Physics 235 Spring 2005

Quasilinear Theory – Vlasov Plasma

i.) Motivation and Overview

→ linear theory determines 'instantaneous stability' of plasma

i.e. $\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int dv \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

⇒ growth/damping rate $\gamma_k = \gamma_k[\langle f\rangle]$

but $\langle f\rangle$ evolves... If $\langle f\rangle$ evolves slowly:

i.e. "slowly" ⇒ $\frac{1}{\langle f\rangle}\frac{\partial\langle f\rangle}{\partial t} < \gamma_k$

can consider: $\gamma_k = \gamma_k[\langle f(t)\rangle]$ → {evolution driven by instabilities

⇒ Physics: mean distribution evolution.... driven by relaxation.

→ quasilinear theory is concerned with describing and understanding the slow evolution of $\langle f\rangle$....

③ quasilinear theory is "mindless mean field theory", i.e.

$\langle f\rangle = \langle f(v,t)\rangle$ where $\to \langle\ \rangle$ eliminates spatial dependence

$\to$ f understood "slow"

so if:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

then Q.L. equation is simply: (upon avg.)

$$\frac{\partial\langle f\rangle}{\partial t} + \frac{\partial}{\partial v}\left\langle \frac{q}{m}\tilde{E}\tilde{f}\right\rangle = 0$$

i.e. generic mean field equation (for $\langle f\rangle$) for mean of conserved order parameter ($\langle f\rangle$)

$$\frac{\partial\langle f\rangle}{\partial t} + \frac{\partial}{\partial v}J_v = 0$$ $\to$ phase space continuity equation

$$J_v = \Gamma_v = \left\langle \frac{q}{m}Ef\right\rangle$$

$$= \frac{q}{m}\langle\tilde{E}\tilde{f}\rangle$$

for: $E = \tilde{E}$

$$f = \langle f\rangle + \tilde{f}$$

elementary closure problem

i.e. relate $\langle f\rangle$ to $\langle\tilde{E}\tilde{f}\rangle$ $\to$ hierarchy!

How close?

simplest example of moment closure.

then Q.L.T. simply takes form:

(i) $\tilde{f} \to \tilde{f}_{Linear}$ (i.e. linear response of f)
plug in linear response

i.e. $\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$ Vlasov Eqn.

$$\Rightarrow -i(\omega - kv)\tilde{f}_{k,\omega} = -\frac{q}{m}\tilde{E}_{k,\omega}\frac{\partial \langle f\rangle}{\partial v}$$

so $$J_v = -\frac{q^2}{m^2}\sum_{k,\omega}|\tilde{E}_{k,\omega}|^2\frac{i}{(\omega - kv)}\frac{\partial\langle f\rangle}{\partial v}$$

and with $\omega = \omega(k)$ only (i.e. spectrum of eigenmodes, only)

i.e. contrast approach to criticality in usual phase transition (2nd order)
→ here growth of order parameter in broken symmetry phase ... not noise driven

Q.L. equation is:

$$\left.\begin{aligned}\frac{\partial\langle f\rangle}{\partial t} &= \frac{\partial}{\partial v}D\frac{\partial\langle f\rangle}{\partial v}\\ D &= \frac{q^2}{m^2}\sum_k|\tilde{E}_k|^2\frac{i}{\omega - kv}\end{aligned}\right\}\text{"Q.L. equation"}$$

i.e. mindless mean field theory ....

$$\text{with}\quad\left.\begin{aligned}\epsilon(k,\omega) &= 0\\ \partial_t|E_k|^2 &= 2\gamma_k|E_k|^2\end{aligned}\right\}\to\text{advance fields.}$$

But

Surprisingly: Q.L.T. works quite well!

Key issue: why?

N.B.: In contrast to critical phenomena, external noise ignored → instability driven ...

④ Some questions to keep in mind: deterministic

i) why is QL equation a diffusion equation? When is this valid?

↔ nature of "irreversibility" ....

ii) can Q.L. equation be derived from Fokker-Planck theory?

↔ also "irreversibility" related ...

iii) how does Q.L. equation balance the energy-momentum budgets?

iv) when / how does Q.L. theory fail?

↔ related i) ..... What is "Ginzburg Criterion" for Q.L.T. Can such a criterion be formulated?

v) what is dynamics of quasilinear relaxation?

i.e. physics?

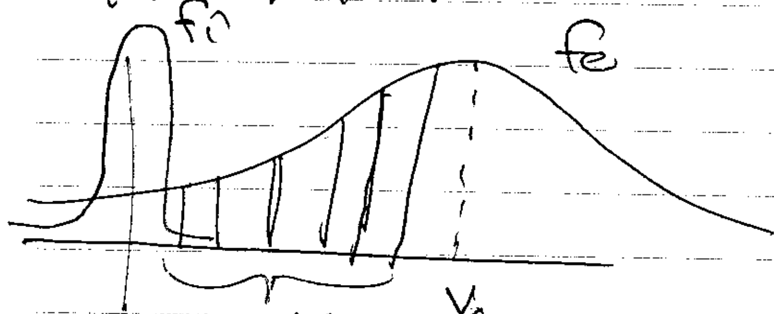
iii) Basic Scales / Regime Definition

① → Generally, Q.L.T. concerned with

i.) 'broad' spectrum of:

ii.) unstable waves

i.e. for current-driven ion-acoustic (C.D.I-A.) turbulence:



②
→ In finite system, k quantized, i.e.

$k_m = m\pi/L$, etc.

- so, have spectrum of phase velocities

$$\omega_m/k_m = \omega(k_m)/k_m = V_{ph\,m}$$

- wave-particle resonance occurs when

$$V = V_{ph,m}$$

then, sin Isaac ⇒

$$m\ddot{x} = \sum_m qE_m \cos(k_m x - \omega_m t)$$

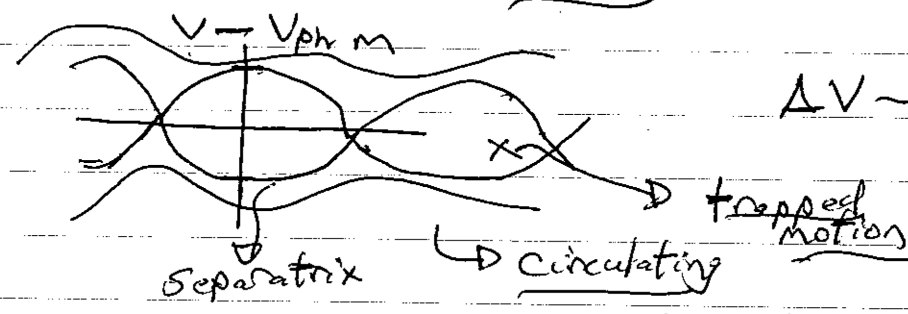
n.b. {deterministic, no RPA}

and 1 resonance dominant ⇒

$$m\ddot{x} \cong qE_{m_0} \cos(k_m x_0 + (k_m v - \omega_m)t)$$

⇒ each resonant velocity defines a phase space island

i.e.

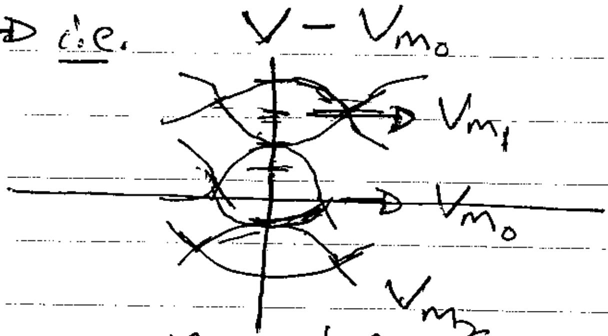


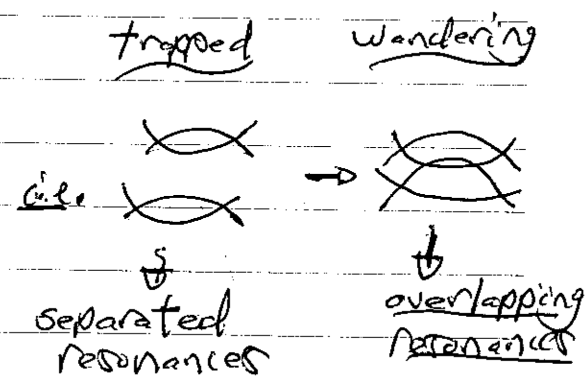
$$\Delta v \sim \left(\frac{q}{m}\phi_0\right)^{1/2}$$

∴ QLT is concerned with the case of:

→ multiple, overlapping resonances → {separatrix proximity ⇒ destruction}

→ i.e.





if overlap,

∴ particle can wander stochastically from resonance-to-resonance, i.e. hopping

⇒ diffusion in v

$$D_v \sim \frac{(\Delta v)^2}{\tau_{ac}}$$

Δv ~ resonance width

τ_{ac} → pattern time

→ what is it?

Overlap condition (B.V. Chirikov):

$$\boxed{\frac{1}{2}\left(\Delta V_m + \Delta V_{m+1}\right) \geq V_{ph_{m+1}} - V_{ph_m}.}$$

$\therefore$ ⇒ particle motion stochastic

→ fundamental irreversibility ⇒ orbit stochasticity (not dissipation, Landau damping ... ⇔ contrast critical phenomena).

→ underpinning of diffusion equation.

③ → But, a swindle !? → use of un-perturbed orbit in estimate!

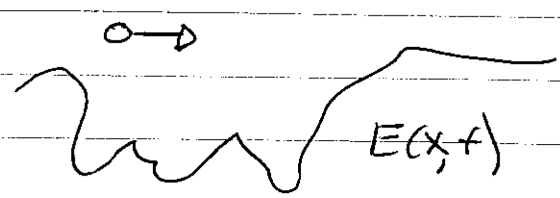
i.e. is $x \to x_0 + vt$ valid ?

Consider: linear, un-perturbed orbit ?!

have: $E(x,t) = \sum_k E_k \exp[i(kx - \omega_k t)]$

$\therefore$ particle "sees" instantaneous pattern of electric field, from modal superposition

i.e.



∴ relevant comparison is:

$\tau_L \rightarrow$ life time of 'instantaneous' pattern

$\tau_b \rightarrow$ 'bounce time' of particle in pattern.

obviously, ① $\tau_L \ll \tau_b \rightarrow$ unperturbed orbit is satisfactory approximation
(pattern changes prior bouncing) $\longleftrightarrow$

② $\tau_L \gg \tau_b \rightarrow$ particle bounces prior pattern changes,

so must consider orbit perturbation...

i.e.

$\tau = 0$ $\quad$ τ_b

vs.

∴ quasilinear theory relevant to evolution when:

① $\rightarrow$ orbits stochastic (Chirikov condition satisfied)

② $\rightarrow$ $\tau_{Life} < \tau_{bounce}$ $\rightarrow$ unperturbed orbits ≈ valid.

3)

But, how relate $T_{lifetime}$, T_{bounce} to physical quantities?

Key point: Superposition patterns disperse!

$E(x,t)$

$$\Leftrightarrow \sum_k E_k e^{i(kx-\omega_k t)}$$

$$= \sum_k E_k \exp\left[i\left(k\left[x-\left(\omega_k/k\right)t\right]\right)\right]$$

$v_{ph}(k)$

$\therefore$ $\Delta(\omega_k/k) \equiv$ spread in phase velocities.

sets dispersal rate

so dispersal rate is $(\text{time})^{-1}$ to disperse by one wavelength

$$1/T_{life} \equiv k\,\Delta(\omega_k/k)$$

$$= k\left(\frac{d\omega_k}{dk}\frac{\Delta k}{k} - \frac{\omega_k}{k^2}\Delta k\right)$$

$(v_g - v_{ph})$

$$= \left(\frac{d\omega_k}{dk} - \frac{\omega_k}{k}\right)\Delta k = \left(v_g(k) - v_{ph}(k)\right)\Delta k$$

n.b. $\begin{cases} T_{life} \to \infty \text{ for non-dispersive waves!} \\ \text{Generally; QLT / weak turbulence encounters trouble for } \begin{cases}\text{non dispersive} \\ \text{weakly dispersive}\end{cases} \text{ waves.} \end{cases}$

How systematize?

Consider: $\langle E(x_1,t_1)\, E(x_2,t_2)\rangle_{x,t} = C$

electric field correlation function

$C = C(x_-, \tau)$, for $\left\{\begin{matrix}\text{homogeneous} \\ \text{stationary}\end{matrix}\right\}$ fluctuations

$x_1 = x_+ + x_-$ $\qquad t_1 = t_+ + t_-$

$x_2 = x_+ - x_-$ $\qquad t_2 = t_+ - t_-$

$\langle\ \rangle_{x,t} = \langle\ \rangle_{x_+,t_+}$

so

$$C(x_-,\tau) = \Big\langle \sum_{k,k'} E_k E_{k'} e^{i(k+k')x_+} e^{-i(\omega_k+\omega_{k'})t_+} \times e^{i(k-k')x_-} e^{-i(\omega_k-\omega_{k'})t_-} \Big\rangle_{x_+,t_+}$$

x_+, t_+ average $\Rightarrow k = -k'$ $\quad \omega_k = -\omega_{k'}$

so

$$C(x_-,\tau) = \sum_k |E_k|^2 e^{ikx_-} e^{-i\omega_k t_-}$$

Now:

→ assume continuous spectrum

→ for simplicity, take model

$$|E_k|^2 = E_0^2 \Big/ \left[\left(\frac{k-k_0}{\Delta k}\right)^2 + 1\right]$$

→ evaluate on u.p.o.

$$x_- = x_{0-} + V\tau$$

$$\therefore \quad \langle E^2 \rangle = \int dk \frac{E_0^2}{\left[\frac{(k-k_0)^2}{\Delta k^2} + 1\right]} e^{ikx_{0-}} e^{i(kV-\omega_k)\tau}$$

integrating:

phase info. - irrelevant

$$\sim E_0^2 e^{ik_0 x_-} e^{-|\Delta k x_{0-}|} *$$

$$e^{i(k_0 V - \omega_{k_0})\tau} e^{-|\Delta(kV-\omega_k)|\tau}$$

oscillation
(→0 on resonance)

→ correlation decay due dispersal and its interplay with resonance.

n.b.: note that spread in doppler-shifted ω is critical now $\Delta(kV-\omega_k) = V\Delta k - V_{gr}\Delta k$

$$= |(V-V_{gr})\Delta k|$$

$V_{gr} = \frac{d\omega}{dk}$

so
$$\langle E^2\rangle = C(x_-, \tau)$$
$$= E_0^2\, e^{ik_0x_-}\, e^{i(k_0V-\omega_{k_0})\tau}\, e^{-|\Delta k\, x_0|}$$
$$* \exp\Big[-\underbrace{|(V-V_{gr})\Delta k|}_{\text{sets lifetime}}\,\tau\Big]$$

$$1/\tau_L = |(V-V_{gr}(k))\Delta k| \equiv (\text{Autocorrelation Time})^{-1}$$
$$\equiv 1/\tau_{ac}.$$

Note:

- for resonant particles, $V = \omega_k/k$

$1/\tau_L = |(V_{ph}-V_{gr})\Delta k|$ $\rightarrow$ recovers earlier!

- can think: $|V\Delta k| \rightarrow 1/\tilde{\tau}_{ac}^{\text{wave-particle}}$

$|V_{gr}\Delta k| \rightarrow 1/\tilde{\tau}_{ac}^{\text{wave}}.$

generally, shorter time dominates, except for non-dispersive waves.

So, can enumerate key time scales

$\tau_{ac} = |\Delta k(v_{ph} - v_{gr})|^{-1}$

$\equiv$ persistance of E pattern ($\langle E^2 \rangle$ autocorrelation) for resonant particles.

$\gamma^{-1} =$ growth/damping time

$\tau_{Tr} = \left(k\sqrt{q\phi/m}\right)^{-1} \equiv$ trapping time

$\tau_{relax} = \left(\frac{1}{\langle f \rangle} \frac{\partial \langle f \rangle}{\partial t}\right)^{-1} \equiv$ avg. distribution relaxation time

So

$\tau_{ac} < \tau_{Tr} \Rightarrow$ u.p.o. valid

$\left.\begin{matrix}\tau_{ac}\\ \gamma^{-1}\end{matrix}\right. < \tau_{relax} \Rightarrow \langle f \rangle$ closure meaningful.

$\tau_{ac} < \gamma^{-1} < \tau_{relax} \Rightarrow$ Q.L.T. valid.

iii.) Energy - Momentum Budgets

→ key Point: There are two ways of implementing the book-keeping and accounting

i.e $\left\{\begin{array}{lll} \text{resonant particles} & \text{vs.} & \text{'waves'} \\ \underline{\text{or}} & & \\ \text{particles} & \underline{\text{vs.}} & \text{fields} \end{array}\right.$

keep in mind: Wave = Field + Non-resonant Particles

i.e for plasma oscillation, $\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

$$\text{Wave Energy} = W = \frac{\partial}{\partial \omega}(\omega \epsilon)\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= \omega \frac{\partial \epsilon}{\partial \omega}\Big|_{\omega_k} \frac{|E|^2}{8\pi}$$

$$= 2 \cdot \frac{|E|^2}{8\pi}$$

field non-resonant particle.

(Show!)

→ Resonant Particles vs. Waves ?

$$\frac{\partial \langle f\rangle}{\partial t} = -\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$\frac{\partial}{\partial t}\int dv\, \frac{mv^2}{2}\langle f\rangle = -\int dv\, \frac{mv^2}{2}\frac{\partial}{\partial v}\frac{q}{m}\langle \tilde{E}\tilde{f}\rangle$$

$$= \int dv\, \cancel{m} v\, \frac{q}{\cancel{m}}\langle \tilde{E}\tilde{f}\rangle$$

Plugging in $\tilde{f}_k^{\,Linear}$ for $\tilde{f}$:

$$\frac{\partial}{\partial t}\mathcal{E}_{kin} = -i\int dv\, \frac{vq^2}{m}\sum_k |E_k|^2\left(\frac{P}{\omega-kv} - i\pi\delta(\omega-kv)\right)\frac{\partial\langle f\rangle}{\partial v}$$

$$\frac{\partial}{\partial t}\mathcal{E}_{kin}^{res} = -\int dv\, \frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\delta(\omega/k - v)\frac{\partial\langle f\rangle}{\partial v}|E_k|^2$$

resonant only

$$= -\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

As resonant particles stabilize/destabilize wave, expect resonant particles conserve energy against waves.

In wave energy evolution:

Recall: $\epsilon = 1 + \frac{\omega_p^2}{k}\int dv \, \frac{\partial\langle f\rangle/\partial v}{\omega - kv}$

$$\epsilon^r(\omega_k + i\gamma_k) + i\epsilon^{IM} = 0$$

$$i\gamma_k = \frac{-i\epsilon^{IM}}{\partial\epsilon^r/\partial\omega} \qquad ; \cancel{i}\gamma_k = -\cancel{i}\frac{\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}$$

$$= -\epsilon_{IM}/\partial\epsilon^r/\partial\omega$$

Now, $W \equiv$ Wave Energy Density

$$W = \sum_k \frac{\partial}{\partial\omega}(\omega\epsilon)\frac{|E_k|^2}{8\pi}$$

$$= \sum_k \omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k}\frac{|E_k|^2}{8\pi}$$

$$\frac{\partial W}{\partial t} = \sum_k 2\gamma_k\,\omega_k \left.\frac{\partial\epsilon^r}{\partial\omega}\right|_{\omega_k}\frac{|E_k|^2}{8\pi} \qquad |E_k|^2 = |E_k^0|^2 e^{2\gamma_k t}$$

$$= \sum_k 2\left(\frac{-\epsilon_{IM}}{\partial\epsilon^r/\partial\omega}\right)\omega_k\frac{\partial\epsilon^r}{\partial\omega}\frac{|E_k|^2}{8\pi}$$

$$= \sum_k -\epsilon_{IM}(k,\omega_k)\,\omega_k\left(\frac{|E_k|^2}{4\pi}\right)$$

$$i\,\epsilon_{IM} = \frac{\omega_p^2}{k}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}\frac{(-i\pi)}{|k|}$$

$(n_0 = 1)$

$$\therefore\ \frac{\partial W}{\partial t} = \sum_k \frac{\pi q^2}{m}\frac{\omega_k \cancel{4\pi}}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial \cancel{k}}\right|_{\omega/k}\frac{|E_k|^2}{\cancel{4\pi}}$$

$$= +\frac{\pi q^2}{m}\sum_k \frac{\omega}{k|k|}\left.\frac{\partial\langle f\rangle}{\partial v}\right|_{\omega/k}|E_k|^2$$

$$\Rightarrow\ \boxed{\partial_t \mathcal{E}_{kinetic}^{resonant} + \partial_t W = 0}$$

Note:

- this is essentially a re-write of the Poynting theorem for plasma waves, i.e.

$$\underbrace{\frac{\partial}{\partial t}W}_{\text{wave energy}} + \underbrace{\nabla\cdot\underline{S}}_{\text{divergence of wave energy density flux}} + Q = 0$$

$Q \rightarrow \langle \tilde{\underline{E}}\cdot\tilde{\underline{J}}\rangle$ coupling

For homogeneous system: $\underline{\nabla}\cdot\underline{S} = 0$

so $\frac{\partial W}{\partial t} + Q = 0$

Q: $\langle E\cdot J\rangle$ mediated by resonant particles (DC field)

$$\Leftrightarrow \boxed{\frac{\partial W}{\partial t} + \frac{\partial}{\partial t}(RPKED) = 0}$$

RPKED: resonant particle kinetic energy density

Energy Thm I

Waves and Resonant particles conserve energy!

? What is the fate of RPKED for saturated waves. What must happen ??

→ Now, can observe:

$$W \equiv NRPKED + FED$$

NRPKED: non-resonant particle kinetic energy density

FED: field energy density

so, simply re-grouping terms:

$$\frac{\partial}{\partial t}(FED) + \frac{\partial}{\partial t}\underbrace{(RPKED + NRPKED)}_{PKED} = 0$$

So $$\boxed{\frac{\partial}{\partial t} FED + \frac{\partial}{\partial t} (PKED) = 0}$$ Energy Thm.

i.e. fields and particles conserve energy.

What is the Physics of all this ?!

$$\underbrace{D}_{\substack{\text{QL} \\ \text{diffusion}}} = \text{re} \sum_k \frac{q^2}{m^2} |E_k|^2 \left(i/\omega - kv \right)$$

for general, weakly non-stationary state ---

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left(|\gamma_k| / \left((\omega - kv)^2 + |\gamma_k|^2 \right) \right)$$

n.b. causality ⇒ no negative diffusion for damped waves

$$\cong \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \underbrace{\pi \delta(\omega - kv)}_{\substack{\text{resonant} \\ \text{diffusion}}} + \underbrace{\frac{|\gamma_k|}{\omega_k^2}}_{\substack{\text{non-resonant} \\ \text{diffusion}}} \right\}$$

Resonant Diffusion → irreversible - resonance overlap is underpinning

→ rooted in particle stochasticity

→ resonant diffusion can be obtained from Fokker-Planck calculation (show this)!

→ in principle, can persist in steady state (but how balance energy....?!)

Non-Resonant Diffusion:

$$D^{NR} = \sum_k \frac{q^2}{m^2}|E_k|^2 \frac{\gamma_k}{\omega_k^2}$$

$$= \frac{1}{2}\partial_t \sum_k |V_k|^2 \qquad \text{where } \underbrace{|V_k|^2 = \frac{q^2}{m^2}\frac{|E_k|^2}{\omega_k^2}}_{\text{ponderomotive energy}}$$

→ corresponds to "sloshing" motion energy of particles in wave

i.e. $D^{NR} \sim \partial_t \, \mathcal{E}_{quiver}$

→ thus reversible, can't be obtained from Fokker-Planck theory → aka' "fake diffusion"

→ vanishes in stationary state

Point is that can "account" non-resonant diffusion as:

$D_{\text{non-resonant}}$ $\left\langle \begin{array}{l} \text{particle sloshing energy part of } \underline{\text{wave}} \text{ energy density} \\ \underline{\underline{\text{or}}} \\ \text{part of total particle kinetic energy density} \end{array} \right.$

so $\underline{\underline{\text{two}}}$ forms of energy conservation !

Note: Physically, the picture of plasma as gas $\begin{cases} \text{- resonant particles} \\ \text{- waves} \end{cases}$ $\underline{\underline{\text{or}}}$ $\underline{\text{equivalently}}$

$\underline{\text{resonant particles}}$ + $\underline{\text{quasi-particles}}$

$\downarrow$

$\underline{\text{waves}}$ $\left\langle \begin{array}{l} N(k, x, t) \\ WKE, \text{ etc.} \end{array} \right.$

is appealing and will pervade this course.

N.B.: Direct Proof of $\partial_t(PKED + FED) = 0$

From QL equation:

$$\frac{\partial}{\partial t}(PKED) = -\sum_k \int dv\, \frac{\omega_p^2}{k} kv \frac{|E_k|^2}{4\pi} \frac{i}{\omega - kv} \frac{\partial \langle f \rangle}{\partial v}$$

$$\epsilon(k,\omega) = 1 + \frac{\omega_p^2}{k}\int \frac{dv}{\omega - kv}\frac{\partial \langle f \rangle}{\partial v}$$

so

$$\partial_t (PKED) = -i \sum_k \frac{|E_k|^2}{4\pi}\int dv\, \frac{\omega_p^2}{k}(\underbrace{kv-\omega}_{} + \omega) * \frac{i}{\omega - kv}\frac{\partial \langle f \rangle}{\partial v}$$

$\hookrightarrow$ { cancels denom
residual odd in
k

$$= -i \sum_k \frac{|E_k|^2}{4\pi}\int dv\, \frac{\omega_p^2}{k}\frac{\omega}{\omega - kv}\frac{\partial \langle f \rangle}{\partial v}$$

using $\epsilon(k,\omega) = 0$

$$= i \sum_k \frac{|E_k|^2}{4\pi}\omega_k$$

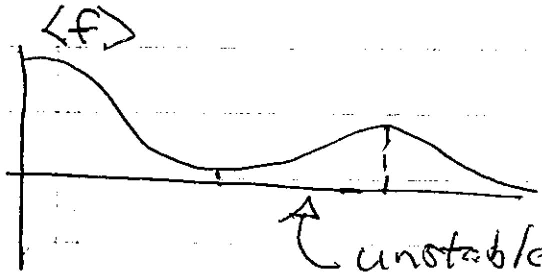
$\omega_k = \omega_k^r + i\gamma_k$

$$= -\sum_k \frac{|E_k|^2}{8\pi}(2\gamma_k)$$

$$= -\partial_t (FED) \checkmark$$

iv.) Applications of Quasilinear Theory

→ Bump on Tail



↑ unstable phase velocities. (bump on tail)

$\omega_k = \omega_{pe}(1 + \frac{3}{2} k^2 \lambda_D^2)^{1/2}$

Quasi-linear Equations:

$$\epsilon(k, \omega_k) = 0 \quad \Rightarrow \omega(k), \gamma(k) \quad \text{from } \langle f \rangle$$

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D \frac{\partial \langle f \rangle}{\partial v}$$

$$D = D^R + D^{NR}$$

$$= \sum_k \frac{q^2}{m^2} |E_k|^2 \left\{ \pi \delta(\omega_k - kv) + \frac{\gamma_k}{\omega_k^2} \right\}$$

$$\frac{\partial}{\partial t} \left(|E_k|^2 / 8\pi \right) = 2\gamma_k |E_k|^2 / 8\pi$$

Observe: - resonant diffusion describes dynamics of tail particles

- non-resonant diffusion describes dynamics of bulk Maxwellian

Expect: - tail flattening
with
- adjustment of core/bulk profile (i.e. effective "temperature")

Now first consider resonant particles (i.e. on bump):

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v} \qquad * \langle f \rangle \text{ and } \int \text{by p} \Rightarrow$$

$\Rightarrow$

$$\frac{\partial}{\partial t} \int_{res} \frac{\langle f \rangle^2}{2} = - \int_{res} dv \, D^R \left(\frac{\partial \langle f \rangle}{\partial v} \right)^2$$

$\left\{ \begin{array}{l} \text{generalization} \Rightarrow \\ \text{Zeldovich Thm.} \end{array} \right.$

stationarity $\Rightarrow$

$$D^R \left(\partial \langle f \rangle / \partial v \right)^2 = 0$$

Now "res" $\rightarrow$ some finite interval of phase velocities

So

stationarity $\Rightarrow$ $D^R = 0$; i.e. fluctuations decay and damp

or

$\partial \langle f \rangle / \partial v = 0$; plateau forms, removing growth

N.B.: – In 1D $\rightarrow$ plateau
– can generalize

To resolve:

$$D^R = \frac{8\pi^2 q^2}{m^2} \sum_k \frac{|E_k|^2}{8\pi} \delta(\omega - kv)$$

$$\cong \frac{16\pi^2 q^2}{m^2} \int dk \, \mathcal{E}_F(k) \, \delta(\omega - kv)$$

$$D^R = \frac{16\pi^2 q^2}{m^2 v} \mathcal{E}_F(\omega_{pe}/v)$$

so

$$\partial_t D^R = \frac{16\pi^2 q^2}{m^2 v} \left(2\gamma_{\omega_{pe}/v}\right) \mathcal{E}(\omega_{pe}/v)$$

Now, $\gamma_k = -\epsilon_{IM} \Big/ \left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega_k}$

$$\gamma_k = \gamma_{\frac{\omega_{pe}}{v}} = \pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}$$

so
$$\frac{\partial D^R}{\partial t} = \frac{16\pi^2 q^2}{m^2 v}\left(2\pi v^2 \omega_p \frac{\partial \langle f \rangle}{\partial v}\right) \mathcal{E}(\omega_p/v)$$

$$= \left(\pi \omega_p v^2 \frac{\partial \langle f \rangle}{\partial v}\right) D^R$$, using D^R defn.

So

$$D^R(v,t) = D^R(v,0) \exp\left[\pi \omega_{pe} v^2 \int_0^t dt' \frac{\partial \langle f \rangle}{\partial v}\right]$$

and:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^R \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial t} \frac{\partial}{\partial v}\left[\frac{D^R}{\pi \omega_{pe} v^2}\right]$$ $\left\{\begin{array}{l}\text{using } \gamma_k, D \\ \text{definitions}\end{array}\right.$

So

$$\langle f(v,t)\rangle - \langle f(v,0)\rangle = \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]$$

∴ have:

$$\boxed{\begin{aligned} D^R &= D^R(v,0)\exp\left[\pi\omega_p v^2\int_0^t dt' \frac{\partial\langle f\rangle}{\partial v}\right] \\ \langle f(v,t)\rangle &= \langle f(v,0)\rangle + \frac{\partial}{\partial v}\left[\frac{D^R(v,t) - D^R(v,0)}{\pi\omega_p v^2}\right]\end{aligned}}$$

Now, recall seek to know if:

i.) $D^R \to 0 \Rightarrow \partial\langle f\rangle/\partial v\big|_{t\to\infty} < 0$ (fluctuations damp)

ii.) $\partial\langle f\rangle/\partial v \to 0 \Rightarrow$ finite D^R, distribution plateaus.

Now, if $D^R \to 0$,

$$\langle f(v,t)\rangle = \langle f(v,0)\rangle - \frac{\partial}{\partial v}\left[\frac{D^R(v,0)}{\pi\omega_p v^2}\right]$$

$$D^R(0) = \frac{16\pi^2 q^2}{m^2 v}\,\mathcal{E}(\omega_p/v, 0)$$

fluctuation energy

but $\frac{16\pi^2 q^2}{m^2 v} \frac{\mathcal{E}(0)}{\pi\omega_p v^2} = 2E_F(0) / (n m v_b^2/2)$

$\ll 1$, as $n \gg n_b$

$\therefore$ $\langle f(v,t)\rangle \cong \langle f(v,0)\rangle$, to good approx.

but, for resonant velocities,

$\rightarrow$ linear instability $\Rightarrow$ $\partial\langle f\rangle/\partial v > 0$

$\rightarrow$ $D^R \underset{t\to\infty}{\rightarrow} 0$ $\Rightarrow$ $\partial\langle f\rangle/\partial v < 0$

but have (for $D^R \to 0$) $\langle f(t)\rangle = \langle f(0)\rangle$!

$\therefore$ $\boxed{\text{contradiction follows from assumption of } D^R(v,t) \to 0}$

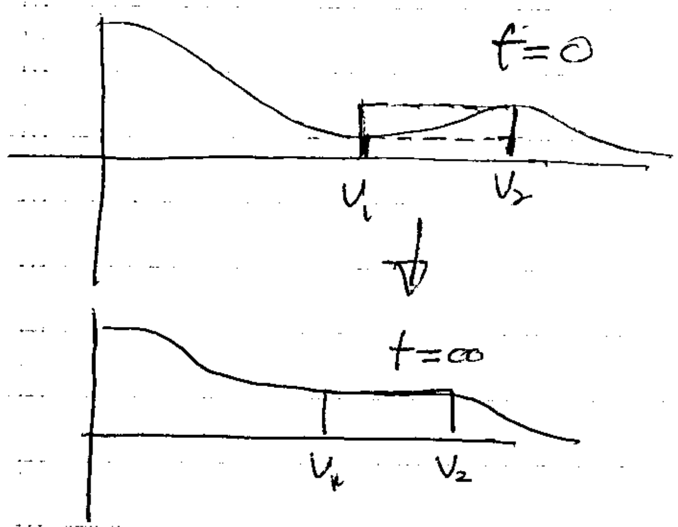
$\therefore$ have established that

$\partial\langle f\rangle/\partial v \big|_{res} \rightarrow 0$ $\Rightarrow$ plateau forms!

For plateau formation, can immediately determine saturation levels from

$$\frac{\partial}{\partial t}(\text{RPKED}) + \frac{\partial}{\partial t}(\text{WED}) = 0$$

i.e.



$t \to \infty$

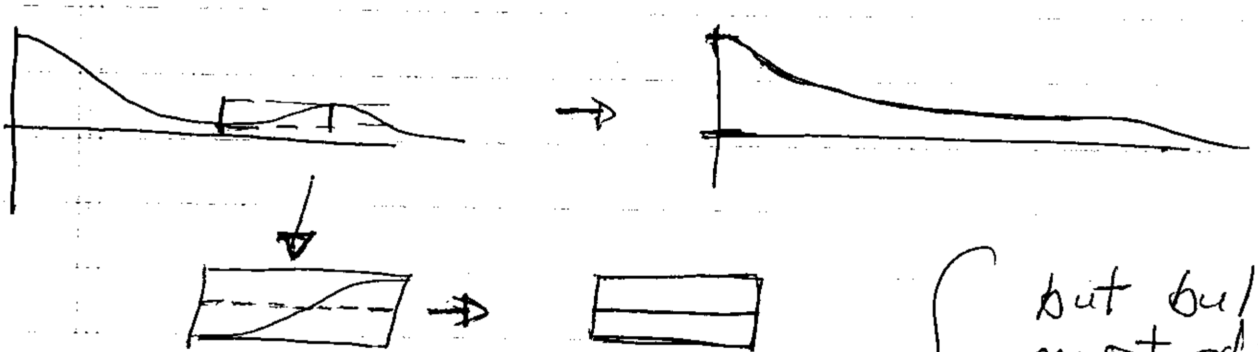
$k = \omega_p / v$

$$\therefore \quad \Delta\left(\int_{v_1}^{v_2} \frac{mv^3}{2}\langle f\rangle\right) = -\Delta\int_{k_1}^{k_2} W_k \, dk$$

but $W_k = 2\,\mathcal{E}(k)$

$$\Rightarrow \quad \Delta\left(\int_{v_1}^{v_2} dv\, \frac{mv^3}{2}\langle f\rangle\right) = -2\Delta\int_{k_1}^{k_2} \mathcal{E}(k)\, dk$$

→ can estimate A (RPKED) analytically, via construction



i.e. beam slows down

but bulk must adjust to conserve momentum!

i.e. bulk spreads outward, to conserve momentum as beam slows (bump flattened inward)

Now, for non-resonant particles:

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} D^{NR} \frac{\partial \langle f \rangle}{\partial v}$$

$$= \frac{\partial}{\partial v} \frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\gamma_k}{(\omega - kv)^2} \frac{\partial \langle f \rangle}{\partial v}$$

$$\cong \frac{8\pi q^2}{m^2} \int dk\, \mathcal{E}(k) \frac{\gamma_k}{\omega_{p0}^2} \frac{\partial^2 \langle f \rangle}{\partial v^2}$$

so, using γ definition:

$$\frac{\partial\langle f\rangle}{\partial t} = \left(\frac{1}{nm}\frac{\partial}{\partial t}\int dk\, \mathcal{E}(k)\right)\frac{\partial^2\langle f\rangle}{\partial v^2}$$

now define $T(t) = \frac{2}{n_e}\int dk\, \mathcal{E}(k,t)$

so

$\Rightarrow$

$$\frac{\partial\langle f\rangle}{\partial T} = \frac{1}{2m}\frac{\partial^2\langle f\rangle}{\partial v^2}$$

thus, for initial Maxwellian:

$$\langle f\rangle = \left[m/2\pi\left[T+T(t)-T(0)\right]\right]^{1/2}\exp\left[\frac{-mv^2/2}{\left[T+T(t)-T(0)\right]}\right]$$

Thus, for non-resonant particles

- at saturation

$$T/2 \to T/2 + \frac{1}{n}\int dk\left[\mathcal{E}(k,\infty)-\mathcal{E}(k,0)\right]$$

i.e. electrons 'heated' by net increase in field energy ....

– can also note:

$$\frac{\partial}{\partial t}(RPKED) + \frac{\partial}{\partial t}(WED) = 0$$

for plasma waves,

$$\frac{\partial}{\partial t}(RPKED) = -2\frac{\partial}{\partial t}(FED)$$

so $\Delta(RPKED) = -2\,\Delta(FED)$

but

$$\Delta(PKED) = -\Delta(FED)$$

so $\Delta(RPKED) = +2\,(\Delta(PKED))$

$$\Rightarrow \quad 0 = \Delta(RPKED) + 2\Delta(NRPKED)$$

and

$$\Delta(PKED) - \Delta(RPKED) = -\Delta(FED) - (-2)\Delta(FED)$$

$$\boxed{\Delta(NRPKED) = \Delta(FED)}$$

as shown above!

→ heating is one-sided, to conserve momentum.