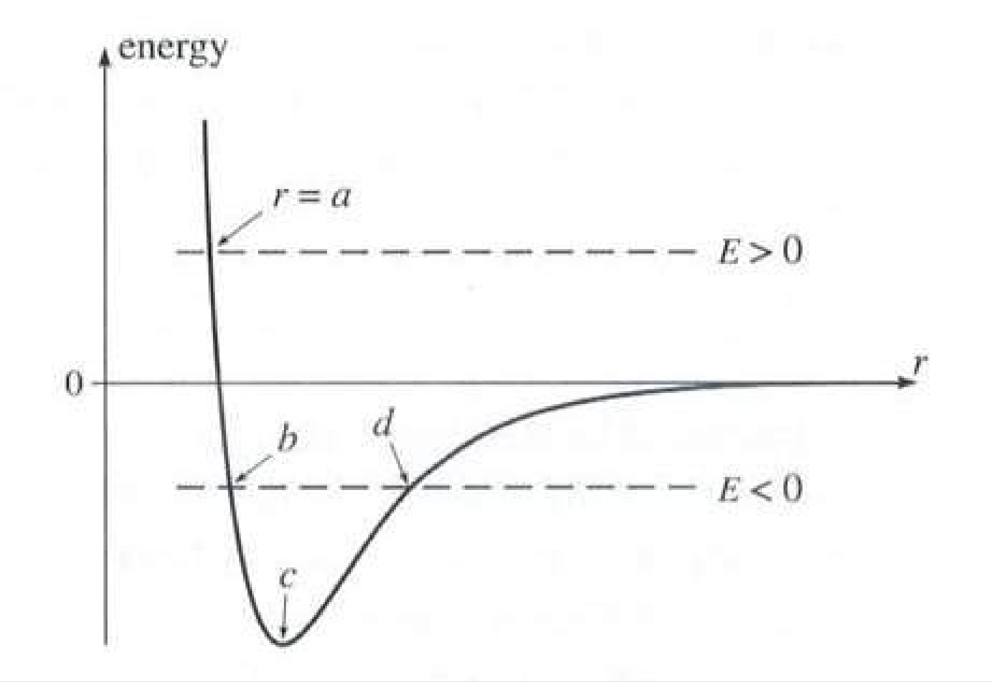
Today's Lecture

Lecture 15: Chapter 8,
Review of Energy Diagrams
Review of Work-Energy Thm

Chapter 10,
System of Particles

Energy Diagram - Typical Diatomic Molecule



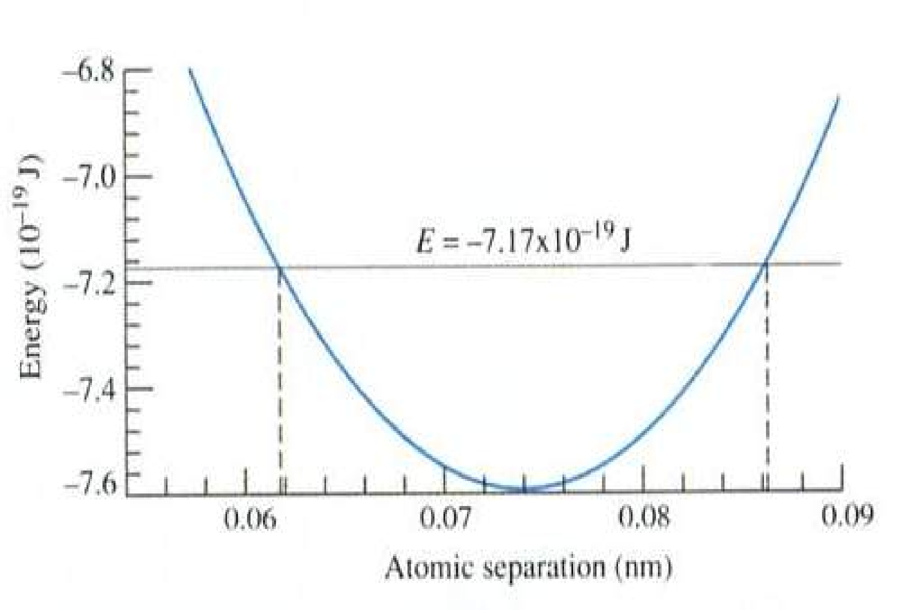
If $\boldsymbol{E > 0}$, then the atoms cannot approach closer than $\boldsymbol{r = a}$. The atoms can approach arbitrarily large separations – the molecule is not bound.

If $\boldsymbol{E < 0}$, then the atoms are trapped between the turning points at $\boldsymbol{b}$ and $\boldsymbol{d}$. The atoms form a bound system.

The equilibrium separation occurs at $\boldsymbol{r = c}$ where $\boldsymbol{dU/dr = 0}$.

What is the force between the atoms at $\boldsymbol{r = c}$?

Energy Diagram - Typical Diatomic Molecule



Near the bottom of the potential well, $\boldsymbol{r = r_c}$ the potential energy is given approximately by:

$$U = U_0 + a(r - r_c)^2, \quad U_0 = -7.6 \times 10^{-19} J,$$
$$a = 2.86 \times 10^{-16} J/nm^2, \quad r_c = .0741 nm.$$

What range of atomic separations is allowed if the total energy is ***E = -7.17x10^{-19} J***?

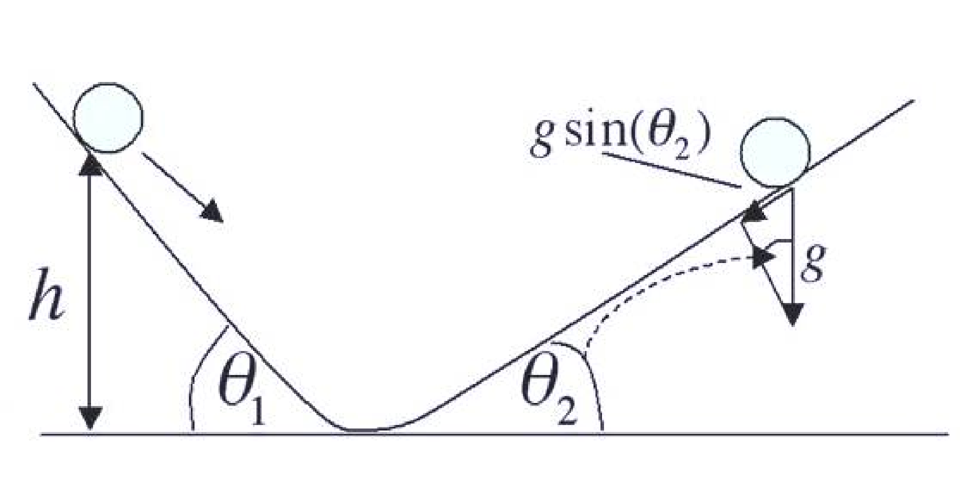
Kinetic energy vanishes at the turning points, ***E = U*** there. Solving for $\boldsymbol{r - r_c}$:

$$r - r_c = \pm\sqrt{(E - U_0)/a} = \pm\sqrt{.44 \times 10^{-19}/2.86 \times 10^{-16}} = \pm 1.24 \times 10^{-2} nm$$

Hence:

$$r_{\min} = .0741 - .0124 = .0617 nm$$
$$r_{\max} = .0741 + .0124 = .0865 nm$$

Energy Conservation on a Track



If a ball is released from a height ***h*** on a frictionless track find the expression for the ratio of the time spent on either side during the oscillatory motion.

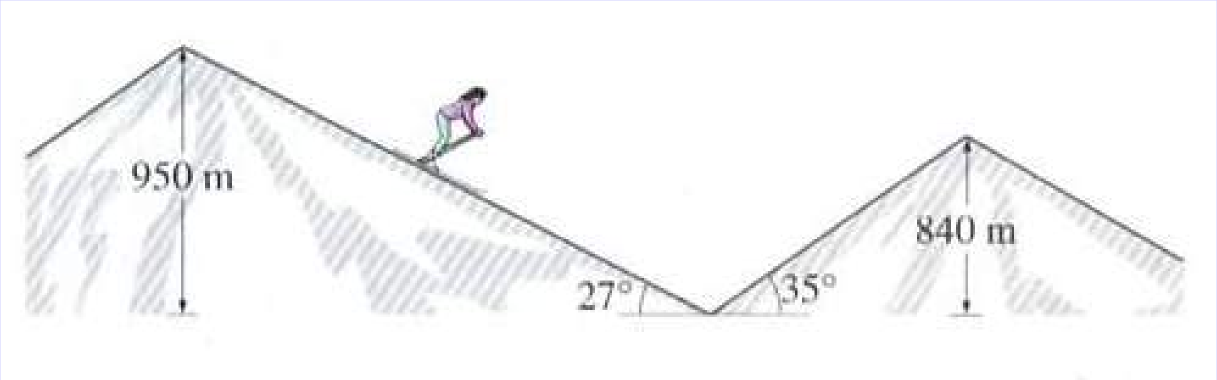
From the conservation of energy the ball will rise to the same height on either side. However the accelerations will be different!

The velocities at the bottom are the same, while their respective accelerations (uniform) are $\boldsymbol{a_i = g \sin\theta_i}$. Hence:

$$v_{bot} = a_i t_i \rightarrow \frac{t_1}{t_2} = \frac{a_2}{a_1} = \frac{g\sin\theta_2}{g\sin\theta_1} = \frac{\sin\theta_2}{\sin\theta_1}$$

As expected the side with the steeper angle has the shorter time. Kinematics could also solved this problem, albeit more complicated.

Skier and Two Peaks



If a skier starts from rest at the top of the left-hand peak, what is the maximum value of the coefficient of kinetic friction, $\boldsymbol{\mu_k}$, that would allow the skier to reach the second peak?

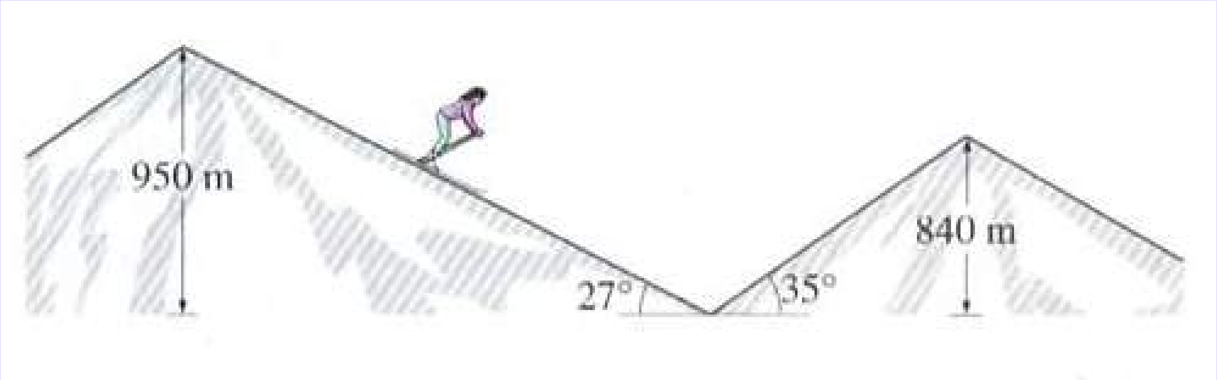
From the work-energy theorem: $$\Delta K + \Delta U = W_{nc}$$

The kinetic energy is zero at both peaks. This means that the loss in potential energy (height) is dissipated into friction. With the use of some trigonometry:

$$-mg\Delta h = -\mu_k mg\cos\theta_1 d_1 - \mu_k mg\cos\theta_2 d_2$$

$$\Delta h = \mu_k(d_1\cos\theta_1 + d_2\cos\theta_2) = \mu_k(h_1\cot\theta_1 + h_2\cot\theta_2)$$

Skier and Two Peaks



If a skier starts from rest at the top of the left-hand peak, what is the maximum value of the coefficient of kinetic friction, $\boldsymbol{\mu_k}$, that would allow the skier to reach the second peak?

Solving for $\boldsymbol{\mu_k}$:

$$\mu_k = \frac{\Delta h}{h_1 \cot\theta_1 + h_2 \cot\theta_2} = \frac{110}{950\cot 27^\circ + 840\cot 35^\circ} = .036$$

To reiterate, we could have found the acceleration down the slopes and used kinematics to find this result. However, the work-energy theorem greatly facilitated obtaining the solution.

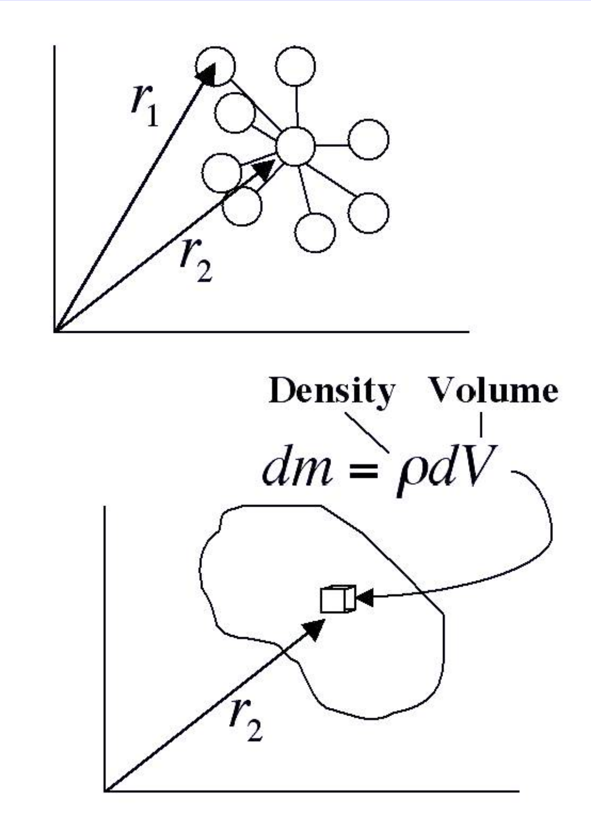
Chapter 10

System of Particles

Center of Mass

Mathematically we define the center of mass as the average of the mass weighted vector displacement of the individual particles. Defining the total mass as ***M***.

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{N} m_i$$



Allows us to define the center of mass as:

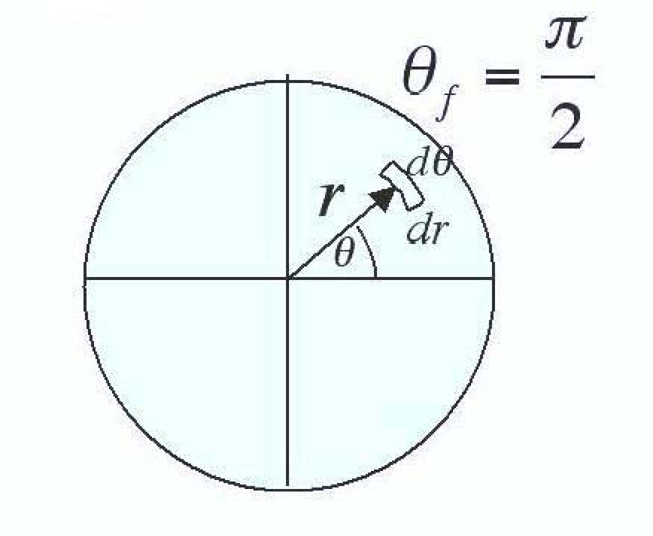
$$\vec{R}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots) = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

For continuous media
both sums become integrals:

$$M = \int dm, \quad \text{and} \quad \vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Example: Center of Mass for a Quarter Plate

A uniform flat disk of radius ***R*** and thickness ***l << R*** is cut into four quadrants. What is the radial location from the center of the original circle for the center of mass of the quadrants?



The center of mass is given by the integral:

$$\vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

First we need to determine ***dm***. For a uniform disk ***dm = ρ dV = ρ ldA,*** where $\rho = M/V = M/(\pi R^2 l/4) = 4M/(\pi R^2 l)$. In general for polar coordinates the area element and the position vector are:

$$dA = r\,dr\,d\theta, \quad \text{and} \quad \vec{r} = \vec{x} + \vec{y} + \vec{z} = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}$$

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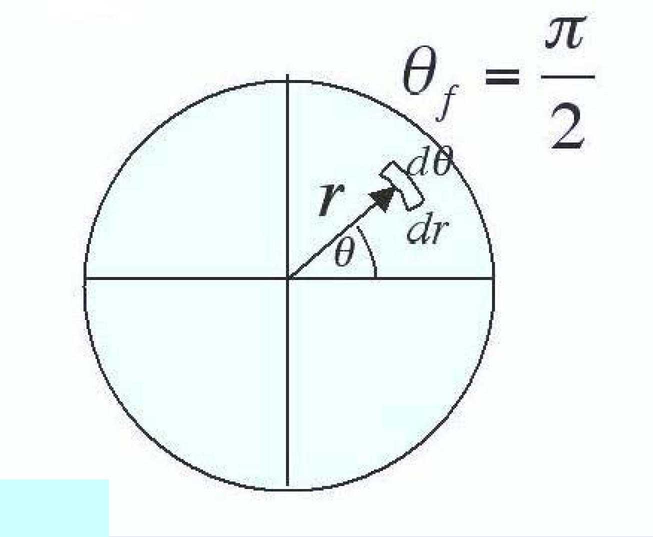

The center of mass integral becomes:

$$\vec{R}_{cm} = \frac{1}{M}\int_0^R\int_0^{\pi/2}\int_0^l \left(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}\right)\rho r\,dr\,d\theta\,dz$$

$$\vec{R}_{cm} = \frac{1}{V}\int_0^R\int_0^{\pi/2}\int_0^l \left(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + z\hat{k}\right) r\,dr\,d\theta\,dz$$

From symmetry (or the integral over ***z***) $\boldsymbol{Z_{cm}}$ is simply ***l*/2**. The integrals for $\boldsymbol{X_{cm}}$ and $\boldsymbol{Y_{cm}}$ are:

$$X_{cm} = \frac{l}{V}\int_0^R r^2dr\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}\frac{R^3l}{V} = \frac{1}{3}\frac{R^3l}{\pi R^2l/4} = \frac{4}{3\pi}R$$

$$Y_{cm} = \frac{l}{V}\int_0^R r^2dr\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{3}\frac{R^3l}{V} = \frac{1}{3}\frac{R^3l}{\pi R^2l/4} = \frac{4}{3\pi}R$$

Why are they equal?

Example: Center of Mass for a Quarter Plate

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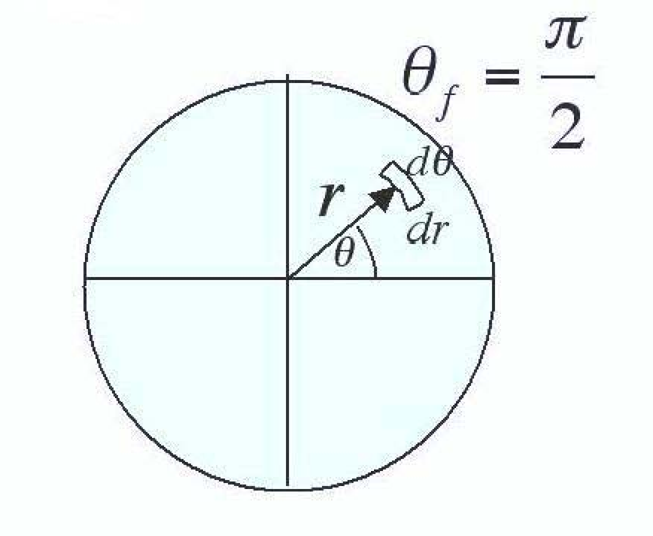


The center of mass for the quadrant is:

$$\vec{R}_{cm} = \frac{4}{3\pi}R\hat{i} + \frac{4}{3\pi}R\hat{j} + \frac{1}{2}L\hat{k}$$

The radial location for the quadrants center of mass is:

$$R_{cm} = \sqrt{X_{cm}^2 + Y_{cm}^2} = \frac{4\sqrt{2}}{3\pi}R \simeq .6R$$

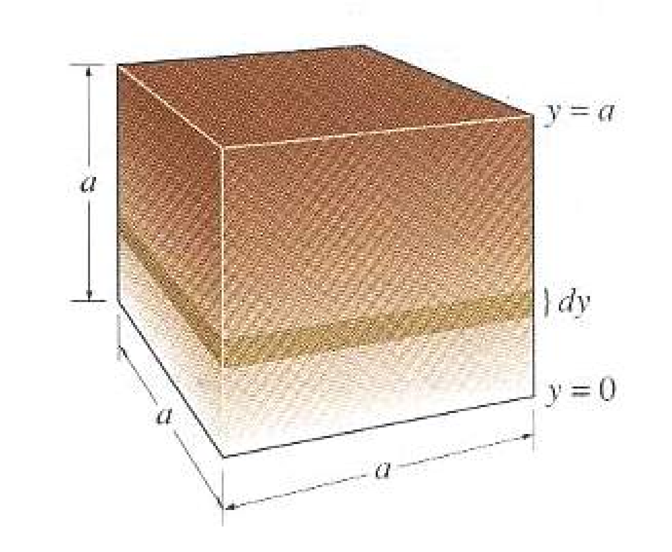
Does this make sense?

Example: A Cube with Varying Density

A solid cube of side ***a*** has a density that varies linearly from zero to $\boldsymbol{\rho_o}$ at the top.

(a) What is the mass of the cube?

(b) What is the vertical coordinate of its CM?



(a) The density varies as $\boldsymbol{\rho = \rho_o\, y/a}$. The volume element is of the cube is $\boldsymbol{dV = A\, dy = a^2\, dy}$. The mass is:

$$M = \int dm = \int \rho dV = \int_0^a \rho_o \frac{y}{a} a^2 dy = \rho_o a \int_0^a y dy = \rho_o a \frac{a^2}{2} = \frac{1}{2} \rho_o V$$

(b) The integral for the center of mass is:

$$y_{cm} = \frac{1}{M} \int y dm = \frac{\rho_o a}{M} \int_0^a y^2 dy = \frac{2a}{V} \frac{1}{3} a^3 = \frac{2}{3} a$$

Does this make sense?

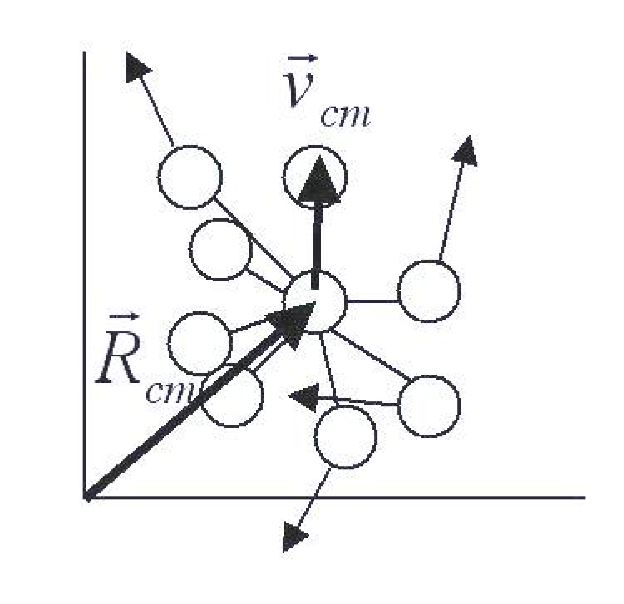
Motion of the Center of Mass

The total momentum of a system of particles is equal to the momentum of the center of mass. In the absence of any **net external force** this momentum is **conserved**.

To see this consider the time derivative of the center of mass:

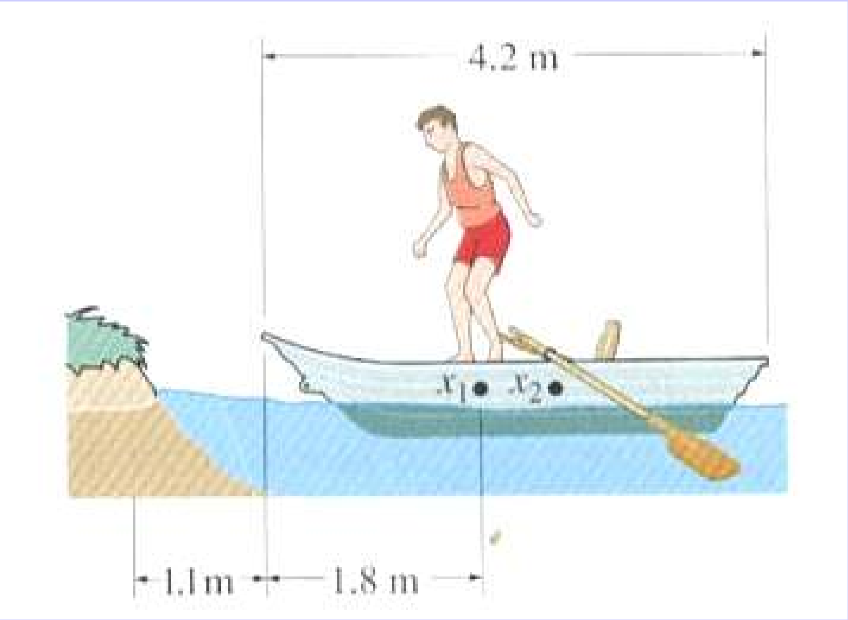
$$\frac{d}{dt}\vec{R}_{cm} = \vec{v}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

$$\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i$$



Even though individual particles may be moving relative to the center of mass, the center of mass maintains a uniform velocity.

Example: Motion of the Center of Mass



A ***70kg*** man is standing ***1.8m*** from the shoreward end of a ***150kg*** boat that is ***4.2m*** long. The boat is ***1.1m*** from the shore. The man walks to the shore end of the boat from which he leaps to shore. Assume that boat's CM is at its center.

(a) How far does the man have to leap?

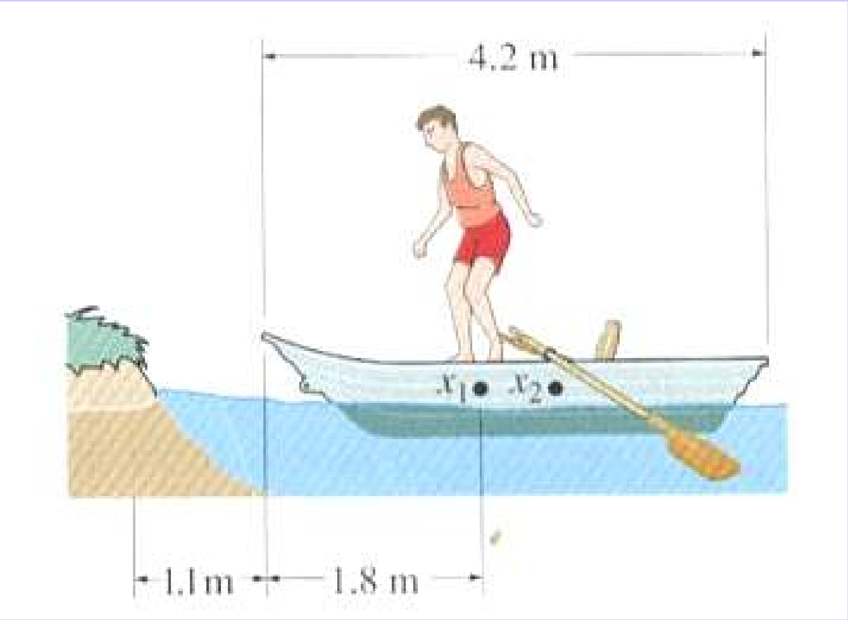
(b) Where is the boat at the instant you reach the shore?

(a) In the absence of external forces the CM of the system remains unchanged. Measuring distances from the shore to the front of the boat:

$$[Mx_{cm}]_i = m_m(1.1 + 1.8) + m_b(1.1 + 2.1) = 70(2.9) + 150(3.2) = 683$$

$$[Mx_{cm}]_f = m_m(x) + m_b(x + 2.1) = 315 + 220x = 683 \rightarrow \boxed{x = 1.67m}$$

Example: Motion of the Center of Mass



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(a) How far does the man have to leap?

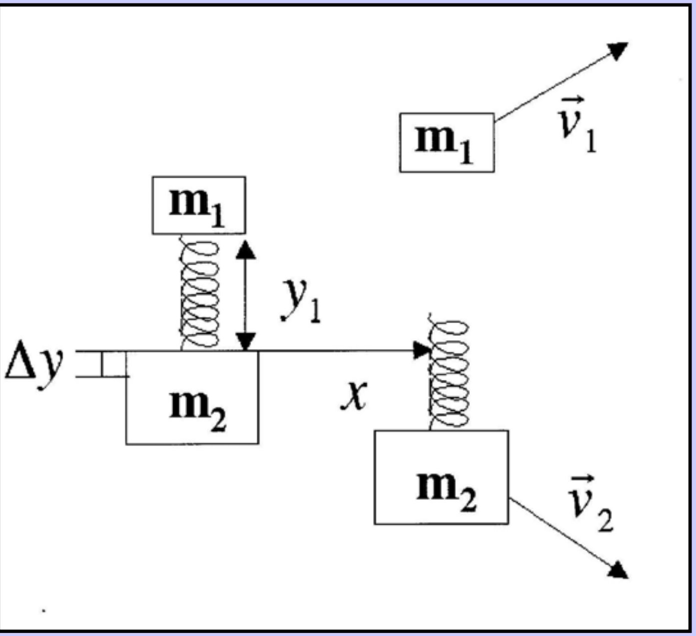
(b) Where is the boat at the instant you reach the shore?

(b) Again the absence of external forces the CM of the system remains unchanged. Measuring distances from the shore to the front of the boat:

$$[Mx_{cm}]_i = 683, \quad [Mx_{cm}]_f = m_b(x + 2.1) = 315 + 150x = 683 \rightarrow \boxed{x = 2.45m}$$

The boat will continue to drift away from shore after the mans lands. Yet the man is stationary! **What gives?**

Example: Motion of the Center of Mass



Two objects are moving in the *x* direction connected by a compressed spring in the *y* direction. Derive an expression for the velocities of the objects after the spring is released.

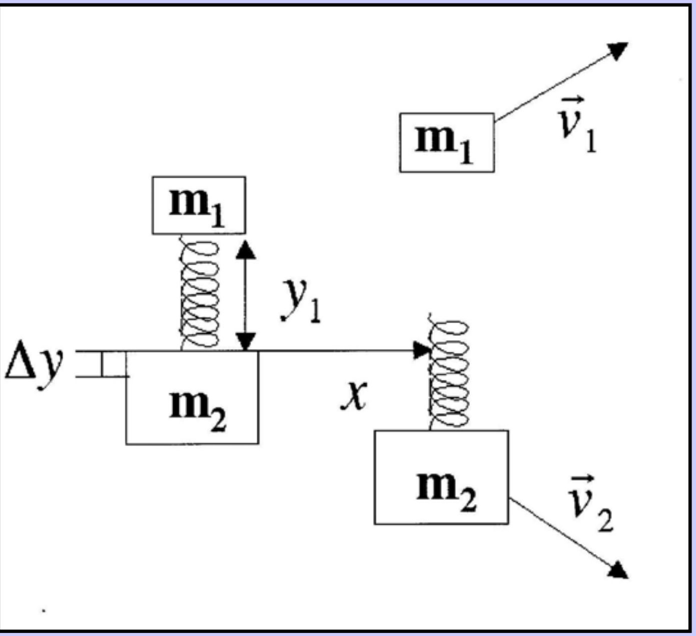
In the absence of external forces the motion of the center of mass remains unchanged. Also the mechanical energy of the system is conserved.

$$\vec{P}_{cm} = (m_1 + m_2)\vec{v}_o = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$E = \frac{1}{2}(m_1 + m_2)v_o^2 + \frac{1}{2}k\Delta y^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

This is three equations with four unknowns, $\boldsymbol{v_{1x}}$, $\boldsymbol{v_{1y}}$, $\boldsymbol{v_{2x}}$, $\boldsymbol{v_{2y}}$. However the ***x*** components of the velocities remain constant. The ***y*** components of the velocities can be solved with the 2 independent equations that are left.

Example: Motion of the Center of Mass



Two objects are moving in the *x* direction connected by a compressed spring in the *y* direction. Derive an expression for the velocities of the objects after the spring is released.

Conservation of the ***y*** momentum:

$$\boxed{m_1 v_{1y} + m_2 v_{2y} = 0}$$

The conservation of energy:

$$\frac{1}{2}(m_1 + m_2)v_o^2 + \frac{1}{2}k\Delta y^2 = \frac{1}{2}m_1(v_o^2 + v_{1y}^2) + \frac{1}{2}m_2(v_o^2 + v_{2y}^2)$$

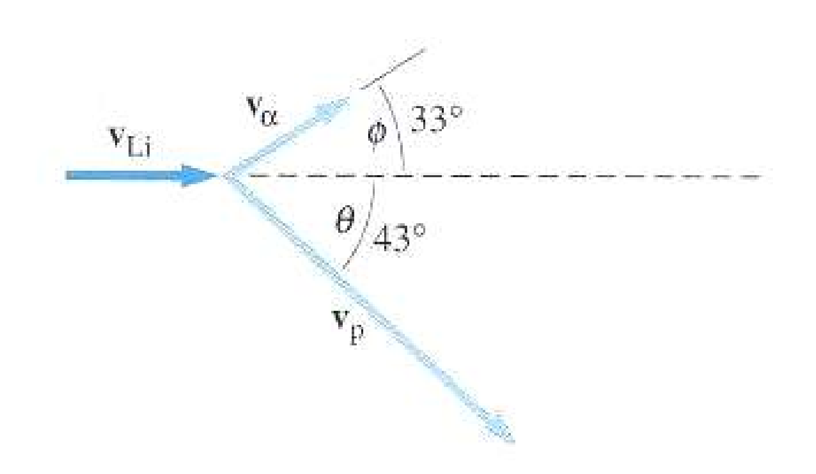
$$\boxed{k\Delta y^2 = m_1 v_{1y}^2 + m_2 v_{2y}^2}$$

Solving for the ***y*** velocity components:

$$k\Delta y^2 = m_1 v_{1y}^2 + m_2 \frac{m_1^2}{m_2^2} v_{1y}^2 = m_1\left(1 + \frac{m_1}{m_2}\right)v_{1y}^2$$

$$v_{1y}^2 = \frac{m_2}{m_1}\frac{k}{m_1 + m_2}\Delta y^2, \quad v_{2y}^2 = \frac{m_1^2}{m_2^2}v_{1y}^2$$

Radioactive Decay



A lithium-5 nucleus with a velocity ***v* = 1.6x10^6 *m/s*** decays into a proton and an α particle. The α particle has a speed ***v* = 1.4x10^6 *m/s*** with an angle **33º** to the original direction. Where and how fast is that proton going?

The momentum of the CM is conserved. Since the lithium can be consider a point particle, we merely have to conserve momentum (both components).

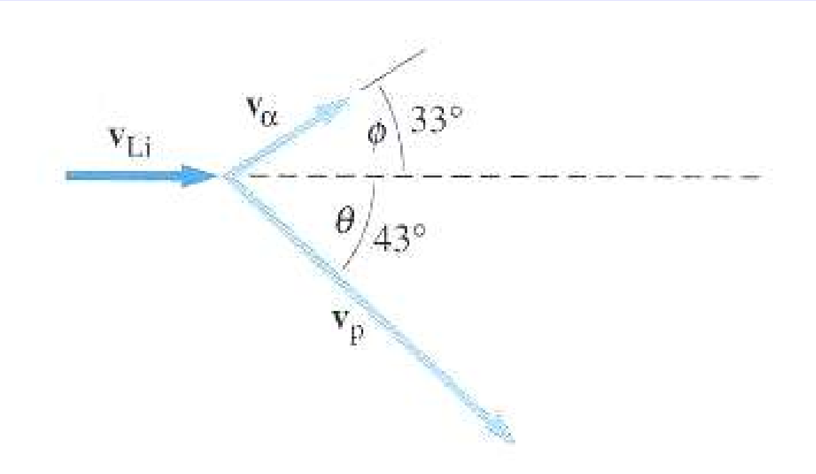
$$\begin{aligned} x: \quad m_{Li}v_{Li} &= m_\alpha v_{\alpha x} + m_p v_{px} \\ y: \quad 0 &= m_\alpha v_{\alpha y} - m_p v_{py} \end{aligned}$$

Solving for v_{px} we find:

$$v_{px} = \frac{m_{Li}v_{Li} - m_\alpha v_{\alpha x}}{m_p} = \frac{m_{Li}v_{Li} - m_\alpha v_\alpha \cos 33^\circ}{m_p}$$

$$v_{px} = \frac{5(1.6\times 10^6) - 4(1.4\times 10^6)\cos 33^\circ}{1} = 3.3\times 10^6 m/s$$

Radioactive Decay



A lithium-5 nucleus with a velocity ***v* = 1.6x10⁶ *m/s*** decays into a proton and an ***α*** particle. The ***α*** particle has a speed ***v* = 1.4x10⁶ *m/s*** with an angle **33°** to the original direction. Where and how fast is that proton going?

Solving for $\boldsymbol{v_{py}}$:

$$v_{py} = \frac{m_\alpha v_{\alpha y}}{m_p} = \frac{m_\alpha v_\alpha \sin 33^\circ}{m_p}$$

$$v_{py} = \frac{4(1.4\times 10^6)\sin 33^\circ}{1} = 3.05\times 10^6 m/s$$

The speed of the proton:

$$v_p = \sqrt{3.3^2 + 3.05^2}\,10^6 = 4.5\times 10^6 m/s$$

The direction of the proton:

$$\theta = \tan^{-1}\frac{3.05}{3.3} = .746 rad = 43^\circ$$

External Forces and the Center of Mass

The sum of all the net forces on each of the particles determines the acceleration of the center of mass.

$$\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{v}_i = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{i-net}$$

However, we need to consider the sum of the forces on each of the particles. Some of the forces on the ***i^th^*** particle are due to external forces (e.g. external gravitational field). There are also forces between the particles themselves (at least a gravitational attraction). This could make the problem virtually intractable, but Newton's 3rd comes to the rescue. It is the basis for recognizing that **the sum of the internal forces over all of the particles cancel!** It is only the sum of all the external forces that induce an acceleration of the center of mass.

$$\boxed{\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} \vec{F}_{i-ext}}$$

External Forces and the Center of Mass

A fire hose delivers water at a rate ***45kg/s***. The water hits the window with a horizontal velocity of ***v = 32m/s***. What is the horizontal force on the window?

The rate of change in the momentum of the water stream is:

$$\frac{dP}{dt} = -(45kg \times 32m/s)/s = -1400kg \cdot m/s^2$$

If the system is the water, then the rate of momentum loss means there is an external force acting on the water. This is the normal force of the window on the water. From Newton's third the water is exerting an equal and opposite force of **1400*N*** on the window.

Kinetic Energy of a Many Particle System

The total kinetic energy of a system of particles is simply the sum of the energies of the constituent particles:

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

The velocity of a particle can be written as the vector sum of center of mass velocity and a velocity relative to the center of mass velocity:

$$\vec{v}_i = \vec{V} + \vec{u}_i$$

The total kinetic energy can now be written as:

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\vec{V} + \vec{u}_i\right) \cdot \left(\vec{V} + \vec{u}_i\right) = \sum_{i=1}^{N} \frac{1}{2} m_i V^2 + \sum_{i=1}^{N} m_i \vec{V} \cdot \vec{u}_i + \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

$$K = \frac{1}{2} M V^2 + \vec{V} \cdot \sum_{i=1}^{N} m_i \vec{u}_i + \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2$$

Kinetic Energy of a Many Particle System

The total momentum of a system of particles is simply the sum of the momentum of the constituent particles. We found this to be the momentum of the center of mass:

$$\vec{P}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} m_i \left(\vec{V} + \vec{u}_i \right) = M\vec{V} + \sum_{i=1}^{N} m_i \vec{u}_i$$

This implies that:

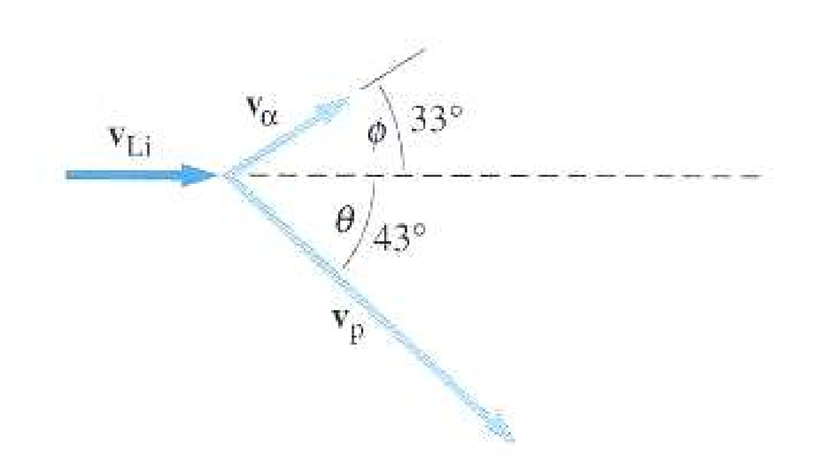
$$\sum_{i=1}^{N} m_i \vec{u}_i = 0!$$

The total kinetic energy can now be written as:

$$K = \frac{1}{2} M V^2 + \sum_{i=1}^{N} \frac{1}{2} m_i u_i^2 = K_{cm} + K_{int}$$

The total kinetic energy is the sum of the **center of mass kinetic energy** and the **internal kinetic energy** which is the kinetic energy measured in the frame of the center of mass!

Radioactive Decay



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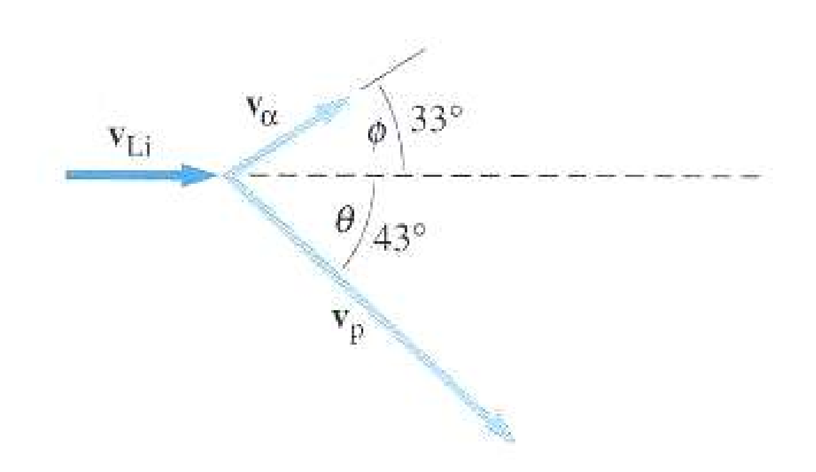
The CM motion does not change after the decay. Hence the center of mass kinetic energy is:

$$K = \frac{1}{2}MV^2 = \frac{1}{2}\frac{5\times 10^{-3}}{6.022\times 10^{23}}\left(1.6\times 10^6\right)^2 = 1.06\times 10^{-14}J$$

The internal energy is **zero** before the decay. After the decay the velocities of the particles were found to be:

$$\vec{v}_\alpha = \left(1.4\cos 33^\circ\hat{i} + 1.4\sin 33^\circ\hat{j}\right)\times 10^6 m/s = \left(1.174\hat{i} + .762\hat{j}\right)\times 10^6 m/s$$
$$\vec{v}_p = \left(3.30\hat{i} + 3.05\hat{j}\right)\times 10^6 m/s$$

Radioactive Decay



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The velocities relative to the center of mass are:

$$\vec{u}_\alpha = \left((1.17 - 1.6)\hat{i} + .762\hat{j}\right) \times 10^6 m/s = \left(-.43\hat{i} + .762\hat{j}\right) \times 10^6 m/s$$
$$\vec{u}_p = \left((3.30 - 1.6)\hat{i} + 3.05\hat{j}\right) \times 10^6 m/s = \left(1.70\hat{i} + 3.05\hat{j}\right) \times 10^6 m/s$$

The internal energy after the decay is:

$$K_{int} = \frac{1}{2}\sum_{i=1}^{N} m_i u_i^2 = \frac{1}{2} m_\alpha (u_{\alpha x}^2 + u_{\alpha y}^2) + \frac{1}{2} m_p (u_{px}^2 + u_{py}^2)$$
$$K_{int} = 1.27 \times 10^{14} J$$

What is the source of this energy?