

PHYSICS 110A : CLASSICAL MECHANICS
PROBLEM SET #1

[1] Consider the equation

$$\dot{u} = (u - 1)(u - 2)(u - 3)(u - 4) . \quad (1)$$

- (a) Sketch the right hand side of the above equation.
(b) Identify and classify all the fixed points.

For each of the following initial conditions, determine the asymptotic value $u(\infty)$ to which u flows:

(c) $u_0 = \frac{1}{2}$ (d) $u_0 = \frac{3}{2}$ (e) $u_0 = \frac{5}{2}$ (f) $u_0 = \frac{7}{2}$ (g) $u_0 = \frac{9}{2}$.

[2] Consider the harmonic oscillator equation

$$\ddot{x} + \Omega^2 x = 0 . \quad (2)$$

(a) Show that this may be written as the dynamical system $\dot{\varphi} = M\varphi$, where φ is a two-component column vector and M is a 2×2 matrix. Determine φ and M .

Hint: This is done in the notes!

(b) The formal solution to the dynamical system is $\varphi(t) = \exp(Mt)\varphi(0)$. Compute the matrix exponential $\exp(Mt)$ by the following method. First, write

$$\exp(Mt) = 1 + Mt + \frac{1}{2!}M^2t^2 + \frac{1}{3!}M^3t^3 + \dots . \quad (3)$$

Next, show that M^2 is a multiple of the identity matrix. This means that every term M^{2k} can be written as a scalar times the identity matrix, and every term M^{2k+1} can be written as a scalar times M . Separate the even and odd terms in the above Taylor series, and show they can be resummed to give trigonometric functions, multiplied by either the identity or by M . Finally, check that your solution agrees with the correct answer,

$$x(t) = x_0 \cos(\Omega t) + \frac{v_0}{\Omega} \sin(\Omega t) \quad (4)$$

$$v(t) = v_0 \cos(\Omega t) - \Omega x_0 \sin(\Omega t) . \quad (5)$$

[3] (Strogatz problem 3.6.6) Fluid systems out of equilibrium give rise to a dazzling array of patterns, which emerge via subcritical or supercritical pitchfork bifurcations from a spatially uniform state. Near the bifurcation, the dynamics of the amplitude of the pattern obeys

$$\dot{A} = \varepsilon A - g A^3 \quad (6)$$

in the supercritical case (*i.e.* $g > 0$), and

$$\dot{A} = \varepsilon A - g A^3 - k A^5 \quad (7)$$

in the subcritical case (*i.e.* $g < 0$, with $k > 0$).¹

(a) The supercritical bifurcation that arises in Rayleigh-Bénard convection was studied experimentally in the 1970's. It was observed that the steady state amplitude depends on ε according to the power law, $A_* \propto \varepsilon^\beta$, where $\beta = 0.50 \pm 0.01$. What is the prediction of the Landau equation?

(b) The equation $\dot{A} = \varepsilon A - g A^3 - k A^5$ is said to undergo a *tricritical bifurcation* when $g = 0$; this case is borderline between the supercritical and subcritical cases. Find the relation between A_* and ε when $g = 0$.

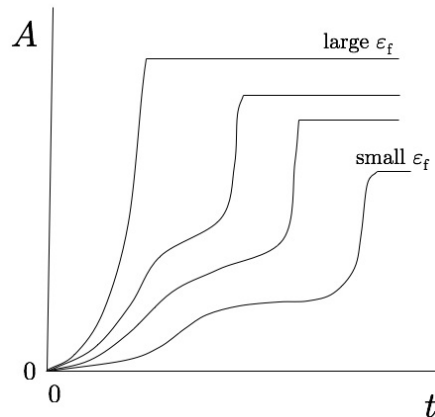


Figure 1: Amplitude *versus* time for the imperfect subcritical case $g < 0$, $h \neq 0$, where ε is switched suddenly from a negative initial value to a positive final value ε_f (problem 1d).

(c) In some experimental settings, both ε and g can independently be varied as control parameters. Consider the case where g varies continuously from negative to positive values. Furthermore assume that a small symmetry breaking term is present, so that the amplitude dynamics is given by

$$\dot{A} = h + \varepsilon A - g A^3 - k A^5, \quad (8)$$

where $k > 0$ is fixed. Assuming $h > 0$, and that h is very small, sketch the bifurcation diagram A_* *vs.* ε in the three cases (i) $g > 0$, (ii) $g = 0$, and (iii) $g < 0$.

¹The equation $\dot{A} = rA - A^3$ is also called the *Landau equation*.

(d) In the imperfect subcritical case, with $g < 0$ and $h \neq 0$, the parameter ε was switched suddenly at time $t = 0$ from a negative value to a positive value ε_f . The evolution of $A(t)$ was observed to follow the curves in Fig. 1. Note that the large ε_f rise very rapidly to their steady state value, while the small ε_f curves rise first to an intermediate plateau before quickly shooting to their final value. Can you explain this behavior?

Hint: Sketch \dot{A} versus A for different values of ε_f .

[4] (Strogatz problem 6.4.3) For at least three of the following systems, find and classify the fixed points, sketch the neighboring trajectories, and try to fill in the rest of the phase portraits.

- (a) $\dot{x} = x - y$, $\dot{y} = x^2 - 4$
- (b) $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$
- (c) $\dot{x} = \sin y$, $\dot{y} = \cos x$
- (d) $\dot{x} = \sin y$, $\dot{y} = x - x^3$
- (e) $\dot{x} = y + x - x^3$, $\dot{y} = -y$
- (f) $\dot{x} = xy - 1$, $\dot{y} = x - y^3$.

[5] (Strogatz problem 6.4.5) Consider the competition model

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2 \tag{9}$$

$$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2 , \tag{10}$$

where all the coefficients r_i and b_i are positive.

(a) By rescaling the populations and time, adimensionalize this model and reduce it to a form with a minimum of control parameters.

(b) Show that there are two qualitatively different kinds of phase portrait, depending on the size of K_1 . One way to proceed is to consider the *nullclines*, which are the sets along which $\dot{N}_1 = 0$ or $\dot{N}_2 = 0$.