Physics 215A – Problem Set #2
Wednesday Oct. 31, 2007

Read Srednicki 3,4,23, 33-40

1. The commutator of a real scalar field with itself is given by

$$[\phi(x), \phi(y)] \equiv i\Delta(x-y),$$

where

$$\Delta(x) \equiv \Delta^+(x) - \Delta^+(-x)$$

and

$$i\Delta^+(x) \equiv \int \frac{d^3\vec{k}}{(2\pi)^3(2\omega_{\vec{k}})} e^{-ik\cdot x} \ .$$

(a) Show that

$$i\Delta^+(x) = \frac{1}{2\pi^3} \int d^4k \ \delta(k^2 - m^2) \ \theta(k^0) \, e^{-ik\cdot x},$$

is manifestly Lorentz invariant for proper Lorentz transformations, i.e. $\Delta^+(\Lambda x) = \Delta^+(x)$. What about for improper Lorentz transformations?

(b) Prove that the real scalar field satisfies

$$[\phi(x), \phi(y)] = 0,$$

for spacelike separations $(x-y)^2 < 0$.

2. A massive vector particle with three-momentum $\vec{k}$ has three polarizations $e^\mu(\vec{k}, \sigma)$. Show that the polarization sum

$$P^{\mu\nu}(\vec{k}) \equiv \sum_{\sigma=\pm 1,0} e^\mu(\vec{k}, \sigma) e^{\nu\,*}(\vec{k}, \sigma) = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2},$$

for arbitrary $\vec{k}$, where m is the mass of the particle.

3. A massless spin-1 (vector) particle has only two polarizations $\sigma = \pm 1$. Determine these polarization vectors for a massless vector boson moving in the $\hat{z}$ direction with three-momentum $\vec{k}$. What is the polarization sum $P^{ij}(\vec{k})$, for spatial indices $i, j = 1, 2, 3$?

4. The Dirac or non-relativistic representation of Dirac γ matrices is defined by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which is related to the Weyl representation by

$$\gamma^\mu_{\text{Dirac}} = U \gamma^\mu_{\text{Weyl}} U^\dagger,$$

$$\psi_{\text{Dirac}}(x) = U \psi_{\text{Weyl}}(x),$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(a) Find the Dirac spinors $u(\vec{0}, \sigma)$ and $v(\vec{0}, \sigma)$ for a massive fermion in the Dirac representation.
(b) Find the spinors $u(k_z, \sigma)$ and $v(k_z, \sigma)$ in the Dirac representation and verify that they are helicity eigenstates.

5. Under charge conjugation, a Dirac field transforms as

$$\mathcal{C}\, \psi(x)\, \mathcal{C}^{-1} = \eta_{\mathcal{C}}^*\, \psi^c(x).$$

In class, it was alleged that

$$\psi^c(x) = \left(-i\gamma^2\right)\ \psi^*(x),$$

which is true so long as the spinors u and v satisfy

$$u^*(\vec{k}, \sigma) = -i\gamma^2 v(\vec{k}, \sigma),$$

$$v^*(\vec{k}, \sigma) = -i\gamma^2 u(\vec{k}, \sigma).$$

(a) Verify that these spinor equations are satisfied when $\vec{k} = \vec{0}$ using the explicit spinors given in class.
(b) Construct a proof for the general case when $\vec{k}$ does not vanish.