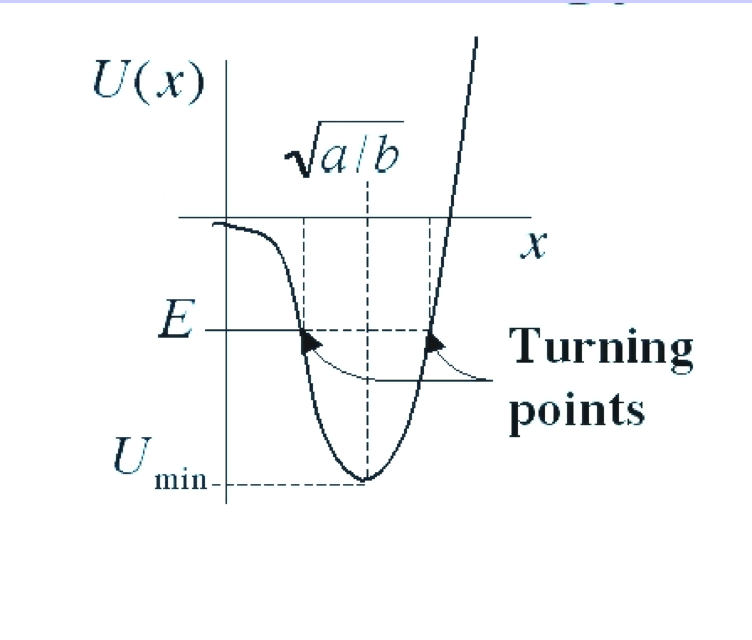
Today's Lecture

Lecture 14: Chapter 8,
Energy Diagrams
Conservation of Energy

Chapter 10,
System of Particles

Example: Force and Potential Energy



(a) Derive an expression for the potential energy of an object subject to a force, $\boldsymbol{F = ax - bx^3}$, where $\boldsymbol{a = 5\ N/m}$ and $\boldsymbol{b = 2\ N/m^3}$ assuming that $\boldsymbol{U(0) = 0}$. (b) Derive the turning points when $\boldsymbol{E = -1J}$.

The potential energy is found from the integral:

$$U(x) - U(0) = -\int_0^x F(x)dx = -\int_0^x (ax - bx^3)dx$$

$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

First we note that the force vanishes at:

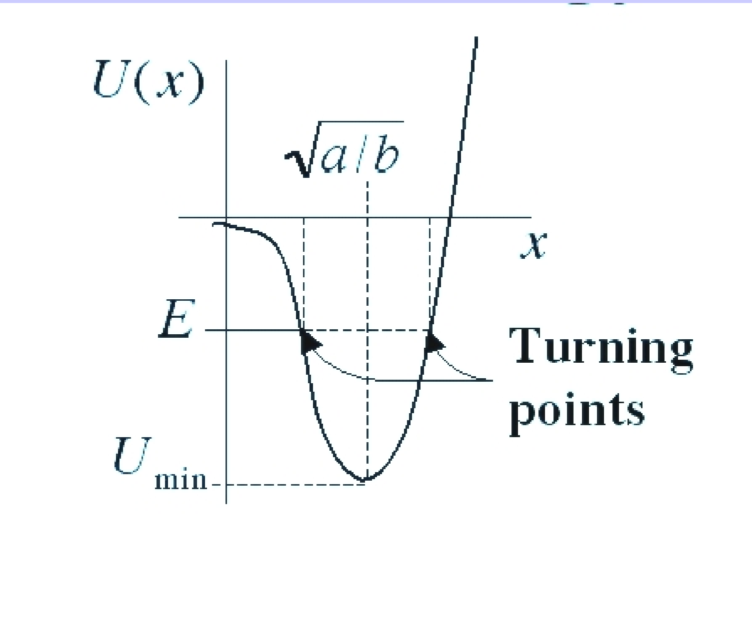
$$x = 0 \text{ and } x = \pm\sqrt{a/b}$$

This potential is double welled with a local maximum at $\boldsymbol{x = 0}$. The minimum value of the potential is:

$$U\left(\sqrt{a/b}\right) = -\frac{1}{2}a\frac{a}{b} + \frac{1}{4}b\frac{a^2}{b^2} = -\frac{1}{4}\frac{a^2}{b}$$

$$U\left(\sqrt{5/2}\right) = -\frac{25}{8} = -3.125J$$

Example: Force and Potential Energy



Derive an expression for the potential energy of a particle subject to a force, ***F = ax-bx³***, where ***a = 5 N/m*** and ***b = 2 N/m³*** assuming that ***U(0) = 0.*** Derive the turning points when ***E = -1J***.

$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

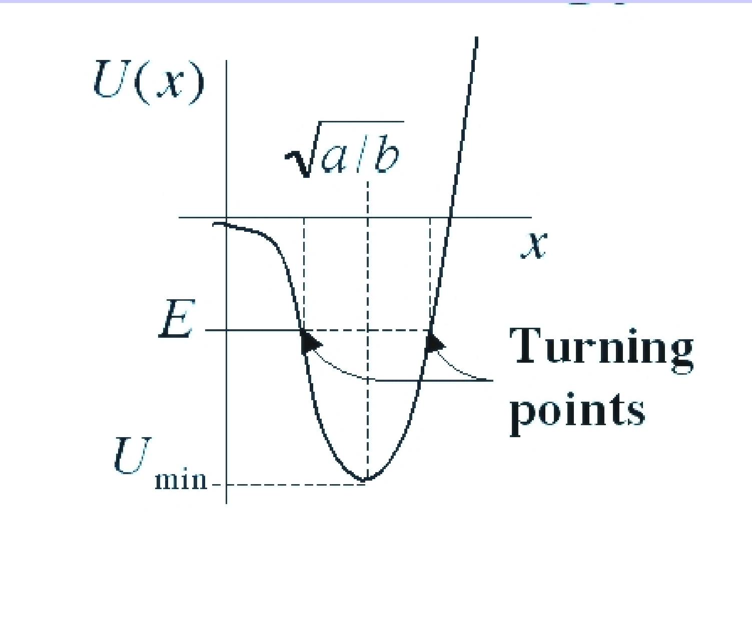
Since ***0 > E > U***

$$E = K + U = \frac{1}{2}mv^2 + U \rightarrow v = \sqrt{2(E-U)/m}$$

The maximum velocity occurs when $\boldsymbol{U = U_{min}}$. The turning points occur when $\mathbf{v = 0}$ or $\boldsymbol{E = U}$. Solving the quartic yields:

$$-1 = -\frac{5}{2}x^2 + \frac{1}{2}x^4 \rightarrow x = \pm.662,\ \ \pm 2.14$$

Example: Force and Potential Energy



Derive an expression for the potential energy of a particle subject to a force, ***F = ax-bx*** ^{3}, where ***a = 5 N/m*** and ***b = 2 N/m*** ^{3} assuming that ***U(0) = 0.*** Derive the turning points when ***E = -1J***.

$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

The turning points are:

$$x = \pm .662, \quad \pm 2.14$$

The turning points for ***x*** > 0 are the ones shown in the figure. What about the turning points for ***x < 0*** ?

The particle oscillates between these two points!
What if ***E*** > **0**?

Conservation of Energy

From the Work-Energy Theorem the work done on an object is equal to the change in its kinetic energy:

$$W = m\int \frac{d\vec{v}}{dt}\cdot d\vec{r} = \int \vec{F}_{net}\cdot d\vec{r} = \Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

If we consider separately the work done by conservative forces, W_c, and non-conservative forces, $\boldsymbol{W_{nc}}$:

$$\Delta K = W_c + W_{nc}$$

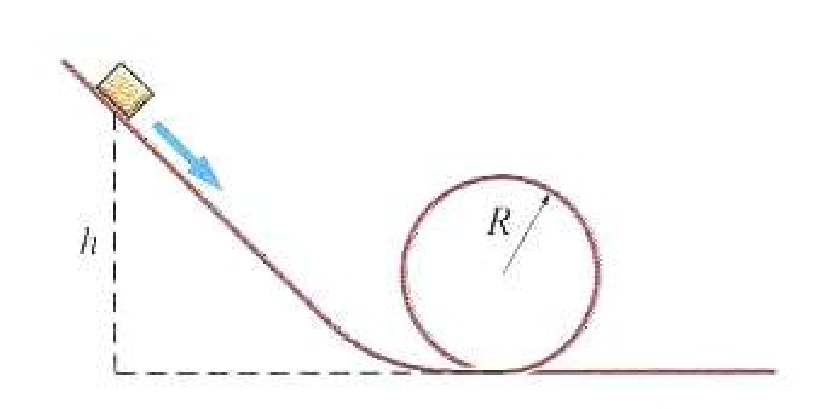
Potential energy was defined as the negative of the work done by conservative forces: $\boldsymbol{\Delta U = -W_c}$. Hence:

$$\Delta K + \Delta U = W_{nc}$$

In the absence of non-conservative forces, the total mechanical energy is conserved!

$$\boxed{\Delta K + \Delta U = 0 \rightarrow \frac{1}{2}mv_i^2 + U_i = \frac{1}{2}mv_f^2 + U_f = E}$$

Example 1: Conservation of Energy



What is the minimum height, ***h***, for which the block can start from rest and make it around the loop?

From the conservation of energy the velocity of the block at the top of the loop is:

$$E = mgh = \frac{1}{2}mv^2 + 2mgR$$

We have already determined from Newton's 2nd that at the top of the loop:

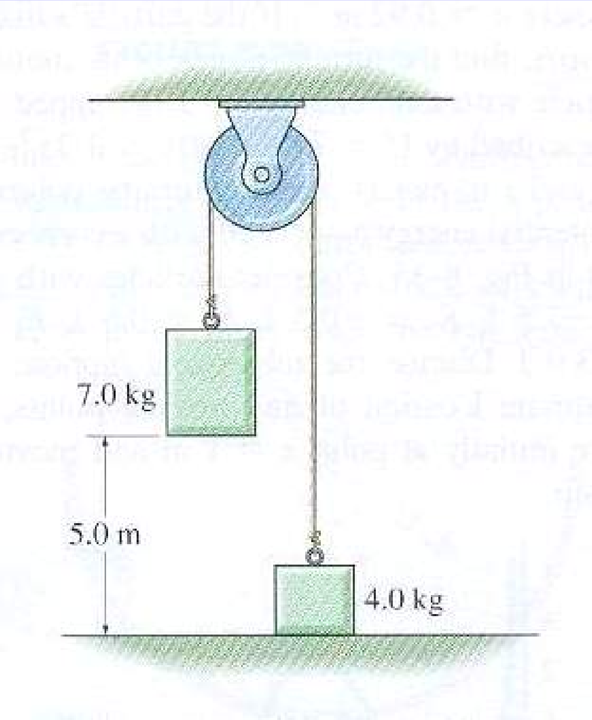
$$mg + N = mv^2/R,\ N = 0 \rightarrow v^2 = gR$$

If ***v*** is less than this the block will fall off the loop before it reaches the top! Solving for ***h***:

$$gh = \frac{1}{2}v^2 + 2gR = \frac{1}{2}gR + 2gR = \frac{5}{2}gR$$

$$h = \frac{5}{2}R$$

Example 2: Conservation of Energy



Consider the Atwood machine with masses, $m_1 = 7kg$ and $m_2 = 4kg$. (a) Find the velocity of m_1 just as it hits the floor. (b) Find the maximum height reached by m_2. (c) Find the fraction of the total mechanical energy lost when m_1 hits the floor.

(a) From the conservation of energy, the velocity of both masses just as m_1 hits the floor is:

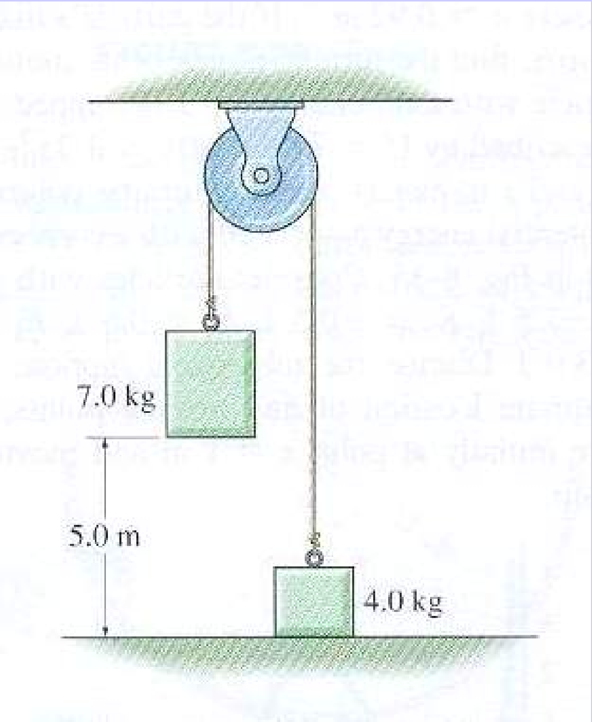
$$E = m_1gh = m_2gh + \frac{1}{2}(m_1 + m_2)v^2$$

$$v^2 = \frac{m_1 - m_2}{m_1 + m_2}2gh \rightarrow v = \sqrt{\frac{3}{11}98} = 5.17m/s$$

Does this make sense? Remember, for the Atwood machine

$$a = (m_1 - m_2)g/ (m_1 + m_2)$$

Example 2: Conservation of Energy



Consider the Atwood machine with masses, $\boldsymbol{m_1 = 7kg}$ and $\boldsymbol{m_2 = 4kg}$. (a) Find the velocity of $\boldsymbol{m_1}$ just as it hits the floor. (b) Find the maximum height reached by $\boldsymbol{m_2}$. (c) Find the fraction of the total mechanical energy lost when $\boldsymbol{m_1}$ hits the floor.

(b) Energy is not conserved for the system after $\boldsymbol{m_1}$ hits the floor. However, energy is conserved for $\boldsymbol{m_2}$. Hence the max height for $\boldsymbol{m_2}$ is:

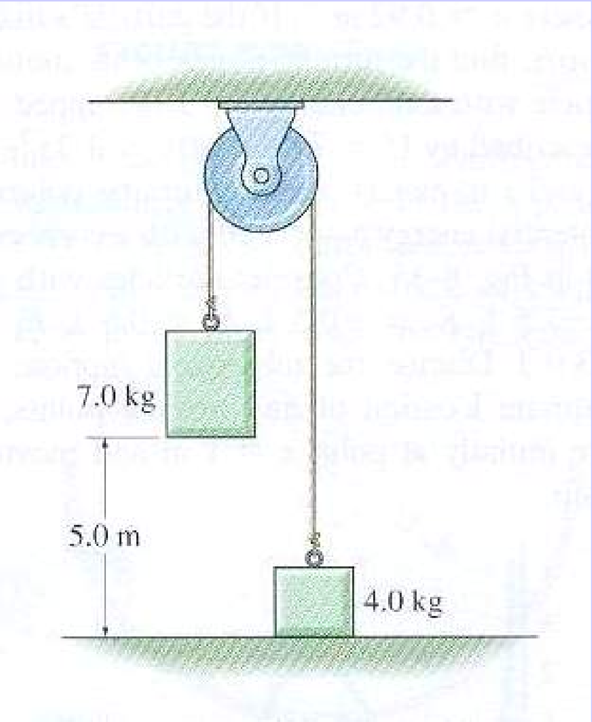
$$E_i = m_2 gh + \frac{1}{2} m_2 v^2 = m_2 gh + \frac{1}{2} m_2 \frac{m_1 - m_2}{m_1 + m_2} 2gh$$

$$E_f = m_2 gy = \left(1 + \frac{m_1 - m_2}{m_1 + m_2}\right) m_2 gh = \frac{2m_1}{m_1 + m_2} m_2 gh$$

$$y = \frac{2m_1}{m_1 + m_2} h = \frac{14}{11} 5 = 6.36m$$

Does this make sense? What happens when $\boldsymbol{m_1 = m_2}$ or $\boldsymbol{m_1 >> m_2}$?

Example 2: Conservation of Energy



Consider the Atwood machine with masses, $\boldsymbol{m_1 = 7kg}$ and $\boldsymbol{m_2 = 4kg}$. (a) Find the velocity of $\boldsymbol{m_1}$ just as it hits the floor. (b) Find the maximum height reached by $\boldsymbol{m_2}$. (c) Find the fraction of the total mechanical energy lost when $\boldsymbol{m_1}$ hits the floor.

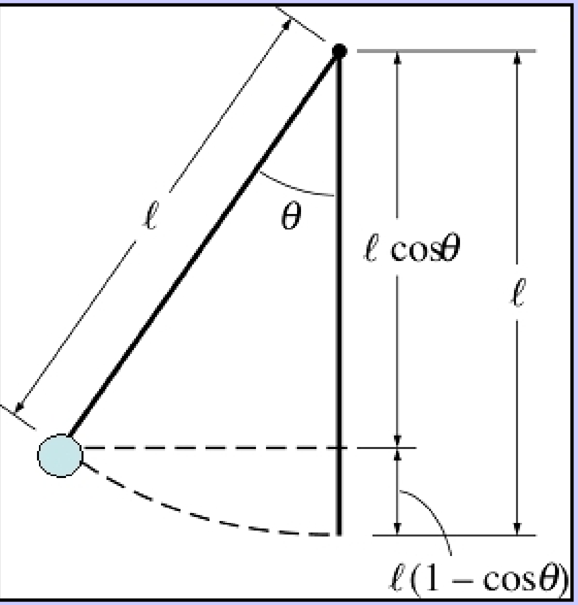
(c) Initially the total mechanical energy is $\boldsymbol{E_i = m_1 gh}$. When $\boldsymbol{m_1}$ hits the floor the total energy of the system is reduced by the kinetic energy of $\boldsymbol{m_1}$. Hence:

$$K_1 = \frac{1}{2} m_1 v^2 = \frac{1}{2} m_1 \frac{m_1 - m_2}{m_1 + m_2} 2gh = \frac{m_1 - m_2}{m_1 + m_2} m_1 gh$$

$$\frac{K_1}{E_i} = \frac{m_1 - m_2}{m_1 + m_2} = 3/11 = 27.2\%$$

Does this make sense? What happens when $\boldsymbol{m_1 = m_2}$ or $\boldsymbol{m_1 \gg m_2}$?

Example 3: Conservation of Energy



The Pendulum is one of the fundamental problems in mechanics. It can be used to understand many more complex problems in oscillatory motion. We will consider the simplest example.

Given an initial velocity, $\boldsymbol{v_o}$, what is the maximum angle, $\boldsymbol{\theta}$, of the swinging pendulum?

From the conservation of energy we know that $\boldsymbol{E_i = E_f}$:

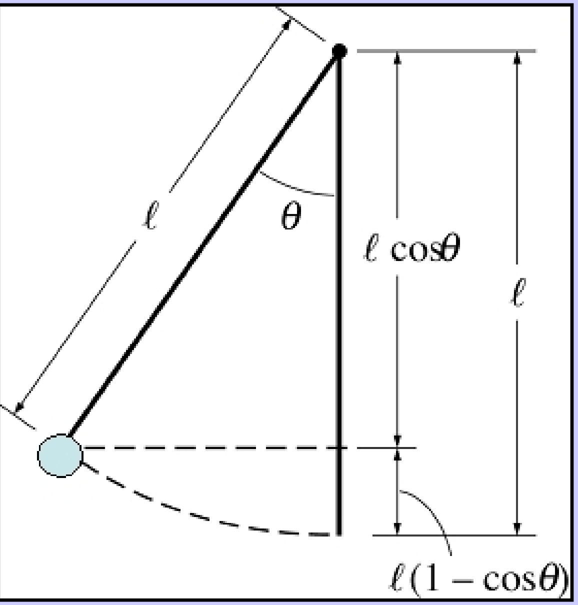
$$E_i = \frac{1}{2}mv_o^2 = E_f = mg\Delta y = mg\ell(1-\cos\theta)$$

$$1-\cos\theta = \frac{1}{2}\frac{v_o^2}{g\ell} \rightarrow \cos\theta = 1-\frac{1}{2}\frac{v_o^2}{g\ell}$$

$$\theta = \cos^{-1}(1-v_o^2/2g\ell)$$

Does this make sense? What happens when $\boldsymbol{v_o^2 = 2gl}$?

Example 3: Conservation of Energy



Tarzan runs and grabs a ***12m*** vine to swing up to Jane who is on a tree branch ***3m*** above the ground. (a) How fast must he run to reach Jane as he comes to a full stop?

From the conservation of energy:

$$E_i = \frac{1}{2}mv^2 = E_f = mgh$$

$$v = \sqrt{2gh} = \sqrt{6(9.8)} = 7.67m/s$$

Note that this is independent of the length of the vine. Does that make sense?

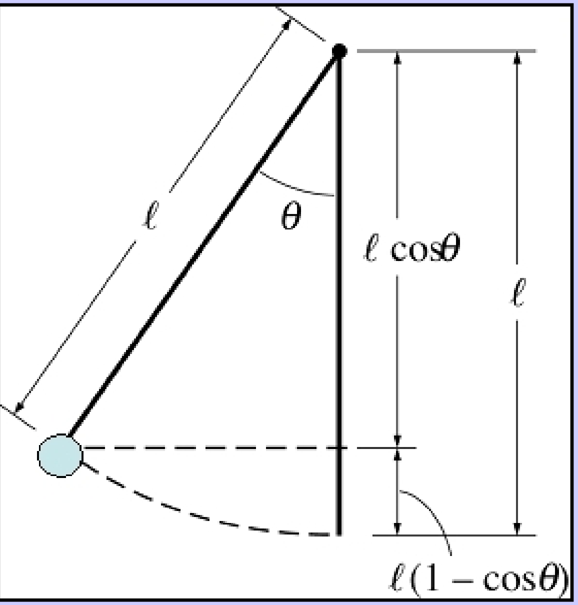
How far must Jane be from the initial position of the rope for Tarzan to land safely?

$$\theta = \cos^{-1}(1 - v_o^2/2g\ell) = \cos^{-1}(1 - 2gh/2g\ell) = \cos^{-1}(1 - h/\ell)$$

$$x = \ell\sin\theta = \ell\sqrt{1-\cos^2\theta} = \ell\sqrt{1-(1-h/\ell)^2} = 12\sqrt{1-.75^2} = 7.9m$$

Note that she had to lie somewhere on the circular arc of the vine.

Example 3: Conservation of Energy



The Pendulum is one of the fundamental problems in mechanics. It can be used to understand many more complex problems in oscillatory motion. We will consider the simplest example.

The conservation of energy for the pendulum is:

$$E = \frac{1}{2}mv^2 + mg\Delta y = \frac{1}{2}mv^2 + mg\ell(1-\cos\theta)$$

Now consider the case for oscillations when $\boldsymbol{\theta_{max}} \ll \mathbf{1}$.

$$E = \frac{1}{2}mv^2 + mg\ell\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) = \frac{1}{2}mv^2 + \frac{1}{2}mg\ell\theta^2$$

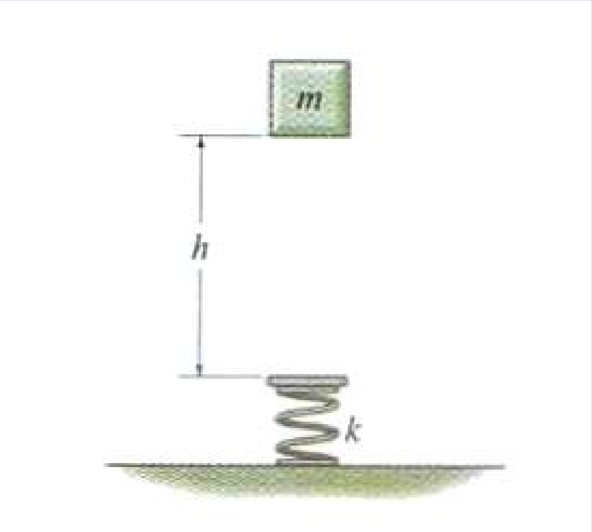
$$E = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}mg\ell\theta^2 = \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}m\ell^2\frac{g}{\ell}\theta^2$$

Compare this expression to the analogous expression for a spring.

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left(\frac{k}{m}\right)x^2$$

Later, how do the period of oscillations compare?

Example 4: Vertical Spring



A mass ***m*** is dropped from a height ***h*** above the top of a spring with spring constant ***k***. What is the maximum compression of the spring?

If zero for the gravitational potential energy is chosen to be the height at the top of the spring, then the conservation of energy for this problem is:

$$E = mgh = -mgx + \frac{1}{2}kx^2$$

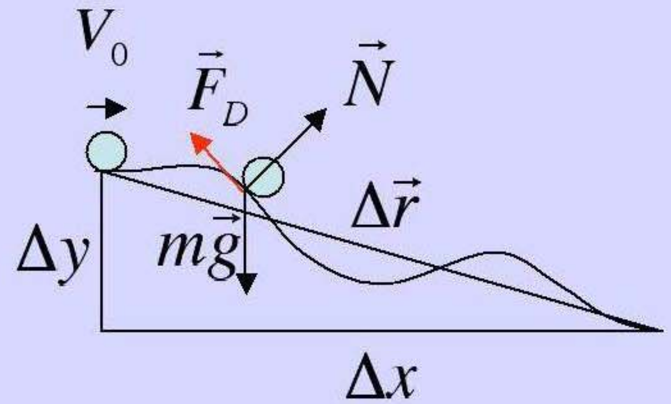
Note that it is important to note that the mass does not come to rest until the spring obtains maximum compression!

Now it a simple problem of solving the quadratic equation:

$$\frac{1}{2}kx^2 - mgx - mgh = 0 \rightarrow x = \frac{mg}{k}\left(1 + \sqrt{1 + 2kh/mg}\right)$$

What is the physical significance of the other root?

With Friction or Drag, these forces always point along path, causing work



A ball rolling down a bumpy hill with drag force also has a component of work done by this force.

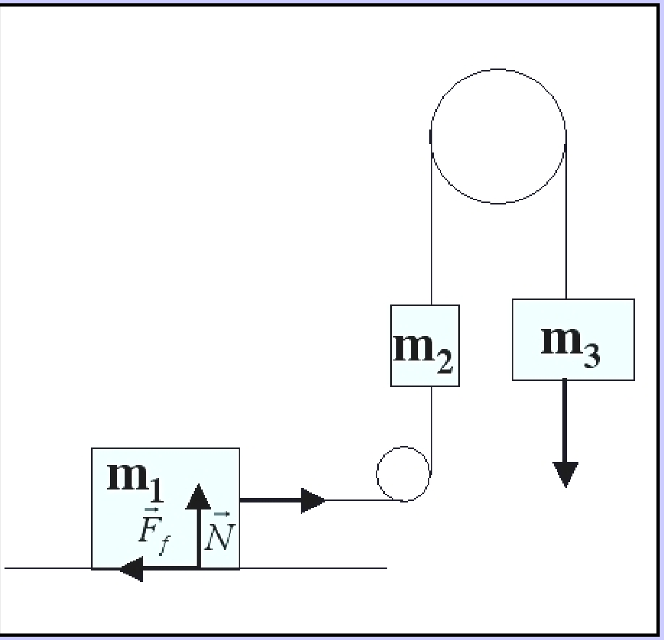
The drag (and/or friction) force is always || to $\vec{dl}$

$$W_D = \int \vec{F}_D \cdot \vec{dl} \neq 0$$

This must be added to the work done by other forces.

Note: the work done on friction and drag WILL in general depend on the path taken.

Example 4: Nonconservative Forces



Three masses are attached to pulleys as shown. Mass m_1 slides on a surface with a coefficient of kinetic friction μ_k. Find the velocity, v, after starting from rest of the objects after traveling a distance Δy.

From the Work-KE Theorem:

$$\Delta K = W = W_c + W_{nc} \rightarrow \Delta K + \Delta U = W_{nc}$$

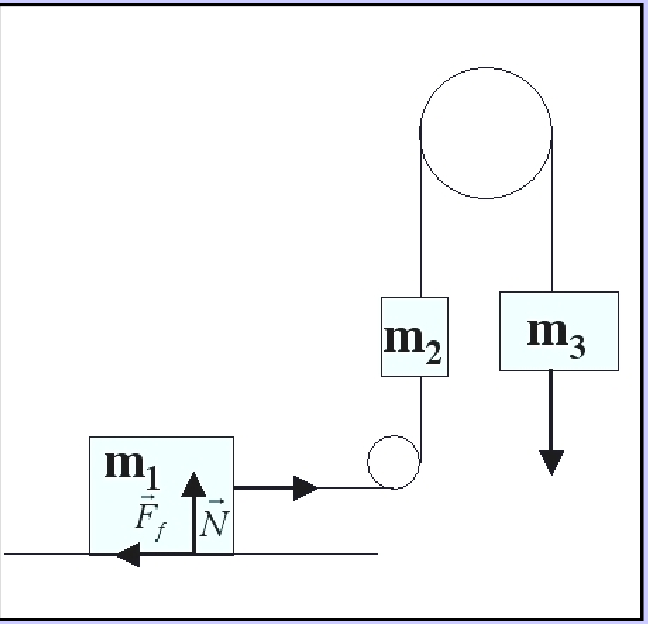
After m_3 falls a distance Δy:

$$\frac{1}{2}(m_1 + m_2 + m_3)v^2 + (m_2 g\Delta y - m_3 g\Delta y) = -\mu_k m_1 g\Delta y$$

Solving for v is straighforward:

$$v = \sqrt{\frac{m_3 - m_2 - \mu_k m_1}{m_1 + m_2 + m_3} 2g\Delta y}$$

Example 4: Nonconservative Forces



Three masses are attached to pulleys as shown. Mass $\boldsymbol{m_1}$ slides on a surface with a coefficient of kinetic friction $\boldsymbol{\mu_k}$. Find the velocity, $\boldsymbol{v}$, after starting from rest of the objects after traveling a distance $\boldsymbol{\Delta y}$.

A free-body diagram for each object yields three equations of motion:

$$m_3 g - T_1 = m_3 a, \quad T_1 - T_2 - m_2 g = m_2 a, \quad T_2 - \mu_k m_1 g = m_1 a$$

Summing these equations and solving for $\boldsymbol{a}$:

$$a = \frac{m_3 - m_2 - \mu_k m_1}{m_1 + m_2 + m_3} g$$

From our work with kinetmatics:

$$v^2 = 2a\Delta y = \frac{m_3 - m_2 - \mu_k m_1}{m_1 + m_2 + m_3} 2g\Delta y$$

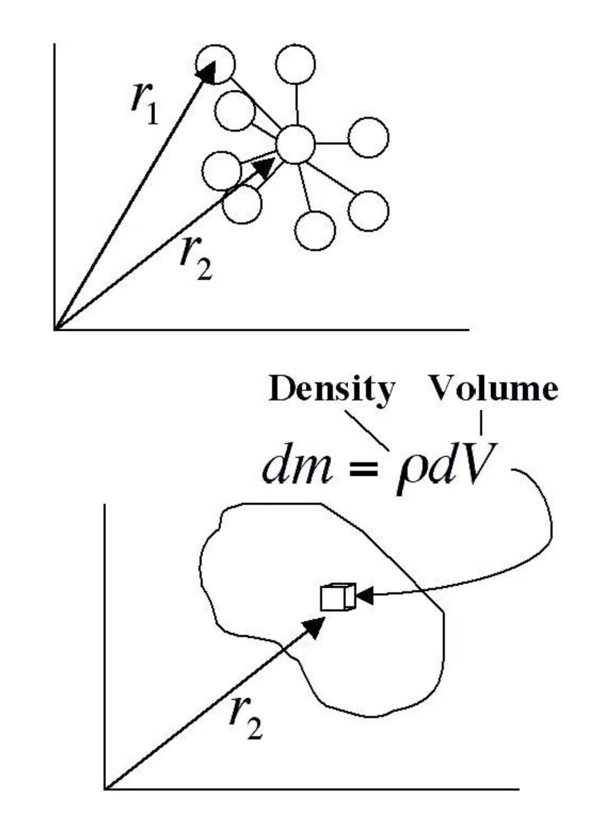
Same!

Chapter 10
System of Particles

When considering multiple particles we summed the kinetic and potential energies of all the masses and set it equal to the work dissipated by friction. This conceptually leads to treating a general system of particles using conservation laws (and Newton's EOM).

To accomplish this we first define a center of mass.

Center of Mass



Mathematically we define the center of mass as the average of the mass weighted vector displacement of the individual particles. Defining the total mass as ***M***.

$$M = m_1 + m_2 + m_3 + \cdots = \sum_{i=1}^{N} m_i$$

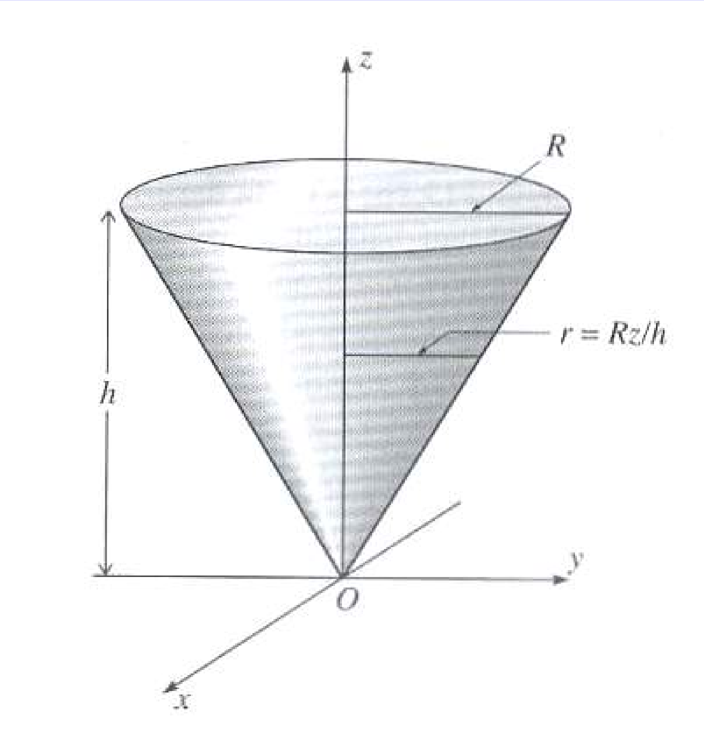
This allows us to define the center of mass as:

$$\vec{R}_{cm} = \frac{1}{M}(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots) = \frac{1}{M}\sum_{i=1}^{N} m_i\vec{r}_i$$

For continuous media both sums become integrals:

$$M = \int dm, \quad \text{and} \quad \vec{R}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

Example: Center of Mass – Uniform Solid Cone



From symmetry considerations the center of mass must lie on the *z* axis. All that is left is to perform the integral to determine Z_{cm}.

$$Z_{cm} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int z\rho\,dV = \frac{1}{V}\int z\,dV$$

At a height *z* (radius *r*) the volume element is:

$$dV = A(z)dz = \pi r^2(z)dz$$

Since *r(z)* satisfies the relationship *r* = *Rz/h* the integral for Z_{cm} becomes:

$$Z_{cm} = \frac{1}{V}\int z\,dV = \frac{1}{V}\int z\pi r^2(z)dz$$

$$Z_{cm} = \frac{1}{V}\int z\pi\frac{R^2}{h^2}z^2dz = \frac{\pi R^2}{h^2(\pi R^2 h/3)}\int_0^h z^3dz$$

$$Z_{cm} = \frac{3}{h^3}\frac{h^4}{4} = \frac{3}{4}h \quad \text{(from the vertex)}$$

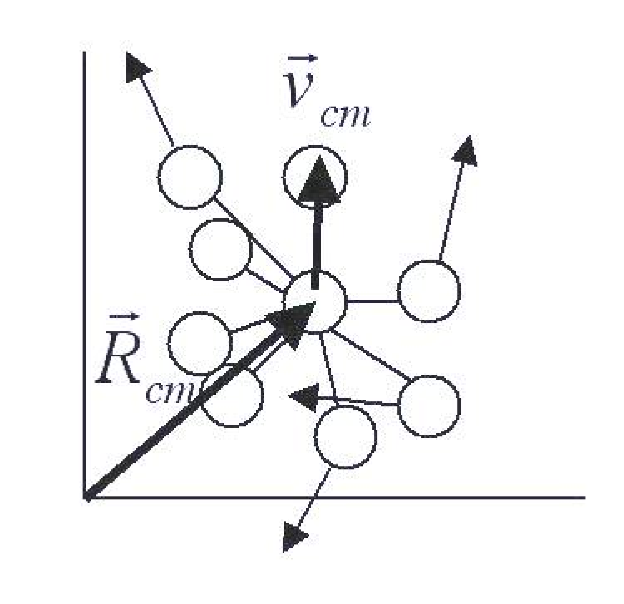
Motion of the Center of Mass

The total momentum of a system of particles is equal to the momentum of the center of mass. In the absence of any **net external force** this momentum is **conserved**.

To see this consider the time derivative of the center of mass:

$$\frac{d}{dt}\vec{R}_{cm} = \vec{v}_{cm} = \frac{1}{M}\sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{v}_i$$

$$\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{r}_i = M\vec{v}_{cm} = \sum_{i=1}^{N} m_i \vec{v}_i$$



Even though individual particles may be moving relative to the center of mass, the center of mass maintains a uniform velocity.

External Forces and the Center of Mass

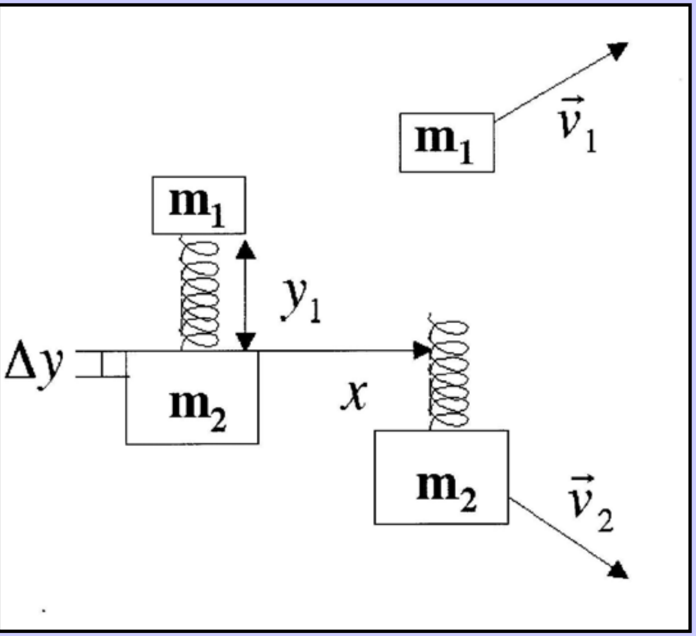
The sum of all the net forces on each of the particles determines the acceleration of the center of mass.

$$\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} m_i \frac{d}{dt}\vec{v}_i = \sum_{i=1}^{N} m_i \vec{a}_i = \sum_{i=1}^{N} \vec{F}_{i-net}$$

However, we need to consider the sum of the forces on each of the particles. Some of the forces on the ***i**^th^* particle are due to external forces (e.g. external gravitational field). There are also forces between the particles themselves (at least a gravitational attraction). This could make the problem virtually intractable, but Newton's 3rd comes to the rescue. It is the basis for recognizing that **the sum of the internal forces over all of the particles cancel!** It is only the sum of all the external forces that induce an acceleration of the center of mass.

$$\boxed{\frac{d}{dt}\vec{P}_{cm} = \sum_{i=1}^{N} \vec{F}_{i-ext}}$$

Example: Motion of the Center of Mass



Two objects are moving in the x direction connected by a compressed spring in the y direction. Derive an expression for the velocities of the objects after the spring is released.

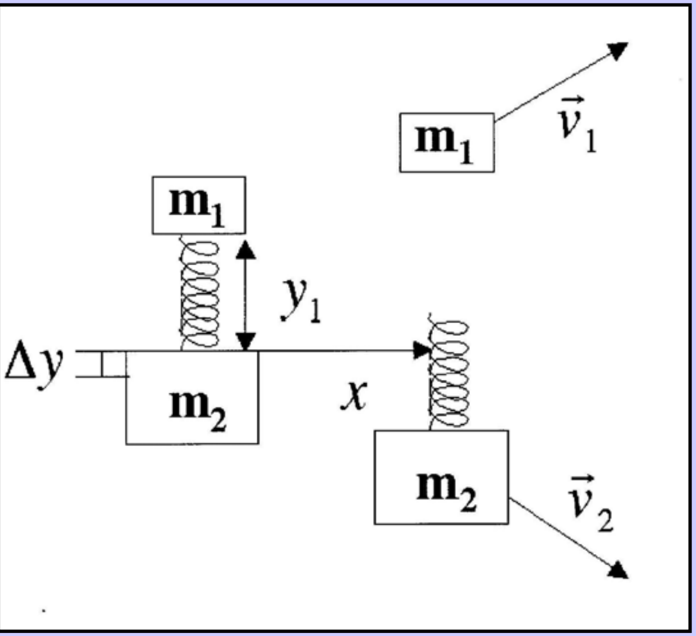
In the absence of external forces the motion of the center of mass remains unchanged. Also the mechanical energy of the system is conserved.

$$\vec{P}_{cm} = (m_1 + m_2)\vec{v}_o = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$E = \frac{1}{2}(m_1 + m_2)v_o^2 + \frac{1}{2}k\Delta y^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

This is three equations with four unknowns, $\boldsymbol{v_{1x}}$, $\boldsymbol{v_{1y}}$, $\boldsymbol{v_{2x}}$, $\boldsymbol{v_{2y}}$. However the x components of the velocities remain constant. The y components of the velocities can be solved with the 2 independent equations that are left.

Example: Motion of the Center of Mass



Two objects are moving in the x direction connected by a compressed spring in the y direction. Derive an expression for the velocities of the objects after the spring is released.

Conservation of the y momentum:

$$\boxed{m_1 v_{1y} + m_2 v_{2y} = 0}$$

The conservation of energy:

$$\frac{1}{2}(m_1 + m_2)v_o^2 + \frac{1}{2}k\Delta y^2 = \frac{1}{2}m_1(v_o^2 + v_{1y}^2) + \frac{1}{2}m_2(v_o^2 + v_{2y}^2)$$

$$\boxed{k\Delta y^2 = m_1 v_{1y}^2 + m_2 v_{2y}^2}$$

Solving for the y velocity components:

$$k\Delta y^2 = m_1 v_{1y}^2 + m_2 \frac{m_1^2}{m_2^2} v_{1y}^2 = m_1\left(1 + \frac{m_1}{m_2}\right)v_{1y}^2$$

$$v_{1y}^2 = \frac{m_2}{m_1}\frac{k}{m_1 + m_2}\Delta y^2, \quad v_{2y}^2 = \frac{m_1^2}{m_2^2}v_{1y}^2$$