Problem Session 9/29

1. $\dot{u} = u(u-1)(u-2)$

i) Sketch the equation & find fixed pts.
ii) classify the fixed pts.
* iii) $u_0 = \frac{3}{2}$. how long will it take to get to w/in 10^{-10} of the asymptotic value?

2. A magnetic system has free energy

$$F(M) = \tfrac{1}{2}aM^2 - \tfrac{1}{3}cM^3 + \tfrac{1}{4}bM^4 \quad ; \quad b, c > 0$$

The evolution of magnetization is given by

$$\frac{dM}{dt} = -\Gamma \frac{\partial F}{\partial M} \quad ; \quad \Gamma > 0$$

i) Rescale M & t to dimensionless variables m, τ s.t.

$$\frac{dm}{d\tau} = -\frac{\partial f}{\partial m}$$

where $f(m,r) = -\tfrac{1}{2}rm^2 - \tfrac{1}{3}m^3 + \tfrac{1}{4}m^4$.

ii) sketch the set of fixed pts m^* as a function of the control parameter r. Label stable and unstable branches. Identify and classify all bifurcations.

3. Consider the two dimensional phase flow

$$\dot{x} = \tfrac{1}{2}x + xy - 2x^3$$

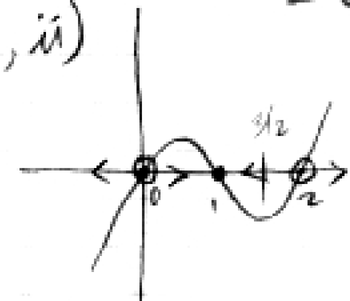
$$\dot{y} = \tfrac{5}{2}y + xy - y^2$$

i) Find and classify all fixed pts.

ii) Sketch the phase flow.

Solution Set

1. i), ii)



iii) $u_0 = 3/2$, $\dot{u} = u(u-1)(u-2)$

$$\frac{du}{dt} = u(u-1)(u-2) \quad \rightarrow \quad dt = \frac{du}{u(u-1)(u-2)}$$

→ method of Partial fractions:

$$\frac{1}{u(u-1)(u-2)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-2}$$

$$= \frac{A(u-1)(u-2) + B(u(u-2)) + C(u(u-1))}{u(u-1)(u-2)}$$

$$= \frac{(A+B+C)u^2 - (3A+2B+C)u + 2A}{u(u-1)(u-2)}$$

$2A = 1$, $3A+2B+C = 0$, $A+B+C = 0$

$A = \frac{1}{2}$, $B = -1$, $C = 1/2$

$$\therefore \quad dt = \frac{du}{2u} - \frac{du}{u-1} + \frac{du}{2(u-2)}$$

$$t(u) - t(0) = \frac{1}{2}\int_{u}^{u_0} \frac{du}{u} - \frac{2du}{u-1} + \frac{du}{u-2}$$

$$= \frac{1}{2} d\ln\left(\frac{u(u-2)}{(u-1)^2}\right)\Bigg|_{u}^{u_0}$$

$$t(u) = \cancel{t(0)}^{\,0} + \frac{1}{2}\ln\left(\frac{u(u-2)}{(u-1)^2}\,\frac{(u_0-1)^2}{u_0(u_0-2)}\right)$$

now, $u_0 = 3/2$, $u = 1 + 10^{-10}$

$$t = \frac{1}{2}\ln\left(\frac{1\cdot(-1)}{10^{-20}} \cdot \frac{(1/2)^2}{3/2(-1/2)}\right) = \frac{1}{2}\ln\left(\frac{1}{3}\times 10^{20}\right)$$

$$= -\frac{1}{2}\ln(3) + 10\ln 10$$

$$t = 22.48\,s$$

2 i) write $M = \alpha m$, $t = \beta \tau$

$$\frac{dm}{d\tau} = -\Gamma a \beta m + \Gamma \alpha \beta c m^2 - \Gamma \alpha^2 \beta b m^3$$

$$= r m + m^2 - m^3$$

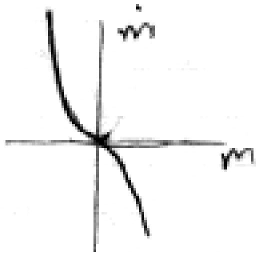
$$\Gamma a \beta = -r, \quad \Gamma \alpha \beta c = 1, \quad \Gamma \alpha^2 \beta b = 1 \quad \text{gcr}$$

$$\alpha = \frac{c}{b}, \quad \beta = \frac{b}{\Gamma c^2}, \quad r = -\frac{ab}{c^2}$$

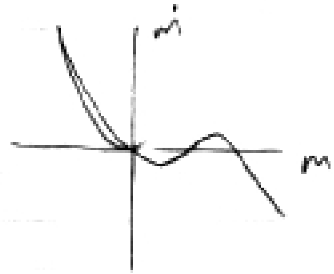
ii) $$\frac{\partial f}{\partial m} = (m^2 - m - r)m = 0$$

$$m^* = 0$$

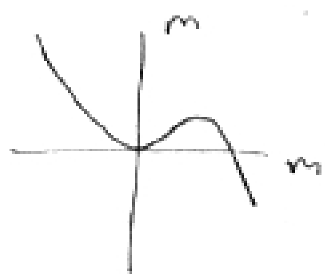
$$m^*_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{1 + 4r}\right)$$



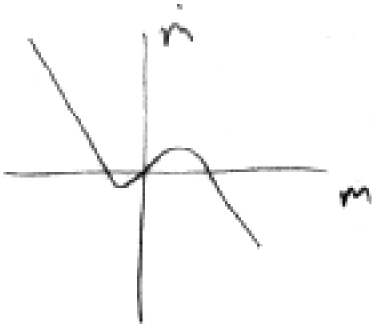
$r < -\frac{1}{4}$



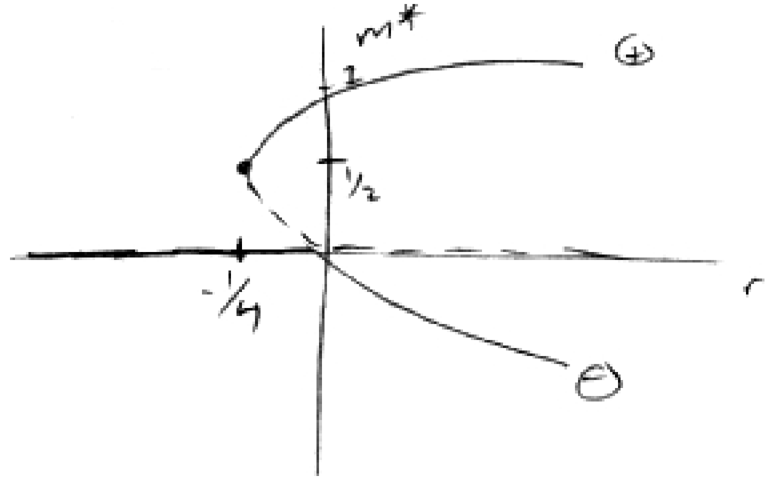
$-\frac{1}{4} < r < 0$



$r = 0$



$r > 0$



3. i)

$$\dot{x} = x\left(\tfrac{1}{2} + y - 2x^2\right) = 0$$

$$\dot{y} = y\left(\tfrac{5}{2} + x - y\right) = 0$$

$$M = \begin{pmatrix} \partial_x V_x & \partial_y V_x \\ \partial_x V_y & \partial_y V_y \end{pmatrix}$$

$$\lambda = \tfrac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$$

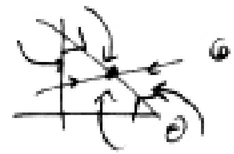
① $D < 0$, $\lambda_- < 0 < \lambda_+$

saddle pt:

transcritical

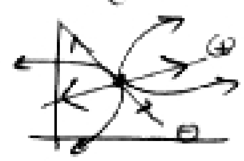
② $0 < D < \frac{1}{4}T^2$

(a) $T < 0$: $\lambda_- < \lambda_+ < 0$



stable

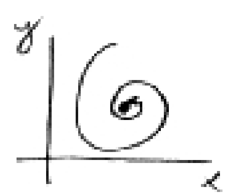
(b) $T > 0$; $0 < \lambda_- < \lambda_+$



unstable

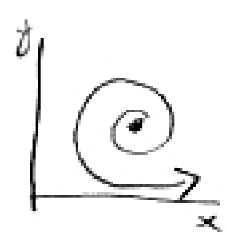
③ $D > \frac{1}{4}T^2$: $\lambda = \frac{1}{2}(T \pm i\nu)$

(a) $T < 0$: $\lambda = -\frac{1}{2}|T| \pm \frac{i}{2}\nu$



stable

(b) $T > 0$: $\lambda = +\frac{1}{2}|T| \pm \frac{i}{2}\nu$



unstable

→ spiral clockwise or CC depends on whether λ is + or −

There are six fixed points

1: $(x, y) = (0,0)$, $D = 5/4$, $T = 3$

$\therefore$ unstable node w/ $\lambda_1 = 1/2$, $\lambda_2 = 5/2$

2: $(x, y) = (0, 5/2)$, $D = -\frac{15}{2}$, $T = 1/2$

$\therefore$ since $D < 0$, saddle point. $\lambda_1 = -5/2$, $\lambda_2 = 3$

3: $(x, y) = (-1/2, 0)$, $D = -2$, $T = 1$

$\rightarrow$ saddle point , $\lambda_1 = -1$, $\lambda_2 = 2$

4: $(x, y) = (\frac{1}{2}, 0)$, $D = -3$, $T = +2$

$\rightarrow$ saddle point. $\lambda_1 = -1$, $\lambda_2 = 3$

5: $(x, y) = (\frac{3}{2}, 4)$, $D = 30$, $T = -13$

$D < \frac{1}{4}T^2$, $T < 0$ $\therefore$ stable node.

$\lambda_1 = -10$, $\lambda_2 = -3$

6: $(x, y) = (-1, 3/2)$, $D = \frac{15}{2}$, $T = -\frac{11}{2}$

$\rightarrow$ stable node , $\lambda_1 = -3$, $\lambda_2 = -5/2$