Chapter 1

A Tutorial on Basic Concepts in MHD Turbulence and Turbulent Transport

P.H. Diamond(1,2), S.-I. Itoh(2), K. Itoh(3)
(1)University of California, San Diego, La Jolla, CA 92093-0319 USA (2)Research Institute for Applied Mechanics, Kyushu University, Kasuga City 816-8580 Japan (3)National Institute for Fusion Science, Toki-Shi 509-5292 Japan

1.1 Introduction

In this paper, we present an eclectic tutorial on some of the basic ideas in MHD turbulence and turbulent transport, with special attention to incompressible and weakly compressible dynamics with a mean magnetic field. The approach throughout is *conceptual* - we emphasize intuition, ideas and basic notions rather than detailed results. The latter are already well documented in the research literature. Also, we place primary emphasis on understanding the case of *magnetized* MHD turbulence, in which an externally fixed, large scale magnetic field breaks symmetry, produces anisotropy and restricts nonlinear interactions. This case should be contrasted with that considered in many (but not all) discussions of MHD turbulence, which focus on weakly magnetized or unmagnetized systems.

This paper is organized into four sections, each of which discusses an essential paradigm in MHD turbulence theory. These four sections should be thought of as four related but distinct vignettes, rather than one continuous narrative. Each attempts to give the reader the chance to peep at some essential ideas in the theory. The four sections are:

1.2.) K41 Beyond Dimensional Analysis - Revisiting the Theory of Hydrodynamic Turbulence

1.3.) Kraichnan-Iroshnikov, Goldreich-Sridhar and all that: A Scaling Theory of MHD Turbulence

1.4.) Steepening of Nonlinear Alfven Waves - a little compressibility goes a long way.....

1.5.) Turbulent Flux Diffusion in 2D MHD - a 'minimal' problem which is not so simple.....

While these four topics are distinct, they do have a common theme, namely the effect of Alfvenically induced 'memory' on turbulence and transport. These four sections are followed by a brief discussion and conclusion (1.6). We indicate some possible future research directions throughout the paper, where appropriate.

1.2 K41 Beyond Dimensional Analysis - Revisiting the Theory of Hydrodynamic Turbulence

Surely everyone has encountered the basic ideas of Kolmogorov's theory of high Reynolds number turbulence[1]! Loosely put, empirically motivated assumptions of

i.) spatial homogeneity - i.e. the turbulence is uniformly distributed in space,

ii.) isotropy - i.e. the turbulence exhibits no preferred spatial orientation,

iii.) self-similarity - i.e. all inertial range scales exhibit the same physics and are equivalent. Here "inertial range" refers to the range of scales ℓ smaller than the stirring scale ℓ_0 but larger than the dissipation scale ($\ell_d < \ell < \ell_0$),

iv.) locality of interaction - i.e. the (dominant) nonlinear interactions in the inertial range are local in scale, i.e. while large scales advect small scales, they cannot distort or destroy small scales, only sweep them around. Inertial range transfer occurs via like-scale straining, *only.*

Assumptions i.) - iv.) and the basic idea of an inertial range cascade are summarized in Figs.1.1 and 1.2. Using assumptions i.) - iv.), we can state that energy thru-put must be constant for all inertial range scales, so

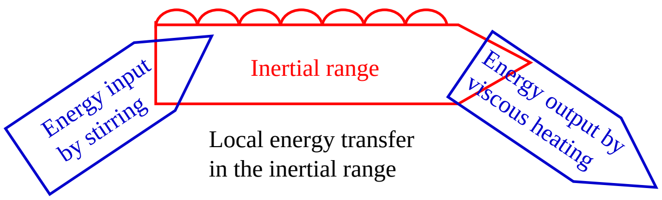

Figure 1.1. Basic cartoon explanation of the Richardson-Kolmogorov cascade. Energy transfer in Fourier-space.

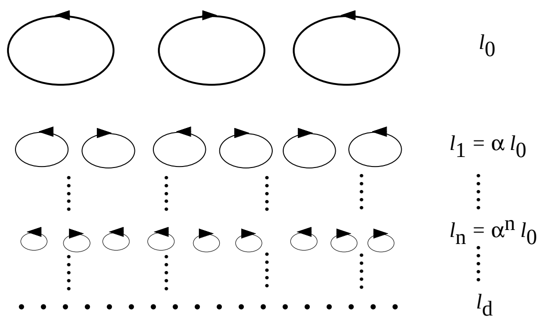

Figure 1.2. Basic cartoon explanation of the Richardson-Kolmogorov cascade. That in real space.

$$\epsilon \sim v_0^3/\ell_0 \sim v(\ell)^3/\ell, \tag{1a}$$

and

$$v(\ell) \sim (\epsilon\ell)^{1/3}, \tag{1b}$$

$$E(k) \sim \epsilon^{2/3}k^{-5/3}, \tag{1c}$$

which are the familiar K41 results. The dissipation scale ℓ_d is obtained by balancing the eddy straining rate $\epsilon^{1/3}/\ell^{2/3}$ with the viscous dissipation rate ν/ℓ^2 to find the Kolmogorov microscale.

$$\ell_d \sim \nu^{3/4}/\epsilon^{1/4} \tag{2}$$

A related and important phenomenon, which also may be illuminated by scaling arguments, is how the distance between two test particles grows in time in a turbulent flow. This problem was first considered by Louis Fry Richardson, who was stimulated by observations of the rate at which pairs of weather balloons drifted apart from one another in the (turbulent) atmosphere[2]. Consistent with the assumption of locality of interaction in scale, Richardson ansatzed that the distance between two points in a turbulent flow increases at the speed set by the eddy velocity on scales corresponding (and comparable) to the distance of separation. Thus, for distance ℓ,

$$\frac{d\ell}{dt} = v(\ell) \qquad (3a)$$

Figs.1.3, 1.4, and 1.5 so using the K41 results gives

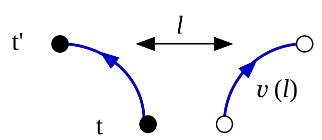

Figure 1.3. Basic idea of the Richardson dispersion problem. The evolution of the separation of the two points (black and white dots) l follows the relation.

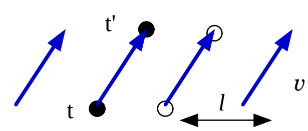

Figure 1.4. If the advection field scale exceeds l, particle pair swept together, so l unchanged.

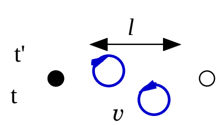

Figure 1.5. If the advection field scale is less than l, there is no effect on pair dispersion.

$$\ell(t) \sim \epsilon^{1/3} t^{3/2}, \qquad (3b)$$

a result which Richardson found to be in good agreement with observations. Notice that the distance of separation grows *super-diffusively*, i.e. $\ell(t) \sim t^{3/2}$, and not $\sim t^{1/2}$, as in the textbook case of Brownian motion. The super-diffusive character of $\ell(t)$ is due to the fact that larger eddys support larger speeds, so the separation process is *self-accelerating*. Note too, that the separation grows as a power of time, and not exponentially, as in the case of a dynamical system with positive Lyapunov exponent. This is because for each separation scale ℓ, there is a *unique* corresponding separation velocity $v(\ell)$, so in fact there is a *continuum* of Lyapunov exponents (one for each scale) in the case of a turbulent flow. Thus, $\ell(t)$ is algebraic, not exponential! By way of contrast, the exponential rate of particle pair separation in a smooth chaotic flow is set by the largest positive Lyapunov exponent. We also remark here that while intermittency corrections to the K41 theory based upon the notion of a dissipative attractor with a fractal dimension less than three have been extensively discussed in the literature[3], the effects of intermittency in the corresponding Richardson problem have received relatively little attention. This is unfortunate, since, though it may seem heretical to say so, the Richardson problem is, in many ways, more fundamental than the Kolmogorov problem, since unphysical effects due to sweeping by large scales are eliminated by definition in the Richardson problem. An exception to the lack of advanced discussion of the Richardson problem is the excellent review article by Falkovich, Gawedski and Vergassola, 2001[4].

Of course, 'truth in advertising' compels us to emphasize that the scaling arguments presented here contain no more physics than that which was inserted ab initio. To understand the *physical mechanism* underpinning the Kolmogorov energy cascade, one must consider *structures* in the flow. As is well known, the key mechanism in 3D Navier-Stokes turbulence is *vortex tube stretching*, schematically shown in Fig. 1.6. There, we see that alignment of strain $\underline{\nabla v}$ with vorticity $\underline{\omega}$ (i.e. $\omega \cdot \underline{\nabla v} \neq 0$) *generates* small scale vorticity, as dictated by angular momentum conservation in incompressible flows. The enstrophy (mean squared vorticity) thus diverges as

$$\langle \omega^2 \rangle \sim \epsilon/\nu, \tag{4}$$

for $v \to 0$. This indicates that *enstrophy is produced in 3D turbulence*, and suggests that there may be a *finite time singularity* in the system, an issue to which we shall return later. By finite time singularity of enstrophy, we mean that the enstrophy diverges within a finite time (i.e. with a growth rate which is faster than exponential). In a related vein, we note that finiteness of ϵ as $\nu \to 0$ constitutes what is called an *anomaly* in quantum field theory. An anomaly occurs when symmetry breaking (in this case, breaking of time reversal symmetry by viscous dissipation) persists as the symmetry breaking term in the field equation asymptotes to zero. The

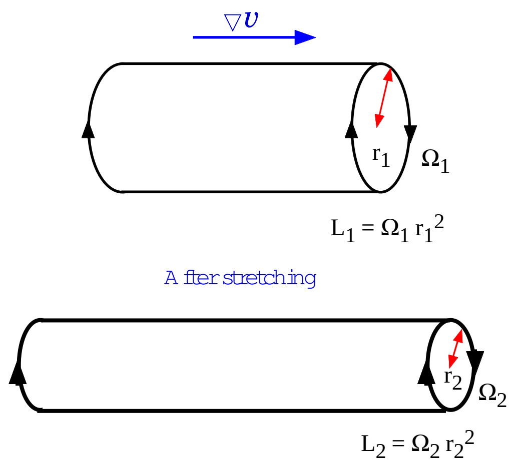

Figure 1.6. The mechanism of enstrophy generation by vortex tube stretching. The vortex tube stretching vigorously produces small scale vorticity.

scaling $\langle \omega^2 \rangle \sim 1/\nu$ is suggestive of this. So is the familiar simple argument using the Euler vorticity equation (for $\nu \to 0$)

$$\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla} v, \tag{5a}$$

$$\frac{d}{dt}\omega^2 \sim \omega^3. \tag{5b}$$

Of course, this "simple argument" is grossly over-simplified, and incorrect. In fact, a mathematical proof of finite time singularity of enstrophy remains an elusive goal, with an as-yet-unclaimed Clay prize of \$1,000,000. In two dimensions $\underline{\omega} \cdot \underline{\nabla v} = 0$, so enstrophy is conserved. As first shown by Kraichnan, this necessitates a *dual cascade*, in which enstrophy *forward cascades* to small scales, while energy *inverse cascades* to large scales[5]. The mechanism by which the dual conservation of energy and enstrophy force a dual cascade in 2D turbulence is shown via the cartoon in Fig. 1.7.

As elegantly and concisely discussed by U. Frisch in his superb monograph "Turbulence-The Legacy of A.N. Kolmogorov", the K41 theory can be systematically developed from a few fundamental hypotheses or postulates. Upon proceeding, the cynical reader will no doubt conclude that the hypotheses H1.)-H4.) stated below are simply restatements of assumptions i.)-iv.). While it is difficult to refute such a statement, we remark here that H1.)-H4.), *are* indeed of value, both for their precise presentation of Kolmogorov's deep understanding and for the insights into his thinking which

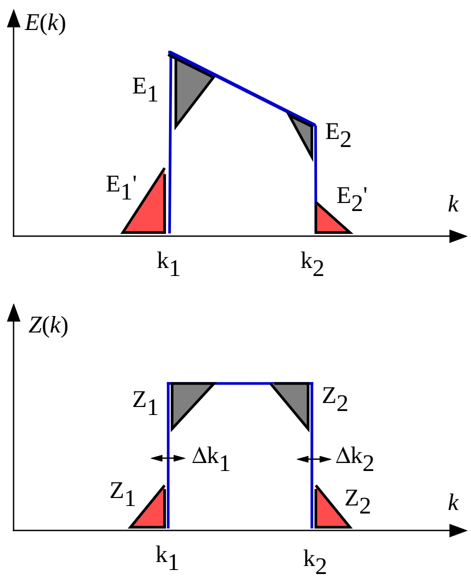

Figure 1.7. A conceptual explanation of the inverse is cascade of energy in two-dimensional turbulence. The energy spectrum $E(k)$ and the enstrophy spectrum is $Z(k)$. In a short time period, cascade events for enstrophy Z_1 and Z_2 occur. Because the enstrophy is conserved, the associated variations in energy satisfies the relation $E_1 - E_1' = -2\Delta k_1 K_1^{-3} Z_1$ and $E_2 - E_2' = 2\Delta k_2 k_2^{-3} Z_2$. As a whole, the energy is transported to lower-k.

they provide. As these postulates involve concepts of great relevance to MHD turbulence, we revisit them here in preparation for our subsequent discussion of MHD turbulence. The first fundamental hypotheses of the K41 theory is:

H1.) As Reynolds number $R_e \to \infty$, all possible symmetries of the Navier-Stokes equation, usually broken by the means of turbulence initiation or production, are restored in a statistical sense at small scales, and away from boundaries.

The reader should note that H1.) is a deceptively simple, and fundamentally quite profound hypothesis! The onset or production of turbulence nearly always involves symmetry breaking. Some examples are:

i.) shear flow turbulence: the initial Kelvin-Helmholtz instability results from breaking of translation and rotation symmetry.

ii.) turbulence in a pipe with a rough boundary: the wall and roughenings break symmetry.

iii.) turbulence in a flushing toilet: the flow has finite chirality and is non-stationary.

Naively, one might expect the turbulent state to have some memory of this broken symmetry. Indeed, the essence of β-model and multi-fractal theories of intermittency is the persistence of some memory of the large, stirring scales into the smallest inertial range scales. Yet, the universal character of K41 turbulence follows directly from, and implies a restoration of, initially broken symmetry at small scales. Assumptions i.) and ii.) really follow from hypothesis H1.).

The second K41 hypothesis is:

H2.) Under the assumptions of H1.), the flow is self-similar at small scales and has a unique scaling exponent h, such that

$$\underline{v}(\underline{r}, \lambda\ell) = \lambda^h v(\underline{r}, \ell).$$

Here, $v(\underline{r}, \ell)$ refers to the velocity wavelet field at position $\underline{r}$ and scale ℓ. Clearly, H2.) implies assumptions iii.) and iv.), concerning self-similarity and locality of interaction.

Hypotheses H1.) and H2.) pertain to flow structure and scaling properties. Two additional postulates pertain to dynamics. These are:

H3.) Given the assumptions of H1.) and H2.), turbulent flow has a finite, non-vanishing mean rate of dissipation per unit mass ϵ, as $\nu \to 0$

and

H4.) In the limit of high but finite R_e, all small-scale statistical properties are uniquely and universally determined by ϵ and ℓ.

Hypothesis H3.) is tacitly equivalent to stating that an anomaly exists in K41 turbulence. Note that ϵ is independent of ν. However, notice also that ϵ, the "mean rate of dissipation per unit mass" is not related to physical, calculable quantities, and is left as a more-than-slightly ambiguous concept. Introduction of fluctuations (which relax the statement 'uniquely' in H4.) in the local dissipation rate (which in reality are usually associated with localized dissipative structures such as strong vortex tubes) and of a statistical distribution of dissipation, leads down the path to intermittency modeling, a topic which is beyond the scope of this paper. The reader is referred to Frisch '95, for an overview, and to seminal references such as Frisch, Sulem, Nelkin '78, She and Leveque '94[6], Falkovich, Gawedski and Vergassola, 2001, and others for an in depth discussion of intermittency modifications to the K41 theory. Finally, hypothesis H4.) relates all statistics to ϵ and ℓ, the only two possible relevant parameters, given H1.), H4.).

1.3 Kraichnan-Iroshnikov, Goldreich-Sridhar and all that: A Scaling Theory of MHD Turbulence

We finally have arrived at the main topic of this paper, namely MHD turbulence in strongly magnetized systems. In this section, the focus will be exclusively on incompressible MHD, which for uniform $\underline{B}_0 = B_0\hat{z}$, is described by the well known equations for the coupled fluid $\underline{\hat{v}}$ and magnetic field $\underline{\hat{B}}$, namely:

$$\frac{\partial \hat{v}}{\partial t} + \underline{\hat{v}} \cdot \underline{\nabla}\, \underline{\hat{v}} = \frac{-\underline{\nabla}\hat{P}}{\rho_0} + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\underline{\hat{B}} + \frac{\underline{\hat{B}} \cdot \underline{\nabla}\underline{\hat{B}}}{4\pi\rho_0} + \nu\nabla^2\underline{\hat{v}} + \underline{\tilde{f}}_v, \qquad (6a)$$

$$\frac{\partial \hat{B}}{\partial t} + \underline{\hat{v}} \cdot \underline{\nabla \hat{B}} = B_0\frac{\partial}{\partial z}\underline{\hat{v}} + \underline{\hat{B}} \cdot \underline{\nabla \hat{v}} + \eta\nabla^2\underline{\hat{B}} + \underline{\tilde{f}}_m. \qquad (6b)$$

Here ρ_0 is constant, and magnetic pressure has been absorbed into p. Equations (6a,b) describe the evolution of two inter-penetrating fluids, which are strongly coupled for large magnetic Reynolds number $R_m \sim v_0\ell_0/\eta$. Equivalently put, $\underline{\hat{B}}$ is 'frozen into' the fluid, up to the resistive dissipation. The system can have two external stochastic forcings $\underline{\tilde{f}}_v$ and $\underline{\tilde{f}}_m$, though we take $\underline{\tilde{f}}_m \to 0$ here. There are two control parameters, Re and Rm, or equivalently Rm and magnetic Prandtl number $P_m = \nu/\eta$.

For a strongly magnetized system, we are concerned with small scale turbulence consisting of amplitude fluctuations with ($|\delta\underline{B}| < B_0$) and which are isotropic in the plane perpendicular to $\underline{B}_0$. Forcing is taken to be localized to large scales, and *assumed* to result in a mean dissipation rate ϵ. Note that in contrast to the corresponding hydrodynamic system, MHD turbulence has two components or constituents, namely

i.) shear Alfven waves, with frequency $\omega_{\underline{k}} = k_{||}v_A$, where $v_A^2 = B_0^2/4\pi\rho_0$. Note that a shear Alfven wave is an *exact* solution of the incompressible

MHD equations. In the absence of dissipation or non-Alfvenic perturbations, then, an Alfven wave will simply persist ad-infinition.

and,

ii.) 'eddys', namely zero frequency hyrdodynamic and magnetic cells, which do *not* bend magnetic field lines (i.e. have $\underline{k} \cdot \underline{B}_0 = 0$). Eddys are characterized by a finite self-correlation time or lifetime $\tau_{\underline{k}}$. For strong B_0, $k_{||} v_A > 1/\tau_{\underline{k}}$, which is equivalent to $|\delta \underline{B}| << B_0$.

Note that in MHD, the waves are high frequency with respect to fluid eddys. Thus, as first recognized by Kraichnan and Iroshnikov, *two Alfven waves must beat together and produce a low frequency virtual mode, in order to interact with fluid eddy turbulence*[7,8]. Such interaction is necessary for *any* cascade to small scale dissipation. Indeed, the generation of such non-Alfvenic perturbations is a key to the dynamics of MHD turbulence!

At this point, it is instructive to discuss an analogy between magnetized MHD turbulence and Vlasov turbulence, the latter system a paradigm universally familiar to plasma physicists. Like MHD, Vlasov turbulence also consists of two constituents, namely collective modes or 'waves', and 'particles'. For example, ion acoustic turbulence consists of ion-acoustic waves and ions. The analogue in MHD of the 'collective mode' is the Alfven wave, while the analogue of the 'particle' is the eddy. In both cases, the dispersive character of the collective modes (N.B.: Alfven waves are dispersive via anisotropy, since $k_{||} = \underline{k} \cdot B_0 / |B_0|$. Most plasma waves of interest are also dispersive.) implies that strong nonlinear interaction occurs when two waves interact to generate a low frequency 'beat' or virtual mode. In the case of Vlasov turbulence, such a low frequency beat wave may resonate and exchange energy with the particles, even if the primary waves are non-resonant (i.e. have $\omega >> kv$). This occurs via the familiar process of *nonlinear Landau damping*, which happens when:

$$\omega_k - \omega'_{k'} = (k - k')v. \tag{7}$$

In the case of MHD, the frequency and wave number matching conditions for Alfven wave interaction require that:

$$\underline{k}_1 + \underline{k}_2 = \underline{k}_3, \tag{8a}$$

$$k_{||_1} v_A + k_{||_2} v_A = k_{||_3} v_A. \tag{8b}$$

Thus, the only way to generate higher $|\underline{k}_\perp|$, and thus smaller scales, through the coupling with vortical motion at $\omega \sim 0$ as in a cascade, is to have

$k_{||_1} k_{||_2} < 0$, which means that the *two primary waves must be counter-propagating*! Note that counter-propagating waves necessarily generate low frequency modes, which resemble the quasi-2D eddys or cells referred to earlier. Indeed, for $k_{||_3} v_A \lesssim 1/\tau_{k_3}$, the distinction between these two classes of fluctuations is lost. Hence, in strongly magnetized MHD turbulence, interaction between counter-propagating populations generates smaller *perpendicular* scales, thus triggering a cascade. Note that parallel propagating packets *cannot* interact, as each Alfven wave moves at the same speed and is, in fact, an exact solution of the incompressible MHD equations. Instead, Alfven populations must pass thru one another for cascading to occur (see Figs. 1.8 and 1.9. This seminal insight is due to Kraichnan and Iroshnikov.

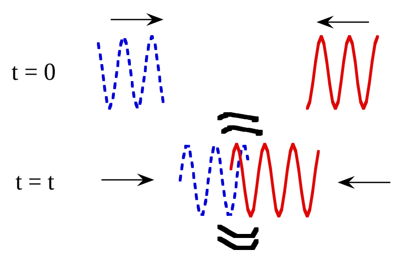

Figure 1.8. Counter-propagating Alfven wave streams interact.

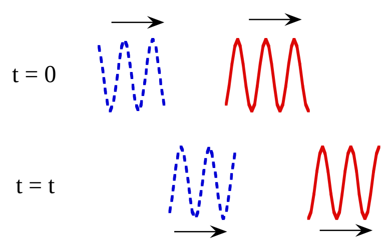

Figure 1.9. Parallel propagating wave streams do not interact.

We note here that the requirement of counter-propagating populations constrains the cross-helicity of the system. The Elsasser variables $Z_{\underline{+}}$, where

$$Z_{\underline{+}} = \underline{v} \underline{+} \underline{B}, \tag{9a}$$

each correspond to one of the two Elsasser populations. The net imbalance in the two population densities is thus

$$N_+ - N_- = \underline{Z}_+ \cdot \underline{Z}_+ - \underline{Z}_- \cdot \underline{Z}_- = 4\underline{v} \cdot \underline{B}, \tag{9b}$$

where the total cross helicity is just

$$H_c = \int d^3x \underline{v} \cdot \underline{B}. \tag{9c}$$

Thus, for a system with counter propagating populations of equal intensity, H_c necessarily must vanish. Similarly, maximal cross helicity ($|\underline{v} \cdot \underline{B}| = (|v|^2|B|^2)^{1/2}$) implies that either $N_+ = 0$ or $N_- = 0$, meaning that no Alfven wave cascade can occur. Hereafter in this section, we take $H_c = 0$.

We now present a heuristic derivation of the MHD turbulence spectrum produced by the Alfven wave cascade[9,10]. As in the K41 theory, the critical element is the lifetime or self-correlation time of a particular mode $\underline{k}$. Alternatively put, we seek a time scale $\tau_{\underline{k}}$ such that

$$(\underline{v} \cdot \nabla \underline{v})_{\underline{k}} \sim v_{\underline{k}}/\tau_{\underline{k}}. \tag{10}$$

This is most straightforwardly addressed by extracting the portion of the nonlinear mixing term which is *phase coherent* with the 'test mode' of interest. Thus, we wish to determine

$$v_{\underline{k}}/\tau_{\underline{k}} = \underline{k} \cdot \sum_{\underline{k}'} \widehat{v}_{-\underline{k}'} \, \widehat{v}^{(2)}_{\underline{k}+\underline{k}'}, \tag{11a}$$

where $v^{(2)}_{\underline{k}+\underline{k}}$ is determined via perturbation theory by solving:

$$\Delta\omega''_k \underline{\widehat{v}''_{k}}^{(2)} - \frac{ik''_{||}}{4\pi\rho_0} B_0 \underline{\widehat{B}}^{(2)}_{\underline{k}+\underline{k}'} = \underline{\widehat{v}}^{(1)}_{\underline{k}'} \cdot \underline{k\widehat{v}}^{(1)}_{\underline{k}}, \tag{11b}$$

$$\Delta\omega_{\underline{k}''} \underline{\widehat{B}}^{(2)}_{\underline{k}''} = B_0 ik''_{||} \underline{\widehat{v}}^{(2)}_{k''}. \tag{11c}$$

Here $\underline{k}'' = \underline{k} + \underline{k}'$, $\Delta\omega_{\underline{k}''}$ is the self-correlation rate of the best mode, and nonlinearities other than $\underline{v} \cdot \underline{\nabla v}$ are ignored. This results in no loss of generality, as all nonlinear couplings are of comparable strength in the case of nonlinear Alfven interaction. Most important of all, we take the $\underline{k}''$ virtual mode to be low frequency since, as discussed above, such interactions maximize the power transfer to small scales. Equations (11a, b, c) then yield:

$$1/\tau_{\underline{k}} = \sum_{\underline{k}'} |\underline{k} \cdot \widehat{v}_{\underline{k}'}|^2 \left[\frac{1/\Delta\omega_{\underline{k}''}}{1 + (k''_{||} v_A/\Delta\omega_{\underline{k}''})^2} \right], \tag{12a}$$

which, for $k_z v_A > \Delta\omega_{\underline{k}}$, reduces to:

$$1/\tau_{\underline{k}} = \sum_{\underline{k}'} |\underline{k} \cdot \widehat{v}_{\underline{k}'}|^2 \pi\delta(k''_{||} v_A). \qquad (12b)$$

Note that Eqn. (12b) is equivalent to the estimate $1/\tau_{\underline{k}} \sim \sum_{\underline{k}'} |\underline{k}\cdot\widehat{v}_{\underline{k}'}|^2 \tau_{ac_{||}}$, where $\tau_{ac_{||}} \sim 1/|\Delta k_{||} v_A$ is the auto-correlation time of the Alfven spectrum. Here, $\Delta k_{||}$ refers to the bandwidth of the $k_{||}$ spectrum. Of course, the need for counter-propagating populations emerges naturally from the resonance condition. Similarly, anistotropy is clearly evident, in that the coupling coefficients, (i.e. $\underline{k}_\perp \cdot \underline{k}'_\perp \times \widehat{z}$), depend on $\underline{k}_\perp$ while the selection rules depend on $k_{||}$. Finally, the correspondence with nonlinear Landau damping in Vlasov turbulence is also clear. For that process,

$$|E_{\underline{k}}|^2/\tau_k \sim \left(\sum_{\underline{k'}} |E_{\underline{k'}}|^2 F(k,k')\pi\delta(\omega_{\underline{k}} + \omega_{\underline{k'}} - (k+k')v) v_T^2 \frac{\partial\langle f\rangle}{\partial v}\bigg|_{v_b} \right) |E_{\underline{k}}|^2 \qquad (13)$$

where $f(k,k')$ refers to a coupling function and interaction occurs at the beat phase velocity $v_b = (\omega+\omega')/(k+k)'$[11].

Having derived the correlation time, we now can proceed to determine the spectrum. In the interests of clarity and simplicity, we derive a scaling relation, using the expression for $\tau_{\underline{k}}$ given in Eqn. (12a.). Despite the facts that:

i.) there are no apriori theoretical reasons or well documented experimental evidence that energy transfer in MHD turbulence is local in $\underline{k}$,

ii.) there is no reason whatsoever to expect that the (as yet unproven!) anomaly or finite time singularity which underlies the independence of ϵ from dissipation in hydrodynamic turbulence should necessarily persist in MHD,

we plunge ahead and write a cascade energy transfer balance relation. The old proverb, "Fools rush in, where angels fear to tread" comes vividly to mind at this point. However, so does another ancient aphorism, "Nothing ventured, nothing gained". Anticipating the role of anisotropy, the transfer balance relation is:

$$\epsilon = v(\ell_\perp)^2/\tau(\ell_\perp), \qquad (14a)$$

where

$$1/\tau(\ell_\perp) = 1/\tau_{\underline{k}} = \sum_{\underline{k}'} |k \cdot v_{\underline{k}'}|^2 \pi\delta(k''_{||} v_A) \cong \frac{1}{\ell_\perp^2} \frac{v(\ell_\perp)^2}{k_{||} v_A}, \qquad (14b)$$

so

$$\epsilon = \frac{v(\ell_\perp)^4}{\ell_\perp^2 k_{||} v_A}. \tag{14c}$$

Those readers who are skeptical of the simple arguments presented in the past few paragraphs can arrive at Eqn. (14c) by the even simpler reasoning that, as is generic in weak turbulence theory, the energy transfer will have the form

$$\epsilon \sim (coupling\ coefficient)^2 * (interaction\ time)$$
$$*(scatter-er\ energy) * (scatter-ee\ energy). \tag{15}$$

Taking the coupling $\sim 1/\ell_\perp$, interaction time $\sim 1/k_{||} v_A$, and scatterer and scatteree energy $\sim v(\ell_\perp)^2$ then yields Eqn. (14c).

In comparison to the familiar (and deceptive) relation $\epsilon = v(\ell)^3/\ell$ for K41 turbulence, Eqn. (14c) contains two new elements, namely:

a.) anisotropy - the distinction between perpendicular and parallel remains,

b.) reduction in transfer note - notice that in comparison to its hydrodynamic counterpart, energy transfer in MHD turbulence is reduced by a factor of $v_\perp/\ell_\perp k_{||} v_A$, the ratio of a parallel Alfven transit time to a perpendicular eddy shearing rate, which is typically much less than unity. The reduction in transfer rate in comparison to hydrodynamic turbulence is commonly referred to as the *Alfven effect*. The Alfven effect is a consequence of the enhanced memory of MHD turbulence, as compared to that of hydrodynamic turbulence. The memory enhancement is due to the reversibility intrinsic to Alfven waves.

It is now possible to consider several related cases and incarnations of the MHD cascade. First, we revisit the original paradigm of Iroshnikov and Kraichnan. Here, we consider a weakly magnetized system, where $Brms >> B_0$. Note that in contrast to hydrodynamics, Alfvenic interaction in MHD is *not constrained* by Galilean invariance. Thus, Equation (14c) applies, with $B_0 \to Brms = \langle \tilde{B}^2 \rangle^{1/2}$. Furthermore, as there is no *large scale* anisotropy (B_0 is negligible!), we can dare to take $k_{||}\ell_\perp \sim 1$, so that the energy transfer balance [Eqn. (14c)] becomes:

$$\epsilon \sim v(\ell)^4/\ell \tilde{v}_A, \tag{16a}$$

where $\tilde{v}_A = v_A$ computed with $\tilde{B}rms$. The value of $Brms$ is dominated by the large eddys, and is sensitive to the forcing distribution and the geometry. In this system, the *rms* field is not straight, but does possess some large scale order. Thus, 'here the Alfven waves' should be thought of as propagating along a large scale field with some macroscopic correlation length and a stochastic component. This in turn immediately gives:

$$v(\ell) \sim \ell^{1/4}(\epsilon\tilde{v}_A)^{1/4}, \tag{16b}$$

and

$$E(k) \sim (\epsilon\tilde{v}_A)^{1/2}k^{-3/2}, \tag{16c}$$

where we use the normalization $\int dkE(k) = Energy$. Equation (16c) gives the famous Iroshnikov-Kraichnan (I.-K.) spectrum for weakly magnetized incompressible MHD turbulence. Concomitant with the departure from $k^{-5/3}$, reconsidering the onset of dissipation when (for $P_m = 1$) $\nu/\ell_d^2 = v(\ell_\perp)/\ell_\perp$ gives the I.-K. dissipation scale $\ell_d = \nu^{2/3}(\tilde{v}_A/\epsilon)^{1/3}$. Interestingly, there is nothing in this argument which is specific to three dimensions! Indeed, since the $\underline{J}\times\underline{B}$ force breaks enstrophy conservation for inviscid 2D MHD, a *forward* cascade of energy is to be expected there, ab initio. Thus, it is not completely surprising that the results of detailed, high resolution numerical simulations of 2D MHD turbulence are in excellent agreement with both the I.-K. spectrum and dissipation scale[12]. The success of the I.-K. theory in predicting the properties of weakly magnetized 3D MHD will be discussed later in this article. Finally, we note that two rather subtle issues have been 'swept under the rug' in this discussion. First, the large scale field $\tilde{B}rms$ is tangled, with zero *mean* direction but with a *local coherence length* set by the turbulence integral scale. Thus, while there is no *system averaged* anisotropy, it seems likely that strong *local* anisotropy will occur in the turbulence. The theory does not account for this local anisotropy. Second, it is reasonable to expect that some minimum value of $\tilde{B}rms$ is necessary to arrest the inverse energy cascade, characteristic of 2D hydrodynamics, and to generate a forward cascade. The scaling of this $\tilde{B}rms$ and possible dependence on forcing scaling and statistics are as yet unknown. Both of the subtle issues mentioned here are topics of active, ongoing research.

We now turn to the case of strongly magnetized, anisotropic turbulence. In that case, Eqn. (14c) states the energy flux balance condition, which is

$$\epsilon \sim \frac{1}{\ell_\perp^2}\frac{v(\ell_\perp)^4}{k_{||}v_A}. \tag{17}$$

Here again $v(\ell_\perp)/(\ell_\perp k_{||}v_A) < 1$. Now using the normalization for an anisotropic spectrum where ($Energy\, E = \int dk_{||}\int dk_\perp E(k_{||},k_\perp)$), Eqn. (17) directly suggests that

$$E(k_\perp) \sim (\epsilon k_{||}v_A)^{1/2}/k_\perp^2, \tag{18}$$

a steeper inertial range spectrum than that predicted by I.-K. for the weakly magnetized case. Note that consistency with the ordering $|\delta \underline{B}| < B_0$, or equivalently $v(\ell_\perp)/\ell_\perp < k_{||} v_A$, requires that

$$\ell_\perp^{1/3} \epsilon^{1/3} / v_A \leq k_{||} \ell_\perp << 1, \tag{19}$$

symptomatic of the *anisotropic cascade* of Goldreich and Sridhar (G.-S.)[13,14]. It is interesting to note that Eqn. (19) says that the *anisotropy increases as the cascade progresses toward smaller scales*, so that initially spheroidal eddys on integral scales produce progressively more prolate and extended (along B_0) eddys on smaller (cross-field) scales, which ultimately fragment into long, thin cylindrical 'rods' on the smallest inertial range scales. This anisotropic cascade process is compared to the isotropic eddy fragmentation picture of Kolmgorov in Fig. 1.10. Recognition of the intrinsically anisotropic character of the strongly magnetized MHD cascade was the important contribution of the series of papers by Goldreich and Sridhar.

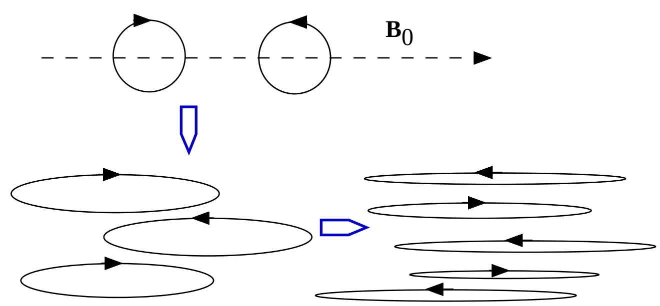

Figure 1.10. Comparison of the isotropic Kolmogorov cascade with the anisotropic Alfven turbulence cascade. In the latter case, anisotropy increases as the cascade progresses.

A particularly interesting limit of the anisotropic MHD cascade is the "critically balanced" or "marginally Alfvenic" cascade, which occurs in the limiting case where $v_\perp(\ell_\perp)/\ell_\perp \sim k_{||} v_A$, i.e. when the parallel Alfven wave transit time thru an (anisotropic) eddy is equal to the perpendicular straining or turn-over time of that eddy. In this limit, Eqn. (17) reduces to $\epsilon \sim v(\ell_\perp)^3/\ell_\perp$, (i.e. back to K41!) albeit with rather different physics. Thus in the critically balanced cascade, $E(k_\perp) \cong \epsilon^{2/3} k_\perp^{-5/3}$ and $k_{||}\ell_\perp \cong \ell_\perp^{1/3} \epsilon^{1/3}/v_A$, so that $k_{||} \sim k_\perp^{2/3} \epsilon^{1/3}/v_A$, which defines a trajectory or 'cone' in $\underline{k}$ space along which the cascade progresses. On this cone (taken dominant here), one has the spectrum $E(k_\perp) \sim k_\perp^{-5/3}$. Note that for $v(\ell_\perp)/\ell_\perp > k_{||} v_A$, the turbulence shearing rate exceeds the Alfven transit rate, so the dynamics are effectively 'unmagnetized' and so the spectrum will approach that of I.-K. in that limit.

We can summarize this zoology of MHD turbulence spectra by considering a magnetized system with fixed $\nu = \eta$ and variable forcing. As the forcing strength increases, so that ϵ increases at fixed B_0, ν, η, the turbulence spectra should transition thru three different stages. These three stages correspond, respectively, to:

i.) first, the anisotropic cascade, with $E(k_\perp) \sim (\epsilon k_{||} v_A)^{1/2}/k_\perp^2$ and $k_{||}\ell_\perp > \ell_\perp^{1/3}\epsilon^{1/3}/v_A$ throughout the inertial range,

then,

ii.) the critically balanced anisotropic cascade, with $E(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}$ and $k_{||} \sim k_\perp^{2/3}\epsilon^{1/3}/v_A$ throughout the inertial range

and finally,

iii.) the weakly magnetized cascade for $Brms > B_0$, with $E(k_\perp) \sim (\epsilon\ \tilde{v}_A)^{1/2} k^{-3/2}$ and $\underline{k}$ isotropic, on average.

Note that the spectral power law index *decreases* with increasing stirring strength, at fixed B_0.

After reading thru all this theory, the patient reader surely is entitled to a discussion of just how well the theory performs when compared to numerical calculations. As discussed before, the weakly magnetized I.-K. cascade theory is quite successful in explaining 2D MHD turbulence at moderate Re with $Pm = 1$. Three numerical calculations for strong B_0 in 3D have recovered results which agree with the predictions of Goldreich and Sridhar, albeit only over intervals of scale of a decade, or less[15,16,17]. Interestingly, the numerical study with the best resolution to date yields a spectrum which appears closer (for strongly magnetized 3D!) to the I.-K.-like prediction of $E(k_\perp) \sim k_\perp^{-3/2}$ than the G.-S. predictions[18]. The deviation from G.-S. scaling may be due to intermittency corrections or to a more fundamental departure from the physical picture of G.-S. In particular, it is tantalizing to speculate that the $E(k_\perp) \sim k_\perp^{-3/2}$ scaling at strong B_0 result suggests that the turbulence assumes a quasi-2D structure consisting of extended columns along B_0. The viability of this speculation is strengthened by the observation of a clear departure from the accompanying $k_{||} \sim k_{\perp}^{2/3}$ scaling also predicted by G.-S., though perpendicular vs. parallel anisotropy clearly remains. In physical terms it seems plausible that the turbulence might form such a quasi-2D state, since:

i.) a state of extended columns aligned with the strong $\underline{B}_0$ is the 'Taylor-Proudman state' for the system. Such a state naturally minimizes the energy spent on magnetic field line bending, which is necessary for Alfven wave generation.

ii.) a state of extended, field-aligned columns which are re-arranged by approximately horizontal eddy motions is also the state in which the translational symmetry along B_0, which is broken by the excitation mechanism, is restored to the maximal extent.

Thus, formation of such a quasi-2D state seems consistent with considerations *of both energetics and of probability.* Further detailed study of the $k_{||}$ and $k_{\perp}$ spectra is required to clarify the extent and causes of the apparent two dimensionalization. This issue is one of the most fundamental ones confronting researchers in MHD turbulence today.

Of course, difficult to believe as it may be, there is a lot more to understanding MHD turbulence than simply computing spectral indexes. The nature of the dissipative structures in 3D MHD turbulence remains a mystery, and the dynamical foundations of intermittency effects are not understood. In 2D, numerical studies suggest that inertial range energy may be dissipated in current sheets, but much further study of this phenomenon is needed. In both 2D and 3D, the structure of the probability distribution function of hydrodynamic and magnetic strain (i.e. $\underline{\nabla}\ \underline{v}$ and $\underline{\nabla}\ \underline{B}$) at high Rm and Re remains terra incognita. Finally, the dependence of the large scale structure of $Brms$ upon stirring properties, geometry, etc. has not been addressed. Note that this structure ultimately is responsible for the breaking of local rotational symmetry and the origin and extent of domains of local anisotropy in 2D MHD turbulence.

1.4 Steepening of Nonlinear Alfven Waves - a little compressibility goes a long way...

At this point, the alert reader may be wondering how the nonlinear evolution of Alfven waves proceeds in the absence of counter-propagating wave streams. This is an important question, since many physical situations and systems *do* involve nonlinear Alfven dynamics but *do not* have counter-propagating wave streams of comparable intensity. Indeed, any situation involving emission of Alfven waves from an astrophysical body (i.e. star) falls into this category. The answer, of course, is that introduction of even modest compressibility (i.e. parallel compressibility, associated with acoustic perturbations) is sufficient to permit the *steepening* of *uni-directional* shear Alfven wave packets[19]! Wave steepening then generates small scales by the familiar process of shock formation. Steepening terminates in either dissipation at small scales, as in a dissipative or collisional shock, or the arrest of steepening by dispersion, as in the formation of a collisionless shock or solitary wave. Alfven wave steepening is thus the 'mechanism of (nature's) choice' for generating small scales in uni-directional wave spectra, and naturally complements the mechanism of low frequency beat generation, which is the key to the Alfvenic wave cascade in counter-propagating wave streams. Quasi-parallel Alfven wave steepening is especially important to the dynamics of the solar wind, since high intensity streams of outgoing Alfven waves are emitted from solar coronal holes. These high intensity wave streams play a central role in generating and heating the 'fast solar wind'.

We now present a simple, physical derivation of the theory of Alfven wave steepening due to parallel compressibility. Just as in the case of shock formation in gas dynamics, Alfven wave trains steepen in response to modulations in density. As in gas dynamics, the density dependence of the wave speed (here the Alfven speed) is the focus of the modulational coupling. So, starting from the Alfven wave dispersion relation

$$\omega = k_{||} v_A = k_{||} B_0 / \sqrt{4\pi(\rho_0 + \tilde{\rho})}, \tag{20a}$$

where a localized density perturbation $\tilde{\rho}$ enters the wave speed. Straightforward expansion gives an 'envelope' equation for the slow space and time variation of the wave function of the perturbation δB, i.e.

$$\frac{\partial \delta B}{\partial t} = -\frac{v_A}{2}\frac{\partial}{\partial z}\left(\frac{\tilde{\rho}}{\rho_0}\delta B\right). \tag{20b}$$

We understand that, in the spirit of reductive perturbation theory, $\tilde{\rho}^{(2)}/\rho_0$ is second order in perturbation amplitude. Here, "perturbation" refers to a modulation of the uni-directional Alfven wave train. We assume that this modulation has parallel scale $L_{||} > 2\pi/k_{||}$. $\tilde{\rho}^{(2)}/\rho_0$ is easily determined by considering of the parallel flow dynamics. In addition to the linear acoustic force, parallel forces are also induced by the gradient of the carrier Alfven wave energy field, i.e. since

$$\underline{v} \cdot \underline{\nabla}\, \underline{v} = \underline{\nabla}\frac{|v|^2}{2} - \underline{v} \times \underline{\omega}, \tag{21a}$$

$$\underline{J} \times \underline{B} = -\nabla \frac{|B|^2}{2} + \underline{B} \cdot \underline{\nabla}\, \underline{B}, \tag{21b}$$

and since $\hat{z} \cdot (\underline{v} \times \underline{\omega}) = \hat{z} \cdot (\underline{B} \cdot \underline{\nabla}\, \underline{B}) = 0$, to second order, we have

$$\frac{\partial}{\partial t}\hat{v}_{||} = -c_s^2 \frac{\partial}{\partial z}\frac{\hat{\rho}}{\rho_0} - \frac{\partial}{\partial z}\left(\frac{|\delta B|^2}{8\pi\rho_0} + \frac{|\delta v|^2}{2}\right). \tag{21c}$$

Note that the parallel gradient of the ponder motive pressure of the Alfven wave train drives the parallel flow perturbation, which then couples to the density perturbation. The loop of couplings is closed by the linearized continuity equation relating $\hat{v}_{||}$ to $\hat{\rho}/\rho_0$, i.e.

$$\frac{\partial}{\partial t}\frac{\hat{\rho}}{\rho_0} = -\frac{\partial}{\partial z}\hat{v}_{||}. \tag{21d}$$

Equations (21c.) and (21d.) may then be combined to obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_s^2 \frac{\partial^2}{\partial z^2}\right)\frac{\hat{\rho}}{\rho_0} = \frac{\partial^2}{\partial z^2}\left(\frac{|\delta B|^2}{4\pi\rho_0}\right), \tag{21e}$$

where we have used the fact that $v_\perp \sim \delta B/\sqrt{4\pi\rho_0}$ for Alfven waves.

At this point, it is convenient to transform to a frame of reference co-moving with the Alfven carrier wave, so that $\hat{\rho} = \hat{\rho}(z - v_A t)$, etc. In this frame, we can simplify Eqn. (21e.) to:

$$\hat{\rho}/\rho_0 = \frac{1}{(1-\beta)}\left(\frac{|\delta B|^2}{B_0^2}\right), \tag{22a}$$

where $\beta = 8\pi P_{th}/B_0^2$. Substituting $\hat{\rho}/\rho_0$ into the wave function equation for δB gives

$$\frac{\partial}{\partial t}\delta B + \frac{\partial}{\partial z}\left[\frac{v_A}{2(1-\beta)}\left(\left|\frac{\delta B}{B_0}\right|^2 \delta B\right)\right] = 0. \tag{22b}$$

As mentioned above, the fast Alfvenic dependence of δB has already cancelled, so this equation almost fully describes the slow dependence of the perturbation envelope. Equation (22b) describes the steepening of an Alfven wave train. One more ingredient is necessary, however - namely a term which represents possible limitation and saturation of the steepening, once it generates sufficiently small scale. This is accomplished by adding a diffusion and/or dispersion term to Eqn. (22b), such as $\eta\partial^2\delta B/\partial z^2$ or $id_i^2\Omega_i\partial^2\delta B/\partial z^2$, respectively. In that case, the envelope equation for δB becomes the well known Derivative Nonlinear Schrodinger (DNLS) equation

$$\frac{\partial}{\partial t}\delta B + \frac{\partial}{\partial z}\left(\frac{v_A}{2(1-\beta)}\left|\frac{\delta B}{B_0}\right|^2 \delta B\right)$$

$$= \eta \frac{\partial^2}{\partial z^2} \delta B + i d_i^2 \Omega_i \frac{\partial^2}{\partial z^2} \delta B. \qquad (23)$$

Here $d_i = c/\omega_{pi}$, the ion inertial scale, and Ω_i is the ion cyclotron frequency[20]. In most expositions and discussions, the resistive dissipation term is dropped, and ion inertial scale physics (associated with Hall currents, etc.) is invoked to saturate Alfvenic steepening by dispersion. Thus, the stationary width of a modulated Alfven wave train is set by the balance of steepening with dispersion, and so the steepened Alfven wave packet is often referred to as a *quasi-parallel Alfvenic collisionless shock*. In contrast to systems with counter-propagating Alfven streams, in a uni-directional wave train modulations can generate small scale via a *coherent* process of wave train steepening, which is ultimately terminated via balance with small scale dispersion.

The physics of the steepening process encapsulated by the back-of-an-envelope (albeit a large one!) calculation presented here can also be described graphically, by a series of cartoons, as shown in Fig. 1.11. The unperturbed Alfven wave train is shown in Fig. 1.11, and its modulation (a parallel rarefaction) is shown in Fig. 1.12. The modulation induces a perturbation in the pondermotive energy field of the wave train, which in turn produces a pondermotive force couple (i.e. dyad) along $\underline{B}_0$, as shown in Fig. 1.13. Note that the resulting parallel flow is yet another example of a Reynolds stress driven flow, though in this case, the flow is *along* $\underline{B}_0$ and a *diagonal* component of the Reynolds stress tensor is at work, symptomatic of the fact that the flow is compressible. The resulting parallel flow re-enforces δB via $\underline{\nabla} \times \underline{v} \times \underline{B}$, as depicted in Fig. 1.14, thus enhancing the initial modulation.

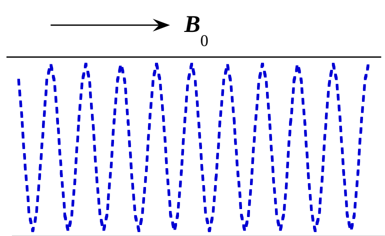

Figure 1.11. Unperturbed wave train and its envelope.

At this point, the alert reader is no doubt wondering about what happens to Eqn. (23) when $\beta \to 1$?! This natural question touches on two interesting issues in the theory of Alfvenic steepening. First, it should be readily apparent that the crucial nonlinear effect in this story is the second order parallel flow, driven by the parallel pondermotive force. Thus, any dissipation, dephasing, etc. such as parallel viscosity, Landau damping,

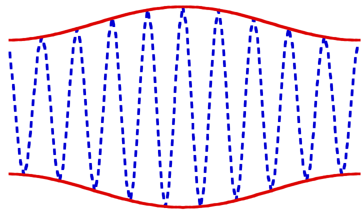

Figure 1.12. Localized modulational perturbation.

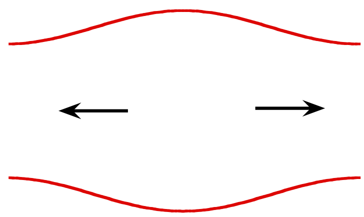

Figure 1.13. Force couple along $\underline{B}_0$.

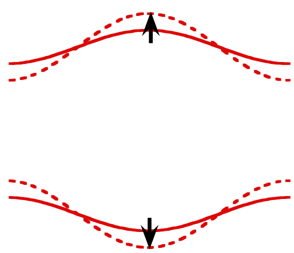

Figure 1.14. Growth of modulation and steepening of initial perturbation.

etc., (which are surely present but not explicitly accounted for) immediately resolves the $\beta \rightarrow 1$ singularity and also can be expected to have an impact on the steepening process for a range of β values. An extensive literature on the important topic of dissipative and kinetic modifications to the DNLS theory exists. One particularly interesting generalization of the DNLS is the KNLS or KDNLS, i.e. the kinetic nonlinear Schroedinger equation or the k-derivative - NLS[21,22,23]. A second point is that for $\beta = 1$, the sound and Alfven speeds are equal, so it no longer makes sense to 'slave' the density perturbation to the Alfven wave. Rather, the acoustic and Alfven dynamics must be treated on an equal footing, as in the analysis by Hada[24].

The DNLS is integrable, via the inverse scattering method. The KNLS, an integro-differential equation, is not so easily tractable, but its numerical solutions seemingly can 'explain' MHD shock phenomena observed in the solar wind, such as rotational discontinuities. The moral of this little story is, then, that one should take care to avoid a tunnel vision focus on only the incompressible theory. Indeed, in this section, we saw that introducing weak compressibility *completely* changed the nonlinear Alfven wave problem, by:

i.) allowing strong nonlinear interaction and wave steepening, leading to the formation of shocks, solitons and other structures.

ii.) allowing a mechanism for the nonlinear evolution of a uni-directional wave train.

Thus, the alert reader should be wary of exclusive reliance upon the I.-K., G.-S. theory and its perturbative fix-ups as a framework for understanding nonlinear Alfven phenomena. Rather, one might more profitably expect that most natural Alfvenic turbulence phenomena will involve some *synergism* between the incompressible dynamics ala' I.-K., G.-S. and the compressible, DNLS-like steepening dynamics. Indeed, recent numerical studies of weakly compressible MHD turbulence have shown *both* a cascade to small scales in the perpendicular direction *and* the formation of residual DNLS-like structures along the field to be at work in the nonlinear dynamics! A theoretical understanding of such weakly compressible MHD turbulence remains elusive.

1.5 Turbulent Flux Diffusion in 2D MHD - a 'minimal' problem which is not so simple.....

Up until now, our discussion has focused primarily on the structure and dynamics of MHD turbulence. In this section, we shift gears somewhat, to discuss the mean field theory of magnetic flux diffusion in two dimensions[25]. This is, no doubt, the simplest, "minimal" problem in the theory of mean field electrodynamics of a turbulent magnetic fluid. However, as we shall see, even the 'simple' problem is not so simple. Indeed, the problem of flux diffusion is a splendid example of the impact of 'dynamical memory' or 'elasticity', both of which are intrinsic to Alfvenic turbulence, upon transport. The upshot of this elasticity in turbulence is the prediction that turbulent diffusion is severely quenched, in comparison to its expected kinematic value. A similar finding is relevant to the alpha effect in three dimensions.

The equations of 2D MHD are

$$\frac{\partial A}{\partial t} + (\nabla\phi \times \widehat{z}) \cdot \nabla A = \eta \nabla^2 A, \tag{24}$$

$$\frac{\partial}{\partial t}\nabla^2\phi + (\nabla\phi \times \widehat{z}) \cdot \nabla\nabla^2\phi = (\nabla A \times \widehat{z}) \cdot \nabla\nabla^2 A + \nu\nabla^2\nabla^2\phi, \tag{25}$$

where A is the magnetic potential ($B = \nabla \times A\widehat{z}$), ϕ is the velocity stream function ($v = \nabla \times \phi\widehat{z}$), η is the resistivity, ν is the viscosity and $\widehat{z}$ is the unit vector orthogonal to the plane of motion. We shall consider the case where the mean magnetic field is in the y-direction, and is a slowly varying function of x. Equations (24) and (25) have non-dissipative quadratic invariants, the energy $E = \int[(\nabla A)^2 + (\nabla\phi)^2]d^2x$, mean-square magnetic potential $H_A = \int A^2 d^2x$ and cross helicity $H_c = \int \nabla A \cdot \nabla\phi d^2x$. Throughout this section, we take $H_c = 0$ *ab initio*, so there is no net Alfvenic alignment in the MHD turbulence considered here.

The basic dynamics of 2D MHD turbulence are well understood[26]. For large-scale stirring, energy is self-similarly transferred to small scales and eventual dissipation via an Alfvenized cascade, as originally suggested by Kraichnan and Iroshnikov, and clearly demonstrated in simulations. Mean square magnetic potential H_A, on the other hand, tends to accumulate at (or cascade toward) large scales, as is easily demonstrated by equilibrium statistical mechanics for non-dissipative 2D MHD. Here, H_c is the second conserved quadratic quantity (in addition to energy), which thus suggests a dual cascade. In 2D, the mean field quantity of interest is the spatial flux of magnetic potential $\Gamma_A = \langle v_x A\rangle$. *An essential element of the physics of* Γ_A *is the competition between advection of scalar potential by the fluid, and the tendency of the flux A to coalesce at large scales.* The former is, in the absence of back-reaction, simply a manifestation of the fact that turbulence tends to strain, mix, and otherwise "chop up" a passive scalar field, thus generating small-scale structure. The latter manifests the fact that A is *not* a passive scalar, and that it resists mixing by the tendency to coagulate on large scales. The inverse cascade of A^2, like the phenomenon of magnetic island coalescence, is ultimately rooted in the fact that like-signed current filaments attract. Not surprisingly then, the velocity field drives a positive potential diffusivity (turbulent resistivity), while the magnetic field perturbations drive a *negative* potential diffusivity. Thus, we may anticipate a relation for the turbulent resistivity of the form $\eta_T \sim \langle v^2\rangle - \langle B^2\rangle$, a considerable departure from expectations based upon kinematic models. A similar competition between mixing and coalescence appears in the spectral dynamics. Note also that η_T vanishes for turbulence at exact Alfvenic equipartition (i.e., $\langle v^2\rangle = \langle B^2\rangle$). Since the presence of even a weak mean magnetic field will naturally convert some of the fluid eddies to Alfven waves, it is thus not entirely surprising that questions arise as to the possible reduction or "quenching" of the magnetic diffusivity relative to expectations based upon kinematics. Also, note that any such quenching is intrinsically a synergistic consequence of both:

i.) the competition between flux advection and flux coalescence intrinsic to 2D MHD,

ii.) the tendency of a mean magnetic field to "Alfvenize" the turbulence.

The close correspondence between the problems of 2D flux diffusion and that of the 3D mean field electromotive force is remarkable. The 3D EMF is central to the theory of the turbulent dynamo. Both seek a representation of a mean product of fluid and magnetic fluctuations in terms of local transport coefficients. In each case, the magnetic dynamics are critically constrained by the conservation, up to resistive dissipation, of magnetic helicity in 3D and of H_A in 2D. Both magnetic helicity and H_A inverse cascade to large scales, and thus produce an interesting dual cascade, since energy flows to small scales in each case. The inverse cascade of magnetic helicity and mean-square potential underpin the appearance of magnetic "back-reaction" contributions to α and η_T. Thus, both tend to vanish for fully Alfvenized turbulence. This trend, then, naturally suggests the possibility of both α-quenching in 3D, and magnetic diffusivity quenching in 2D. Of course, there are crucial *differences* between the two problems. Obviously, in 2D only decay of the magnetic field is possible, whereas 3D admits the possibility of dynamo growth. Furthermore, magnetic helicity and α (the pertinent quantities in 3D) are pseudo-scalars while H_A and η_T are scalars; thus, the effect of helicity conservation on β, the magnetic diffusivity in three dimensions, remains far from clear.

An important element of the basic physics, common to both problems, is the process of "Alfvenization", whereby fluid eddy energy is converted to Alfven wave energy. This may be thought of as a physical perspective on the natural trend of MHD turbulence toward an approximate balance between fluid and magnetic energies, for $P_m \sim 1$. Note also that Alfvenization may be thought of as the development of a *dynamical memory*, which constrains and limits the cross-phase between v_x and A. This is readily apparent from the fact that $\langle v_x A \rangle$ vanishes for Alfven waves in the absence of resistive dissipation. For Alfven waves then, flux diffusion is directly proportional to resistive dissipation, an unsurprising conclusion for cross-field transport of flux which is, in turn, frozen into the fluid up to η. As we shall soon see, the final outcome of the quenching calculation also reveals the explicit proportionality of η_T to η. For small η, then, Γ_A will be quenched. Another perspective on Alfvenization comes from the studies of Lyapunov exponents of fluid elements in MHD turbulence. These showed that as small-scale magnetic fields are amplified and react back on the flow, Lyapunov exponents drop precipitously, so that chaos is suppressed[27]. This observation is consistent with the notion of the development of a dynamical memory.

A key element in our discussion of flux diffusion in 2D MHD is the Zeldovich theorem, which is an expression of the balance between turbulent transport and resistive dissipation for a stationary, 2D MHD system[28]. The Zeldovich theorem is derived by multiplying the magnetic potential equation by A and integrating over space, yielding

$$\frac{1}{2}\left(\frac{\partial}{\partial t}\langle A^2\rangle + \langle\underline{\nabla}\cdot(\underline{v}A^2)\rangle\right) = -\langle v_x A\rangle\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle. \tag{26}$$

We assume a clear-cut separation of scales between mean quantities and fluctuations. For a periodic domain and a stationary state, the relation above reduces to

$$\langle B^2\rangle = -\frac{\langle v_x A\rangle}{\eta}\frac{\partial\langle A\rangle}{\partial x} = \frac{\eta_T}{\eta}\langle B\rangle^2, \tag{27}$$

where we have used Fick's law to represent Γ_A. Equation (27) states the Zeldovich theorem.

The Zeldovich theorem, as expressed in Eqn. (27), has several interpretations and implications. We list these below.

i.) It indicates that the effective turbulent resistivity η_T must scale directly with the collisional resistivity η, in proportion to $\langle B^2\rangle/\langle B\rangle^2$. Note that $\langle B^2\rangle = \langle(\underline{\nabla}A)^2\rangle$ itself is finite as $\eta \to 0$ (consider the I.-K. spectrum, for example), so there is no singularity. This is in distinct contrast to the case of a passive scalar concentration field $c(\underline{x}, t)$, where $\langle(\nabla\tilde{c})^2\rangle$ diverges in the absence of scalar diffusivity.

ii.) It states that the mean square fluctuation level $\langle B^2\rangle$ can be large, even if the mean field $\langle B\rangle$ is weak, i.e. $\langle B^2\rangle/\langle B\rangle^2 \sim R_m >> 1$.

iii.) It may be taken as a statement of Prandtl mixing-length theory for the magnetic potential. This is because Eqn. (27) states an equality between the decay rate of the mean magnetic potential ($\sim \eta_T(\partial\langle A\rangle/\partial x)^2$ - i.e. the rate at which large scales are dissipated) and $\eta\langle B^2\rangle$, the dissipation rate on small scales. Such a balance constitutes an important constraint on the mean magnetic flux transport, Γ_A.

Now we discuss the mean field theory of flux diffusion in 2D. In the discussion of Γ_A, we do not address the relationship between the turbulent velocity field and the mechanisms by which the turbulence is excited or stirred. However, a weak large-scale field (the transport of which is the process to be studied) will be violently stretched and distorted, resulting in the rapid generation of a spectrum of magnetic turbulence. As discussed above, magnetic turbulence will likely tend to retard and impede the diffusion of large-scale magnetic fields. This, of course, is the crux of the matter, as Γ_A depends on the full spectrum arising from the external excitation and the back-reaction of the magnetic field, so, as suggested above, the net imbalance of $\langle v^2 \rangle$ and $\langle B^2 \rangle$ determines the degree of η_T quenching. Leverage on $\langle B^2 \rangle$ is obtained by considering the evolution of mean-square magnetic potential density H_A. In particular, the conservation of $H_A = \int H_A d^2x$ straightforwardly yields the identity

$$\frac{1}{2}\frac{\partial H_A}{\partial t} = -\Gamma_A \frac{\partial \langle A \rangle}{\partial x} - \eta \langle B^2 \rangle, \tag{28}$$

where the surface terms vanish for periodic boundaries. For stationary turbulence, then, this gives

$$\langle B^2 \rangle = -\frac{\Gamma_A}{\eta}\frac{\partial \langle A \rangle}{\partial x} = \frac{\eta_T}{\eta}\left(\frac{\partial \langle A \rangle}{\partial x}\right)^2, \tag{29}$$

which is the well-known Zeldovich theorem discussed earlier. The key message here is that when a weak mean magnetic field is coupled to a turbulent 2D flow, a *large mean-square fluctuation level can result*, on account of stretching iso-A or flux contours by the flow.

To calculate Γ_A, standard closure methods yield

$$\Gamma_A = \sum_{k'} [v_{x-\underline{k}'} \delta A_{\underline{k}'} - B_{x-\underline{k}'} \delta \phi_{\underline{k}'}] = \sum_{k'} \Gamma_A(k'), \tag{30}$$

where $\delta A(k)$ and $\delta \phi(k)$ are, in turn, driven by the beat terms (in (24) and (25)) that contain the mean field $\langle A \rangle$. The calculational approach here treats fluid and magnetic fluctuations on an equal footing, and seeks to determine Γ_A by probing an evolved state of MHD turbulence, rather than a kinematically prescribed state of velocity fluctuations alone. The calculation follows those of Pouquet, *et al.* and yields the result

$$\begin{aligned}\Gamma_A = &-\sum_{k'} \left[\tau_c^{\phi}(\underline{k}')\langle v^2 \rangle_{\underline{k}'} - \tau_c^{A}(\underline{k}')\langle B^2 \rangle_{\underline{k}'}\right] \frac{\partial \langle A \rangle}{\partial x} \\ &-\sum_{k'} \left[\tau_c^{A}(\underline{k}')\langle A^2 \rangle_{\underline{k}'}\right] \frac{\partial}{\partial x}\langle J \rangle. \end{aligned} \tag{31}$$

The magnetic field is expressed in units of velocity (i.e. $\sqrt{4\pi\rho_0} \equiv 1$). Here, consistent with the restriction to a weak mean field, isotropic turbulence is assumed. The quantities $\tau_c^{\phi}(\underline{k})$ and $\tau_c^{A}(\underline{k})$ are the self-correlation times (lifetimes), at k, of the fluid and field perturbations, respectively. These are not at all necessarily equivalent to the coherence time of $v_x(-k')$ with $A(k')$, which determines Γ_A. For a weak mean field, both $\tau_c^{\phi}(k)$ and $\tau_c^{A}(k)$ are determined by nonlinear interaction processes, so that $1/\tau_c^{\phi,A}(k') \geq k'\langle B\rangle$, i.e., fluctuation correlation times are *short* in comparison to the Alfven time of the mean field. In this case, *the decorrelation process is controlled by the Alfven time of the r.m.s. field* (i.e., $[k\langle B^2\rangle^{1/2}]^{-1}$) and the fluid eddy turnover time. Consistent with the assumption of unity magnetic Prandtl number, we take $\tau_c^{\phi}(k) = \tau_c^{A}(k) = \tau_c(k)$, hereafter.

The three terms on the right-hand-side Eqn. (31) correspond respectively to

a.) a positive turbulent resistivity (i.e., Γ_A proportional to flux gradient) due to random fluid advection and straining of flux,

b.) a negative turbulent resistivity symptomatic of the tendency of magnetic flux to accumulate on large scales,

c.) a positive turbulent hyper-resistive diffusion, which gives Γ_A proportional to *current* gradient. Such diffusion of current has been proposed as the mechanism whereby a magnetofluid undergoes Taylor relaxation[29,30].

Note that terms (b) and (c) both arise from $B_x(k)\delta\phi(k')$, and show the trend in 2D MHD turbulence to pump large-scale H_A while damping small-scale H_A. For smooth, slowly varying mean potential profiles, the hyper-resistive term is negligible in comparison with the turbulent resistivity, (i.e., $\langle k'^2\rangle > (1/\langle A\rangle)(\partial^2\langle A\rangle/\partial x^2)$), so that the mean magnetic potential flux reduces to

$$\Gamma_A = -\eta_T \frac{\partial\langle A\rangle}{\partial x}, \tag{32}$$

where

$$\eta_T = \sum_{\underline{k}'} \tau_c(k')\left(\langle v^2\rangle_{\underline{k}'} - \langle B^2\rangle_{\underline{k}'}\right). \tag{33}$$

As stated above, the critical element in determining Γ_A is to calculate $\langle B^2\rangle_{k'}$ in terms of $\langle v^2\rangle_{k'}$, Γ_A itself, etc. For this, mean-square magnetic potential balance is crucial! To see this, note that the Zeldovich theorem states that

$$\langle \tilde{B}^2\rangle = \frac{-\Gamma_A}{\eta}\frac{\partial\langle A\rangle}{\partial x}, \tag{34}$$

assuming incompressibility of the flow. An equivalent, k-space version of Eqn. (34) is

$$\frac{1}{2}\left[\frac{\partial}{\partial t}\langle A^2\rangle_k + T(k)\right] = -\Gamma_A(k)\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_k, \tag{35}$$

where $T(k)$ is the triple correlation

$$T(k) = \langle\nabla\cdot(vA^2)\rangle_k, \tag{36}$$

which controls the nonlinear transfer of mean-square potential, and $\Gamma_A(k) = \langle v_x A\rangle_k$ is the k-component of the flux. Equations (35) and (36) thus allow the determination of $\langle B^2\rangle$ and $\langle B^2\rangle_k$ in terms of $\Gamma_A, \Gamma_A(k)$, $T(k)$, etc.

At simplest, crudest level (the so-called) τ-approximation), a single τ_c is assumed to characterize the response or correlation time in Eqn. (33). In that case, we have

$$\Gamma_A = -\left[\sum_k \tau_c(\langle v^2\rangle_k - \langle B^2\rangle_k)\right]\frac{\partial\langle A\rangle}{\partial x}. \tag{37}$$

For this, admittedly over-simplified case, Eqn. (37) then allows the determination of $\langle B^2\rangle$ in terms of Γ_A, the triplet and $\partial_t\langle A^2\rangle$. With the additional restrictions of stationary turbulence and periodic boundary conditions (so that $\partial\langle A^2\rangle/\partial t = 0$ and $\langle\nabla\cdot(vAA)\rangle = 0$), it follows that $\langle B^2\rangle = -(\Gamma_A/\eta)\partial\langle A\rangle/\partial x$, so that magnetic fluctuation energy is directly proportional to magnetic potential flux, via H_A balance. This corresponds to a balance between local dissipation and spatial flux in the mean-square potential budget. Inserting this into Eqn. (37) then yields the following expression for the turbulent diffusivity:

$$\eta_T = \frac{\sum_k \tau_c\langle v^2\rangle k}{1+\tau_c v_{A0}^2/\eta} = \frac{\eta^k}{1+R_m v_{A0}^2/\langle v^2\rangle}, \tag{38}$$

where η^k refers to the kinematic turbulent resistivity $\tau_c\langle v^2\rangle$, v_{A0} is the Alfven speed of the mean $\langle B\rangle$, and $R_m = \langle v^2\rangle\tau_c/\eta$. It is instructive to note that Eqn. (38) can be rewritten as

$$\eta_T = \frac{\eta\eta^k}{\eta+\tau_c v_{A0}^2}. \tag{39}$$

Thus, as indicated by mean-square potential balance, Γ_A *ultimately scales directly with the collisional resistivity*, a not unexpected result for Alfvenized turbulence with dynamically interesting magnetic fluctuation intensities. This result supports the intuition discussed earlier. It is also interesting to note that for $R_m v_{A0}^2/\langle v^2\rangle > 1$ and $\langle v^2\rangle \sim \langle B^2\rangle$, $\eta_T \cong \eta\langle B^2\rangle/\langle B\rangle^2$, consistent with the Zeldovich theorem prediction.

Equation (38) gives the well-known result for the quenched flux diffusivity. There, the kinematic diffusivity η_T^k is modified by the quenching or suppression factor $[1 + R_m v_{A0}^2/\langle v^2 \rangle]^{-1}$, the salient dependencies of which are on R_m and $\langle B \rangle^2$. Equation (38) predicts a strong quenching of η_T with increasing $R_m \langle B \rangle^2$. Despite the crude approximations made in the derivation, numerical calculations indicate remarkably good agreement between the measured cross-field flux diffusivity (as determined by following marker particles tied to a flux element) and the predictions of Eqn. (38)[31,32]. In particular, the scalings with both R_m and $\langle B \rangle^2$ have been verified, up to R_m values of a few hundred. The quench may be viewed as one consequence of the Alfvenization of turbulence by the stretching of a weak mean magnetic field by the flow.

Limitations of space and time availability force us to leave the fascinating subject of turbulent diffusion of magnetic fields at this point. Truth be told, we have only scratched the surface of the 2D problem, and have not dared to even touch the 3D diffusion or alpha effect issues. In 2D, several aspects of the problem merit further discussion. Perhaps the most important of these is concerned with the effects of a flux or inhomogeneity-driven transport of A^2 upon the Zeldovich theorem balance[33]. If such a process were at work, it would alter the balance between resistive dissipation and turbulent diffusion, and thus change the quench of η_T. This issue is an area of ongoing research.

1.6 Conclusion

This brief pedagogical article has surveyed some of the interesting problems in MHD turbulence theory and has only explored the 'tips' of a few 'icebergs' floating in the 'sea' of that large topical area. The reader is referred to the research literature for further discussion, and for treatments of other related topics. The authors hope that the discussion of key concepts presented in this article will stimulate and facilitate the reader's future explorations.

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