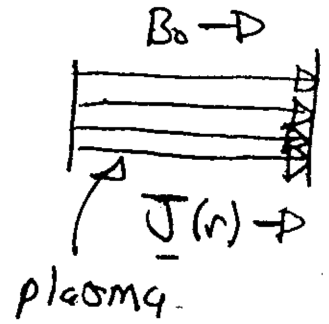
i.) Dynamics of MHD Interchange Instabilities

i.) Heuristics:

$\left\{ \begin{array}{l} \text{First fluid,} \\ \text{then kinetic} \end{array} \right.$

- Consider curved magnetic field

i.e.



$$\Rightarrow \quad \underline{B} = \underline{B}_\theta(r)\hat{\theta} + B_0\hat{z}$$

$$p = p(r)$$

- particle streaming along curved magnetic field experiences centrifugal force

i.e.

$(r, \theta$ slice$)$

$$F = m v_{\parallel}^2 / R_c$$

- thus, in fluid equations, should include external force, i.e.

curvature induced accel.

$$\rho_0 \frac{d\underline{V}}{dt} = -\underline{\nabla}\left(p + B^2/8\pi\right) + \frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi} + \rho g_{\text{eff}}\hat{r}$$

$$g_{\text{eff}} = c_s^2/r$$

i.e. $\left\{ \begin{array}{l} 1/R_c = 1/r \\ \text{inertia} - \text{ions} \\ \text{pressure} - \text{electrons } (T_e \gg T_i) \end{array} \right.$

- then:

a.) $k_\parallel = 0$ ⇒ should recover Rayleigh–Taylor like instability if $d\rho/dr < 0$, i.e. liberate "gravitational" potential energy

b.) $k_\parallel \neq 0$ ⇒ need consider competing processes of
- interchange growth
- energy penalty for exciting shear Alfven waves (analogous vortex line bending)

ii.) Analysis

$(\underline{\nabla} \cdot \underline{V} = 0)$

$$\rho_0 \frac{\partial \tilde{V}}{\partial t} = -\underline{\nabla}\left(\tilde{p} + \frac{\underline{B_0} \cdot \tilde{\underline{B}}}{4\pi}\right) + \frac{\underline{B_0} \cdot \underline{\nabla}\, \tilde{\underline{B}}}{4\pi} + \tilde{\rho}\, g_{eff}\, \hat{r}$$

$$\frac{\partial \tilde{B}}{\partial t} = \underline{B_0} \cdot \underline{\nabla}\, \underline{V}$$

$$\frac{\partial \tilde{\rho}}{\partial t} = -\tilde{V}_r \frac{d\rho_0}{dr} \qquad \left(1/L_\rho = \frac{1}{\rho_0}\frac{d\rho_0}{dx}, \quad k L_\rho \gg 1\right)$$

Hereafter consider slab:
$r \to x$
$r\theta \to y$
$z \to z$

Then:

$$\frac{\partial}{\partial t}\hat{\omega}_z = \left(\frac{\underline{B}_0\cdot\underline{\nabla}}{4\pi\rho_0}\,\underline{\nabla}\times\underline{\tilde{B}}\right)\cdot\hat{z} - \frac{\partial}{\partial y}\, g_{eff}\frac{\tilde{\rho}}{\rho_0}$$

$$\frac{\partial}{\partial t}(\nabla\times B)_z = \underline{B}_0\cdot\underline{\nabla}\,\omega_z$$

$$\frac{\partial\hat{\rho}}{\partial t} = -\hat{V}_r\frac{d\rho_0}{dr}$$

$$\nabla\times\underline{\hat{B}} = \frac{4\pi}{c}\underline{\hat{J}} \quad , \quad \nabla^2\hat{A}_z = -\frac{4\pi}{c}\hat{J}_z \qquad (\underline{\nabla}\cdot\underline{A}=0)$$

$\Rightarrow$

$$\frac{\partial\hat{\omega}_z}{\partial t} = \frac{\underline{B}_0\cdot\underline{\nabla}\,\hat{J}_z}{\rho_0 c} - \frac{\partial}{\partial y}\frac{\tilde{\rho}}{\rho_0}\,g_{eff}$$

— ala RMHD

$$\frac{\partial}{\partial t}\hat{J}_z = \frac{c}{4\pi}\,\underline{B}_0\cdot\underline{\nabla}\,\hat{\omega}_z$$

$\rightarrow$ un-$\nabla_\perp^2$

$$\frac{\partial\hat{\rho}}{\partial t} = -\hat{V}_r\frac{d\rho_0}{dx}$$

$$\therefore\quad \frac{\partial^2\hat{\omega}_z}{\partial t^2} = \frac{(\underline{B}_0\cdot\underline{\nabla})(\underline{B}_0\cdot\underline{\nabla})}{4\pi\rho_0}\,\hat{\omega}_z + \frac{\partial\hat{V}_r}{\partial y}\frac{g_{eff}}{\rho_0}\frac{d\rho_0}{dx}$$

To relate $\underline{V}$, ω:

$$\hat{\underline{V}} = \nabla\hat{\phi} \times \hat{z} \quad , \qquad \hat{\omega}_z = -\nabla_\perp^2 \hat{\phi}$$
$$\hat{V}_r = \frac{\partial \hat{\phi}}{\partial y}$$

finally:

$$\frac{\partial^2}{\partial t^2}(\nabla_\perp^2 \hat{\phi}) = \frac{(\underline{B}_0 \cdot \nabla)^2}{4\pi\rho_0}(\nabla_\perp^2 \hat{\phi}) - \frac{\partial^2 \hat{\phi}}{\partial y^2} \frac{g_{eff}}{L_\rho}$$

inertia — field line bending — interchange drive

Local Theory:

$$\omega^2 k_\perp^2 = k_\perp^2 V_A^2 k_\parallel^2 + \frac{g_{eff}}{L_\rho} k_y^2$$

$$\Rightarrow \quad \omega^2 = k_\parallel^2 V_A^2 - \frac{g_{eff}}{|L_\rho|} \frac{k_y^2}{k_\perp^2}$$

Shear Alfven — interchange (a la' Rayleigh-Taylor)

where – $1/L_\rho < 0$

– $g_{eff} > 0$

i) $k_{\parallel} = 0$ (Flute-Limit)

$$\omega^2 = -\frac{g_{eff}}{|L_p|}\frac{k_y^2}{k_\perp^2} = -\frac{c_s^2}{r L_p}\frac{k_y^2}{k_\perp^2}$$

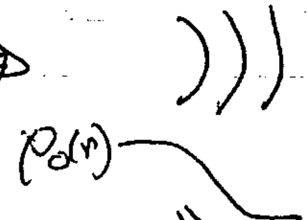
a la' Rayleigh-Taylor.

see: Rosenbluth + Longmire

- instability for $g_{eff}/L_p < 0$

i.e. $1/R_c L_p < 0$ ⇒ as $L_p < 0$ usually

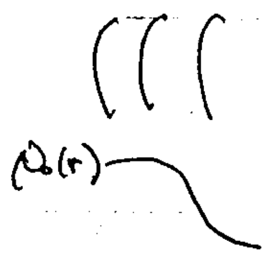
⇒ instability if $R_c > 0$ ⇒



field lines sag outward - "unfavorable" curvature

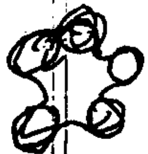
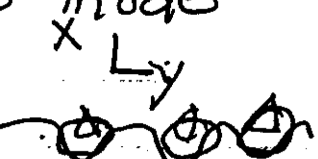
- stability for $g_{eff}/L_p < 0$ (buoyancy oscillations)

i.e.



field lines sag inward - "favorable" curvature (i.e. Mirror + Ioffe Coil)

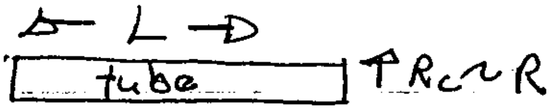
- structure of interchange mode ($k_{\parallel} = 0$)

 → r, θ slice ⇒  $\downarrow g$ $\frac{dp}{dx}\uparrow$

i.e. vortices interchange heavy, light fluid

- introduces concept of stability as limiting β

$\beta \equiv 4\pi P / B_0^2$ (ratio plasma to magnetic pressure)

if: 

$k_\parallel \sim 1/L$
$L_p \sim R$

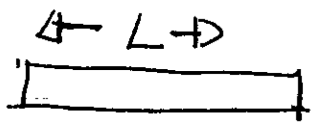
$$\Rightarrow \quad \omega^2 = 0 \quad \Rightarrow \quad \frac{V_A^2}{L^2} \gtrsim \frac{c_s^2}{R^2}$$

$$\Rightarrow \quad \frac{R^2}{L^2} \gtrsim \beta$$

i.e. beta limit $\beta \lesssim \epsilon^2$.

3) Stop Line-Tying

- Consider plasma in cylindrical tube, with metal end plates



- for radially local instability analysis:

$$\omega^2 k_\perp^2 \hat{\phi}(s) = -V_A^2 k_\perp^2 \frac{\partial^2 \hat{\phi}(s)}{\partial s^2} + k_y^2 \frac{g_{eff}}{L_p} \hat{\phi}(s)$$

$s \equiv$ distance along $\underline{B}_0$ field line

$$\therefore \frac{\partial^2 \tilde{\phi}}{\partial s^2} - \left(\frac{k_y^2 g_{eff}}{k_\perp^2 L_\rho V_A^2} - \frac{\omega^2}{V_A^2} \right) \tilde{\phi} = 0$$

– for boundary conditions; recall

$$\underline{V}_\perp = \frac{c}{B_0} \underline{E}_\perp \times \hat{z}$$

Then, $\underline{E}_\perp \Big|_{\text{conducting plate}} = 0 \Rightarrow \phi(s)\Big|_{\pm L} = 0$ (const $\not\to 0$)

$$\therefore \phi(s) = \sum_n \phi_n e^{i(n\pi s/L)} \to \sum_{n\neq 0} \phi_n \sin(n\pi s/L)$$

$$+\frac{n^2\pi^2}{L^2} + \frac{k_y^2 g_{eff}}{k_\perp^2 L_\rho V_A^2} = \frac{\omega_n^2}{V_A^2}, \quad n \neq 0 \text{ (axial variation in } \phi!)$$

$$\omega_n^2 = \frac{n^2\pi^2 V_A^2}{L} - \frac{k_y^2 g_{eff}}{k_\perp^2 |L_\rho|}, \quad n \neq 0$$

– i.e. note:
→ finite geometry + boundary conditions force line-bending stabilization

c) $k_{\parallel} \neq 0$ Line-Bending Effects

$$\omega^2 = k_\parallel^2 V_A^2 - \frac{c_s^2}{r L_p}\frac{k_y^2}{k_\perp^2}$$

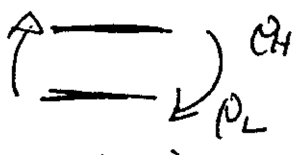
- even for $1/L_p R_0 < 0$, stability if

$$k_\parallel^2 V_A^2 > \frac{c_s^2}{r L_p}\frac{k_y^2}{k_\perp^2}$$

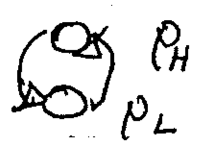
- physical origin is fact that perturbation must now expend energy to "bend" magnetic field lines. Instability if gain beats loss.

i.e. contrast:

a) flute



(side)

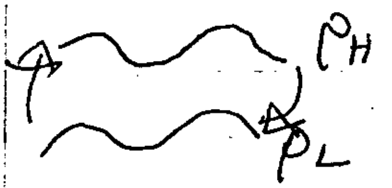


(front) "interchange"

{ Vortex interchanges 2 "tubes" of plasma

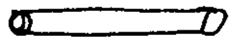
b) with line bending

side



i.e. gain in release of gravitational potential energy must overcome expenditure of energy to bend B_0 lines.

⇒ termed "line tying", as b.c. forces line-bending



↓

no b.c. ⇒ can displace flux tube without bending

↓

with b.c., must "bend" flux tube for interchange motion ⇒ stabilization via line-tying.

4) Effect of Resistive Dissipation

- Consider now effect of resistivity

- note that resistivity damps magnetic perturbation via diffusion of J_z ⇒ resistivity weakens line bending (Alfvenic ⟺ $\vec{B}$) and so results in destabilization!

i.e.

$$-\frac{\partial^2}{\partial t^2}\nabla_\perp^2\hat{\phi} = \frac{B_0\cdot\nabla}{\rho_0 c}\frac{\partial\hat{J}_z}{\partial t} + \frac{\partial^2\hat{\phi}}{\partial y^2}\frac{g_{eff}}{L_\rho}$$

$$\frac{\partial}{\partial t}\hat{J}_z - \eta\underset{\uparrow}{\nabla_\perp^2\hat{J}_z} = \frac{c}{4\pi}B_0\cdot\nabla\left(-\nabla_\perp^2\hat{\phi}\right)$$

resistive diffn. of current ($k_\perp \gg k_\parallel$)

$$\eta = \frac{c^2}{\omega_{pe}^2}\nu_{e,i}$$

Then,

$$-\omega^2 k_\perp^2 \hat{\phi}_{\underline{k}} = \frac{\omega k_\parallel B_0}{\rho_0 c} \hat{J}_{z_{\underline{k}}} - k_y^2 \frac{g}{L_p} \hat{\phi}_{\underline{k}}$$

$$(-i\omega + \eta k_\perp^2) \hat{J}_{z_{\underline{k}}} = \frac{c}{4\pi} i k_\parallel k_\perp^2 \hat{\phi}_{\underline{k}}$$

$\Rightarrow$

$$-\omega^2 k_\perp^2 = -\frac{k_\parallel^2 V_A^2 k_\perp^2}{\left(1 + \frac{i\eta k_\perp^2}{\omega}\right)} - k_y^2 \frac{g}{L_p}$$

$$\omega^2 = +\frac{k_\parallel^2 V_A^2}{1 + \frac{i\eta k_\perp^2}{\omega}} + \frac{k_y^2}{k_\perp^2} \frac{g}{L_p}$$

resistive diffusion
modification of line-bending

Note:

i) $\eta \to \infty \Rightarrow \omega^2 = +\frac{k_y^2}{k_\perp^2} \frac{g}{L_p}$

i.e. for large η, field not frozen into fluid
$\Rightarrow$ field resistively diffuses thru fluid,
so no Alfvenic perturbations

→) Note:

- here, large magnetic dissipation acts as destabilizing effect!. Viscosity still stabilizing, though. (show)

- also, indicates that real system (i.e. with dissipation) may admit instability on small scales more easily

i.e. $\eta = 0$

$\perp$ scale $= a$
$\parallel$ scale $= L$

η finite.

$\perp$ scales: a, $(\eta/\omega)^{1/2}$
$\parallel$ scale: L

characteristic scale introduced by η.

clearly, $k_\perp > (\eta/\omega)^{1/2}$ "more unstable", i.e. resistive interchanges.

Interchanges, cont'd.

ecall, describe wave/instability by:

→ quasi-neutrality

→ $\nabla \cdot J = 0$ (from adding 0^{th} moments of gyrokinetic eqns.)

so, have for (collisional) fluid electrons and fluid ions, i.e.

electrons : $\nu_{e,e}, \nu_{ei} > k_\parallel V_{\parallel e}, \omega$

ions : $\omega > k_\parallel V_{\parallel i}, \nu_{ii}$

$$-i(\omega - \omega_{de} - k_\parallel v_\parallel)\hat{g}_{\underset{\omega}{k}} + C(g) = \frac{i|e|}{T_e}(\omega - \omega_{*e})\langle f\rangle\left(\hat{\phi}_{\underset{\omega}{k}} - \frac{v_\parallel}{c}\hat{A}_{\parallel \underset{\omega}{k}}\right)$$

$$-i(\omega - \omega_{di} - k_\parallel v_{\parallel i})\hat{g}_{\underset{\omega}{k}} = -\frac{i|e|}{T_i}(\omega - \omega_{*i})\langle f\rangle\left(\hat{\phi}_{\underset{\omega}{k}} - \frac{v_\parallel}{c}\hat{A}_{\parallel \underset{\omega}{k}}\right)J_0(k_\perp \rho_i)$$

and

$$\underbrace{\frac{k_\parallel \hat{J}_{\parallel \underset{\omega}{k}}}{n_0|e|}}_{\text{line-bending}} + i\underbrace{\boxed{\int d^3v\, \omega_{di}\hat{g}^i_{\underset{\omega}{k}} J_0(k_\perp\rho_i) - \int d^3v\, \omega_{de}\hat{g}^e_{\underset{\omega}{k}}}}_{\text{interchange - drift driven}}$$

$$= \frac{i|e|}{T_i}\hat{\phi}_{\underset{\omega}{k}}(\omega - \omega_{*i})k_\perp^2\rho_i^2$$

- for flute modes, $k_\parallel = 0$

electrons; l.o. $C(g) = 0$ $(\nu \gg \omega)$

$$g^{(0)} = g^{Max}$$

$$-i(\omega - \omega_{de})\, \tilde{g}^{(0)}_{\underline{k},\omega} + C(g^{(1)})$$

$$= \frac{i|e|}{T_e}(\omega - \omega_{*e}) \langle f \rangle \left(\tilde{\phi}_{\underline{k},\omega} - \frac{v_\parallel}{c} \tilde{A}_{\parallel\, \underline{k},\omega} \right)$$

$$\int d^3v \; -i(\omega - \omega_{de})\, \tilde{g}^{(0)}_{\underline{k},\omega} + \int d^3v\, \cancel{C(g^{(1)})}$$

$$= \frac{i|e|}{T_e}(\omega - \omega_{*e})\, \tilde{\phi}_{\underline{k},\omega}$$

$$g^{(0),e}_{\underline{k},\omega} = -\frac{|e|\tilde{\phi}_{\underline{k}}}{T_e}\left(1 - \frac{\omega_{*e}}{\omega}\right)\langle f \rangle$$

response indep. ν as C # conserving

ions;

$$\tilde{g}^{(0),i}_{\underline{k},\omega} = \frac{|e|\tilde{\phi}_{\underline{k}}}{T_i}\left(1 - \frac{\omega_{*i}}{\omega}\right)$$

Plugging into $\underline{\nabla} \cdot \underline{J} = 0$ gives:

N.B. $k_\parallel = 0 \Rightarrow$ 1 field $\tilde{\phi} \Leftrightarrow$ no Alfven waves.

$$\left[\int d^3v^i\, \omega_{di} \frac{|e|}{T_i} \frac{\vec{\phi}_n}{\omega} \langle f\rangle \left(1-\frac{\omega_{*i}}{\omega}\right)\right.$$

$$\left. + \int d^3v^e\, \omega_{de} \frac{|e|}{T_e} \frac{\hat{\phi}_n}{\omega} \langle f\rangle \left(1-\frac{\omega_{*e}}{\omega}\right)\right]$$

$$= \frac{|e|}{T_i} \frac{\vec{\phi}_n}{\omega} (\omega-\omega_{*i}) k_\perp^2 \rho_i^2$$

now: $\frac{\omega_{di}}{T_i} + \frac{\omega_{de}}{T_e} = 0$ (opposite direction drifts for opposite sign)

$\Rightarrow$

$$-\frac{|e|\vec{\phi}_n}{T_e}\left[\int d^3v^i\, \frac{\omega_{di}\omega_{*i}}{\omega}\langle f\rangle \frac{T_e}{T_i} + \int d^3v^e\, \frac{\omega_{de}\omega_{*e}}{\omega}\langle f\rangle\right]$$

$$= \frac{|e|}{T_e} \frac{\vec{\phi}_n}{\omega} (\omega-\omega_{*i}) k_\perp^2 \rho_s^2$$

so have:

$$-(\omega-\omega_{*i}) k_\perp^2 \rho_s^2 = \int d^3v^i\, \frac{\omega_{di}\omega_{*i}}{\omega}\langle f\rangle \frac{T_e}{T_i} + \int d^3v^e\, \frac{\omega_{de}\omega_{*e}}{\omega}\langle f\rangle$$

$$\frac{T_e}{T_i}\omega_{di}\omega_{*i} \sim \frac{T_e}{T_i} \frac{k_\theta^2\rho_i^2 v_{Ti}^2}{L_n R_0}$$

$$\omega_{de}\omega_{*0} \sim \frac{k_\theta^2 \rho_s^2 c_s^2}{L_n R_c}$$

Taking $\omega > \omega_{*i}$:

$$-\omega^2 = \gamma^2 = \frac{1}{k_\perp^2 \rho_s^2}\left[\frac{k_\theta^2 \rho_s^2 v_{Ti}^2}{L_n R_c} + \frac{k_\theta^2 \rho_s^2 c_s^2}{L_n R_c}\right]$$

$$k_\perp \sim k_\theta \Rightarrow \boxed{\gamma^2 = (v_{Ti}^2 + c_s^2)/L_n R_c}$$

hybrid of curvature, geo. mean scales

recovers interchange growth rate.

If retain finite ω_{*i} :

$$+\ \omega^2 - \omega_{*i}\omega + \gamma_{INT}^2 = 0$$

$$\omega = \frac{\omega_{*i}}{2} \pm \frac{1}{2}\left(\omega_{*i}^2 - 4\gamma_{INT}^2\right)^{1/2}$$

→ high k_θ cut-off on instability (i.e. $\omega_{*i} > 2\gamma_{INT}$ → stable).

→ ω_{*i} stabilizing as constant polarization drift introduced

stable → drift wave.

Astrophysical MHD I: Convection and Magnetic Fields

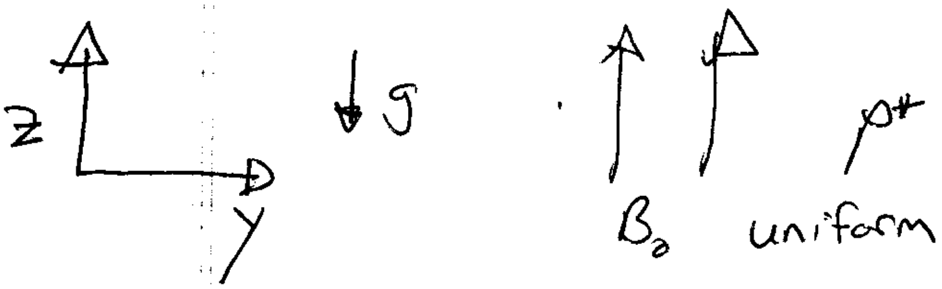
Example:

- → sunspots
- → magnetic field eruptions / solar prominences
- → clumping of matter in ISM

Theme: Buoyancy (Rayleigh - Benard convection) + Magnetic Fields

i.e.
- convection
- convection with $\underline{B}_0 \parallel g$
- exclusion of magnetic field by convective motions
- magnetic buoyancy instability.

→ Effect of Magnetic Field ($\underline{B}_0 \parallel \underline{g}$)



$$\frac{\partial v_y}{\partial t} = -\frac{\nabla_y \hat{p}^*}{\rho_0} + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\tilde{B}_y$$

$$\frac{\partial v_z}{\partial t} = -\frac{\nabla_z p^*}{\rho_0} + \frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\tilde{B}_z - |g_z|\frac{\tilde{\rho}}{\rho_0}$$

$$\frac{\partial \underline{B}}{\partial t} = B_0 \frac{\partial}{\partial z}\underline{V}$$

others as before

$\Rightarrow$

$$\frac{\partial}{\partial t}\tilde{\omega}_x = |g_z|\frac{\partial}{\partial y}\left(\frac{\hat{T}}{T_0}\right) + \frac{c}{4\pi}\frac{B_0}{4\pi\rho_0}\frac{\partial}{\partial z}\tilde{J}_x$$

$$\frac{c}{4\pi}\frac{\partial}{\partial t}\tilde{J}_x = B_0\frac{\partial}{\partial z}\tilde{\omega}_x$$

$$\frac{\partial}{\partial t}\frac{\hat{T}}{T_0} = \frac{\nabla_y \phi}{\gamma}\frac{dS}{dz}$$

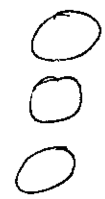
$$\frac{\partial^2}{\partial t^2}(-\nabla^2\phi) = \frac{|g_z|}{\gamma}\frac{\partial^2\phi}{\partial y^2}\frac{dS_0}{dz} + V_A^2\frac{\partial^2}{\partial z^2}(-\nabla^2\phi)$$

$$-\omega^2 k^2 = -k_y^2 N^2 - k_z^2 V_A^2 k^2$$

$$\boxed{\omega^2 = +\frac{k_y^2}{k^2}N^2 + k_z^2 V_A^2}$$

$N^2 < 0$
$\rightarrow$ inst.

$\Rightarrow$ usual R-B. / interchange drive vs. Alfvenic bending criterion.



$\Rightarrow$ here – finite vertical λ_z

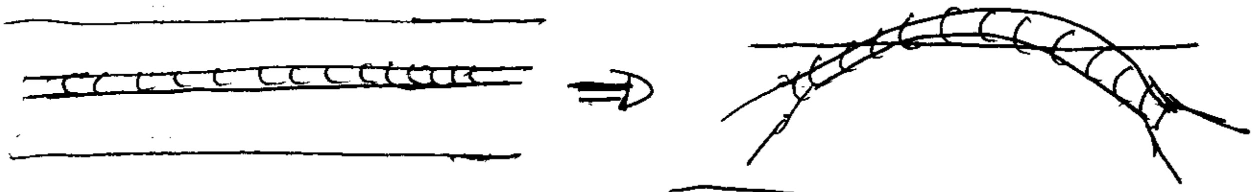
$\Rightarrow$ – field line bending $k_z^2 V_A^2$

i.e. magnetic field stabilizing as vertical cell dimension $\Rightarrow k_z \Rightarrow$ bending

$$\begin{array}{ccc} \text{B-field} & & \text{rotation} \\ \rightarrow V_A^2 k_z^2 & \leftrightarrow & \dfrac{4\Omega^2 k_z^2}{k^2} \end{array} \qquad (4w)$$

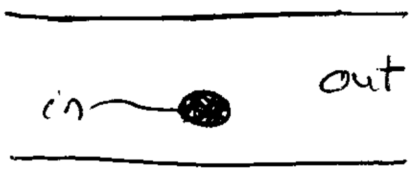
$V_A^2 \to \infty \Rightarrow \partial_z = 0$ a la' Taylor–Proudman Thm.

→ Magnetic Buoyancy Instability



i.e. flux tubes rise! — c.f. picture from Chaudhuri

why? — compare inside tube/outside tube



$$p + \frac{B^2}{8\pi} \cong \text{const.}$$

total pressure balance

$$\left\{ P_{out} = P_{in} + \frac{B^2}{8\pi} \right.$$

$$\therefore \; P_{out} > P_{in}, \quad \text{but} \begin{cases} P = P_0 (\rho/\rho_0)^{\gamma} \\ p = k\rho T \end{cases}$$

buoyancy!

$$\Leftrightarrow \; \rho_{out} > \rho_{in} \, !$$

⇒ tube rises.

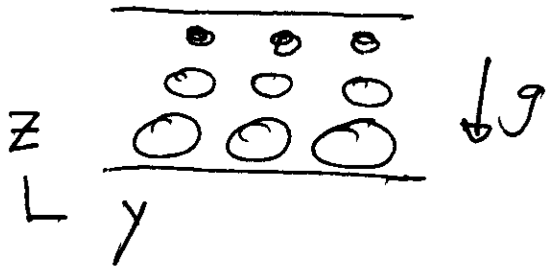
N.B. This is a 'lack of equilibrium', rather than an instability, strictly speaking.

Note: → suggests problem of magnetic buoyancy, i.e.

⇒ convection with magnetic field as the "stuff" to be convected.

The Physics:

a.) Structure



$\frac{dB_{0x}}{dz} < 0$, $\frac{d\rho_0}{dz} < 0$

$\Rightarrow$ stratified magnetic field (vertically)

$\Rightarrow$ $\underline{B}_0 \perp \underline{\tilde{V}}$ $\rightarrow$ interchanges (rolls in y, z; exchanging filled tubes)

$\underline{B}_0 \parallel \underline{\tilde{V}}$ $\rightarrow$ undular instability
(i.e. $B_{0y} \parallel \tilde{V}_y$) (i.e. buoyancy coupled to Alfven wave)

b.) Buoyancy Coupling

- Recall for Rayleigh Benard: $\begin{cases} \omega < k c_s \\ \lambda_z / H_p < 1 \end{cases}$

$\Rightarrow \frac{\delta p}{p_0} \cong 0 \Rightarrow \frac{\hat{\rho}}{\rho_0} = -\frac{\hat{T}}{T_0}$

- with B-field: $\frac{\delta p_{total}}{p_0} \cong 0$ $\Leftrightarrow$ $\omega < k V_{\text{magneto-sonic}}$

$\Rightarrow R(\hat{\rho} T_0 + \hat{T} \rho_0) + \frac{\underline{B}_0 \cdot \underline{\tilde{B}}}{4\pi} \cong 0$

$\frac{\hat{\rho}}{\rho_0} = -\frac{\hat{T}}{T_0} - \frac{\underline{B}_0 \cdot \underline{\tilde{B}}}{4\pi p_0} = -\frac{\hat{T}}{T_0} - \frac{\tilde{p}_m}{p_0}$ $\leftarrow$ magnetic pressure

This obviously suggests that an equation for <u>magnetic</u> <u>pressure</u> would be useful.

c.) B - Pressure evolution

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B})$$

$$\Rightarrow \quad \frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \frac{d\underline{B}}{dt} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

① Now, anelastic approximation

$$\cancel{\frac{\partial \rho}{\partial t}} + \hat{V}_z \frac{\partial \rho_0}{\partial z} = -\rho_0\, \underline{\nabla} \cdot \underline{\hat{V}}$$

$\omega \ll k c_{MS}$

$$\underline{\nabla} \cdot \underline{V} = +\frac{\hat{V}_z}{L_\rho} \qquad \left(1/L_\rho = \frac{-1}{\rho}\frac{d\rho}{dz}\right)$$

② $\underline{B} \cdot \underline{\nabla}\, \underline{V} = 0$ (interchange limit)

$$\therefore \quad \frac{\partial \tilde{B}}{\partial t} + \tilde{V}_z \frac{\partial B_0}{\partial z} = -B_0 \frac{\tilde{V}_z}{L_\rho}$$

$$\frac{\partial}{\partial t} B_0 \cdot \tilde{B} + \frac{B_0^2 \tilde{V}_z}{B_0} \frac{\partial}{\partial z} B_0 = +\frac{B_0^2}{\rho_0} \frac{\partial \rho_0}{\partial z}$$

So, can write:

$$\frac{\partial}{\partial t}\tilde{\rho}_m + \left[\rho_{m,0}\frac{\partial}{\partial z}\ln\left(B_0/\rho_0\right)\right]\hat{v}_z = 0$$

if include resistive dissipation:

$$\frac{\partial}{\partial t}\tilde{\rho}_m - \eta\nabla^2\tilde{\rho}_m = -\hat{v}_z\,\rho_{m,0}\frac{\partial}{\partial z}\left[\ln B_0/\rho_0\right]$$

Now, can proceed with basic equations:

(before)

$$\begin{cases} \frac{\partial}{\partial t}\left(-\nabla^2\phi\right) = |g_z|\frac{\partial}{\partial y}\left(\frac{\hat{T}}{T_0} + \frac{\hat{\rho}_m}{\rho_0}\right) \\ \frac{\partial}{\partial t}\tilde{\rho}_m + \left[\rho_{m,0}\frac{\partial}{\partial z}\ln\left(B_0/\rho_0\right)\right]\hat{v}_z = 0 \quad ; \quad \hat{v}_z = -\nabla_y\phi \end{cases}$$

$$\frac{\partial}{\partial t}\left(\frac{\hat{T}}{T_0} - (\gamma-1)\frac{\hat{\rho}}{\rho_0}\right) + \hat{v}_z\frac{dS_0}{dz} = 0$$

but: $\frac{\hat{\rho}}{\rho_0} = -\frac{\hat{T}}{T_0} - \frac{\rho_m}{\rho_0}$

$$\frac{\partial}{\partial t}\left(\frac{\hat{T}}{T_0} + (\gamma-1)\left(\frac{\hat{T}}{T_0} + \frac{\hat{\rho}_m}{\rho_0}\right)\right) + \hat{V}_z \frac{dS_0}{dz} = 0$$

$$\left\{ \frac{\partial}{\partial t}\left(\cancel{\gamma}\,\frac{\hat{T}}{T_0} + \frac{\cancel{\gamma}-1}{\gamma}\frac{\hat{\rho}_m}{\rho_0}\right) + \frac{\hat{V}_z}{\gamma}\frac{dS_0}{dz} = 0 \right.$$

So can proceed:

$$\Rightarrow \frac{\partial^2}{\partial t^2}\left(-\nabla^2\phi\right) = |g_z|\,\frac{\partial}{\partial y}\left(\frac{\partial}{\partial t}\frac{\hat{T}}{T_0} + \frac{\partial}{\partial t}\frac{\hat{\rho}_m}{\rho_0}\right)$$

$$\frac{\partial}{\partial t}\frac{\hat{T}}{T_0} = -\frac{\left(1-\frac{1}{\gamma}\right)}{\rho_0}\frac{\partial \hat{\rho}_m}{\partial t} - \frac{\hat{V}_z}{\gamma}\frac{dS_0}{dz}$$

$$\frac{\partial}{\partial t}\frac{\hat{\rho}_m}{\rho_0} = -\frac{\rho_{m,0}}{\rho_0}\hat{V}_z\frac{\partial}{\partial z}\ln\left(B_0/\rho_0\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\hat{\rho}_m}{\rho_0} + \frac{\hat{T}}{T_0}\right) = -\hat{V}_z\left\{\left[\frac{\rho_{m,0}}{\rho_0}\frac{d}{dz}\ln\left(B_0/\rho_0\right) + \frac{1}{\gamma}\frac{dS_0}{dz}\right] - \left(1-\frac{1}{\gamma}\right)\left(\frac{\rho_{m,0}}{\rho_0}\frac{\partial}{\partial z}\ln B_0/\rho_0\right)\right\}$$

$$= -\frac{\hat{V}_z}{\gamma}\left[\frac{\rho_{m,0}}{\rho_0}\frac{d}{dz}\ln\left(\frac{B_0}{\rho_0}\right) + \frac{dS_0}{dz}\right]$$

comment → double diffusion

and

$$-\frac{\partial^2}{\partial t^2}\left(+\nabla^2\phi\right) = |g_z| + \frac{\partial^2}{\partial y^2}\phi\left[\frac{1}{\gamma}\left(\frac{P_{m0}}{P_0}\frac{d}{dz}\ln\left(\frac{B_0}{\rho_0}\right) + \frac{dS_0}{dz}\right)\right]$$

$$\Rightarrow \boxed{\omega^2 = +\frac{k_y^2}{k^2}|g_z|\left[\frac{1}{\gamma}\left(\frac{dS_0}{dz} + \frac{P_{m0}}{P_0}\frac{d}{dz}\ln\left(\frac{B_0}{\rho_0}\right)\right)\right]}$$

→ magnetic buoyancy criterion (magnetic Schwarzschild criterion):

$$\boxed{\omega^2 = \frac{k_y^2}{k^2}|g_z|\left[\frac{1}{\gamma}\left(\frac{dS_0}{dz} + \frac{P_{m0}}{P_0}\frac{d}{dz}\ln\left(\frac{B_0}{\rho_0}\right)\right)\right]}$$

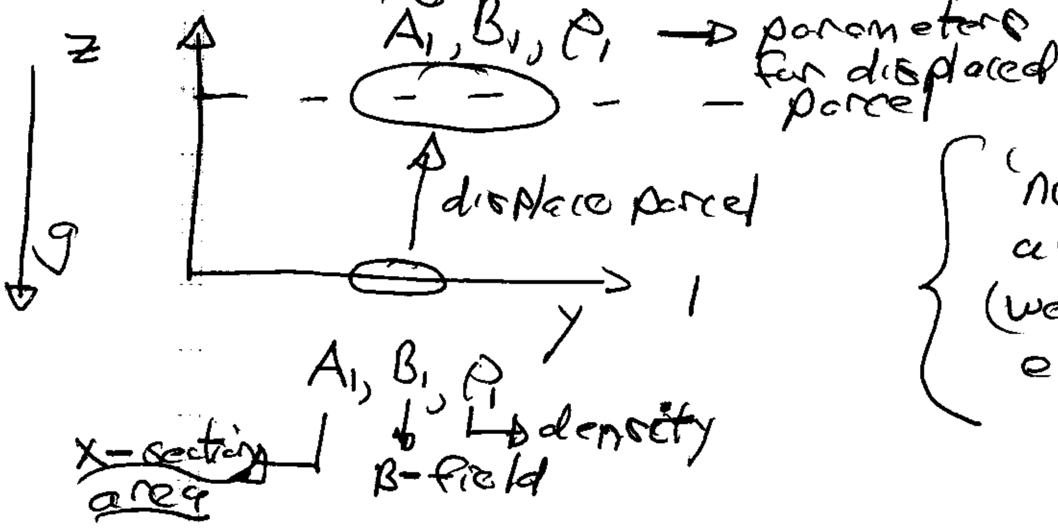
note: for $S_0' = 0$, instability $\Rightarrow \frac{d}{dz}\left(B_0/\rho_0\right) < 0$

for buoyancy instability

→ instabilities are flute / interchanges if $\underline{B}_0 \cdot \underline{\nabla}\, \underline{\tilde{V}} \neq 0 \Rightarrow$ 'undular instability'

→ origin of $(B_0/\rho_0)' < 0$ criterion.

⇒ Reconsider basic story: (ideal interchange)



ρ_2, A_2 (ambient)

'new wrinkle' → A as parameter ⟺ (weak) compressibility essential

so now; in ideal displacement:

mass conserved → $\rho_1 A_1 = \tilde{\rho}_1 \tilde{A}_1$

magnetic flux conserved → $A_1 B_1 = \tilde{A}_1 \tilde{B}_1$

$$\frac{\tilde{A}_1}{A_1} = \frac{B_1}{\tilde{B}_1} = \frac{\rho_1}{\tilde{\rho}_1}$$

$$\Rightarrow B_1/\rho_1 = \tilde{B}_1/\tilde{\rho}_1$$

Now, obviously: stability ⇒ $\tilde{\rho}_1 > \rho_2$ (rise)
instability ⇒ $\tilde{\rho}_1 < \rho_2$ (sink)

Now $\tilde{p}_1 + \frac{\tilde{B}_1^2}{8\pi} = p_2 + \frac{B_2^2}{8\pi}$ (neutral stability)

$\tilde{\rho}_1 = \rho_2 \Rightarrow \tilde{p}_1 = p_2$, by eqn. state

$\therefore \tilde{B}_1 = B_2$ by pressure balance

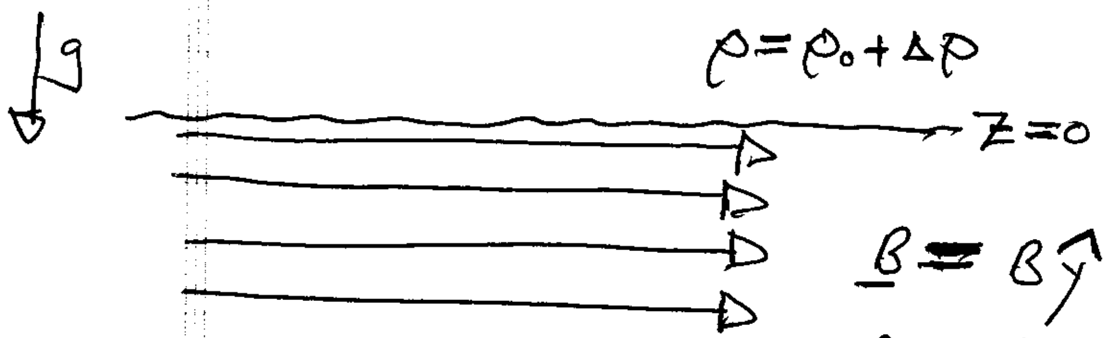
$\therefore$ neutral stability $\Rightarrow$ $\tilde{B}_1/\tilde{\rho}_1 = B_2/\rho_2$

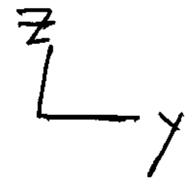
$$\boxed{B_1/\rho_1 = B_2/\rho_2}$$

$\Rightarrow$ instability requires: $\frac{d}{dz}\left(B/\rho\right) < 0$

Related Phenomena / Problem : Interface

Submerged Fields → Stability and Break-up





$T = \text{const.}$

$\underline{B} = B\hat{y}$

$\rho = \rho_0$

$$P_{tot}\Big\}_{z<0} = P_{tot}\Big\}_{z>0}$$

ρ mass-less here

$$\Rightarrow \quad k_B \rho_0 T + \frac{B^2}{8\pi} = k_B(\rho_0 + \Delta\rho)T$$

$$\boxed{\frac{\Delta\rho}{\rho_0} = \frac{B^2/8\pi}{k_B T \rho_0}} = 1/\beta$$

What happens ?

→ Rayleigh-Taylor like instability should occur (modified by bending)

→ bubbles of light fluid (and field) will rise. ⇒ bubble scale ? ↔ eruption scale

Linear Theory:

→ old Rayleigh–Taylor analysis, i.e.

- $\underline{V} = -\nabla\phi$ ⇒ excludes Alfvén waves (rotational)
  $\nabla\cdot\underline{V} = 0$ ⇒ $\nabla^2\phi = 0$
- $\tilde{V}_z$ and δP_{tot} continuous at interface

i.e.

unperturbed interface:

$$\frac{B_0^2}{8\pi} + P_1 = P_2$$

perturbed interface:

$$\tilde{P}_1 + |g|\rho_0\tilde{\eta} + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{4\pi} = \tilde{P}_2 - |g|(\rho_0+\Delta\rho)\tilde{\eta}$$

but: $\tilde{P} = \rho\dfrac{\partial\tilde{\phi}}{\partial t}$

$$\Rightarrow \boxed{\rho_0\frac{\partial\tilde{\phi}_1}{\partial t} - |g|\rho_0\tilde{\eta}_1 + \frac{\underline{B}_0\cdot\tilde{\underline{B}}}{4\pi} = (\rho_0+\Delta\rho)\frac{\partial\tilde{\phi}_2}{\partial t} - |g|(\rho_0+\Delta\rho)\tilde{\eta}_2}$$

where: $$\frac{\partial \tilde{B}_y}{\partial t} = \underline{B_0} \cdot \nabla \tilde{v}_y$$

$$\gamma \tilde{B}_y = i k_y B_0 (-i k_y \hat{\phi}_1)$$

$$\tilde{B}_y = \frac{k_y B_0 \hat{\phi}_1}{\gamma}$$

and $$\gamma \eta = v_z$$

$$\left\{ \begin{array}{l} \phi_1 \sim e^{kz} \quad (\to 0, z = -\infty) \\ k = (k_x^2 + k_y^2)^{1/2} \\ \phi_2 \sim e^{-kz} \end{array} \right.$$

for ①: $\gamma \tilde{\eta}_1 = -k \hat{\phi}_1$

②: $\gamma \tilde{\eta}_2 = +k \hat{\phi}_2$

$$\left\{ \begin{array}{l} \phi_{1,2} = \hat{\phi}_{1,2}\, e^{i \underline{k} \cdot \underline{x}}\, e^{\pm kz} \\ \quad + \to ① \\ \quad - \to ② \end{array} \right.$$

So $$\hat{\phi}_1 + \hat{\phi}_2 = 0$$

$$\gamma \rho_0 \hat{\phi}_1 + \frac{|g| \rho_0 k}{\gamma} \hat{\phi}_1 + \frac{k_y^2 B_0^2}{4\pi \gamma} \hat{\phi}_1 = \gamma (\rho_0 + \Delta\rho) \hat{\phi}_2 - \frac{|g| (\rho_0 + \Delta\rho)}{\gamma} k \hat{\phi}_2$$

$$\Rightarrow \gamma (2\rho_0 + \Delta\rho) \hat{\phi}_1 = \frac{|g| \Delta\rho}{\gamma} k \hat{\phi}_1 - \frac{k_y^2 B_0^2}{4\pi \gamma} \hat{\phi}_1$$

$$\gamma^2 = \frac{g k \Delta\rho}{2\rho_0 + \Delta\rho} - \frac{k_y^2 B_0^2}{4\pi\rho_0\left(1 + \frac{\Delta\rho}{\rho}\right)}$$

$$\boxed{\gamma^2 = \frac{\Delta\rho\, g k}{2\rho_0 + \Delta\rho} - \frac{\rho_0 V_A^2 k_y^2}{2\rho_0 + \Delta\rho}}$$

B-field ↔ "surface tension with direction"

Note:

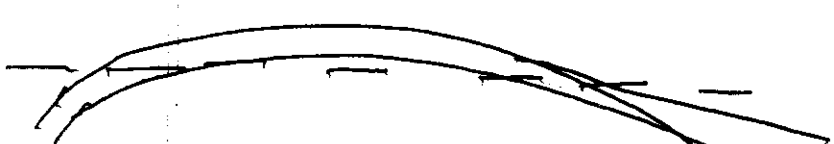
$$\Rightarrow \gamma^2 = \overbrace{\underbrace{A}_{\text{Atwood \#}}\, g k}^{\text{R-T growth}} - \underbrace{\frac{\rho_0}{2\rho_0 + \Delta\rho}}_{\text{bending stabilization}} \left(k_y^2 V_A^2\right)$$

(~ magnetized mass fraction)

→ message is that field and magnetized fluid buoyant
⇒ rising bubbles

$$\Rightarrow \underbrace{1/L_{\parallel}}_{\text{minimum bubble scale, along field}} \sim \frac{\Delta\rho}{\rho}\frac{g}{V_A^2} = \frac{k_y^2}{k} \sim \frac{1}{L}$$

← $L_{\parallel}$ →



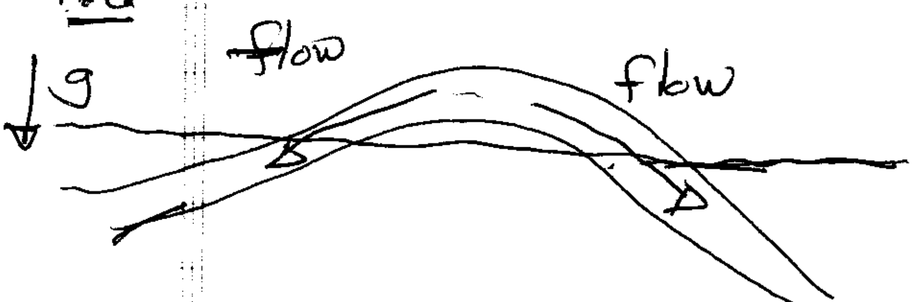
→ easily seen that field 'breaks-up', into structures, even for $k_y = 0$.

⇔ magnetic buoyancy instability generic ⇒ idea underpinning ubiquitous formation of structure in magnetic field.

but → different from Rayleigh Taylor! Field Connection

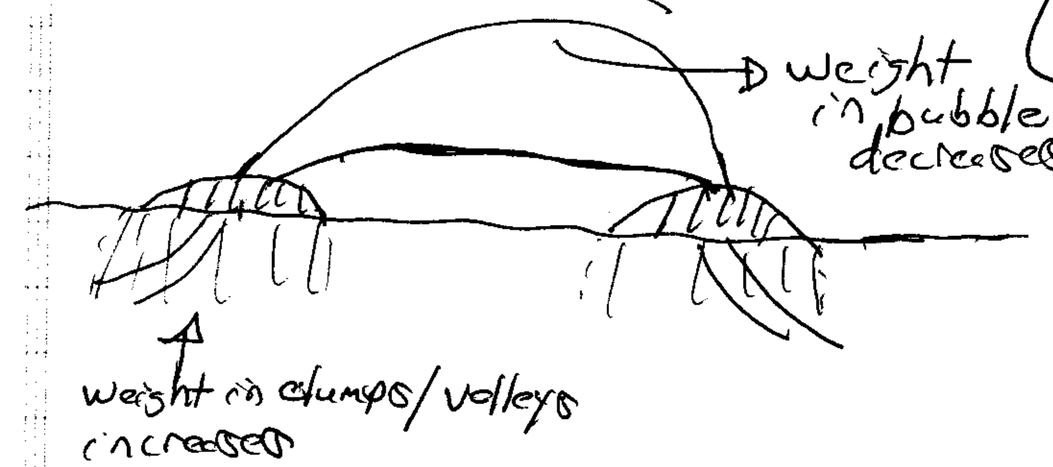
→ furthermore: matter/mass can slide along magnetic field → Parker instability → a/a compressible R-T.

i.e.



⇒ gas slides from peaks → valleys of magnetic undulations

⇒

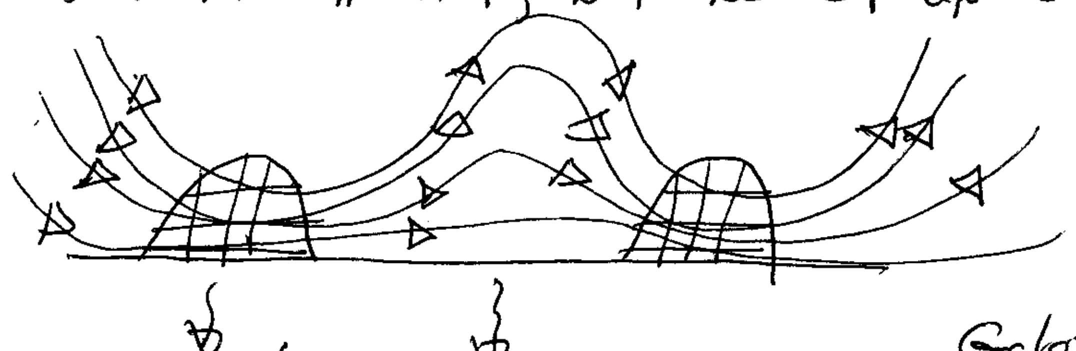


∴ - process is self-reinforcing (i.e. relieving weight at bubble tip allows further rise of tip)

Parker instability → compressible

Key: modes with $k_{\parallel} \neq 0$, ~~~~ line bending penalty offset by reduction in grav. pot. energy.

— matter will form / coagulate in dense clumps, with field attached, but bowed upward



clump/ mass concentration due sliding

upward buoyant undulations

Galaxy as fluid of clumps threaded by field

— B/ρ freezing (i.e. 'sliding' ⟺ refers to field line; i.e. — slide along a given field line)

⇒ B increases (bundle converges) at/in undulation valleys (clumps), as ρ increases.

⇒ B decreases (bundle diverges) in undulation peaks (bubble tips) as ρ decreases.

∴ ⇒ reinforces trend toward energy minimization.

→ Implications for ~~S~~unspots and Prominences (Parker mechanism)

lower ρ

high ρ

tubes twist ⇒ granulation motion

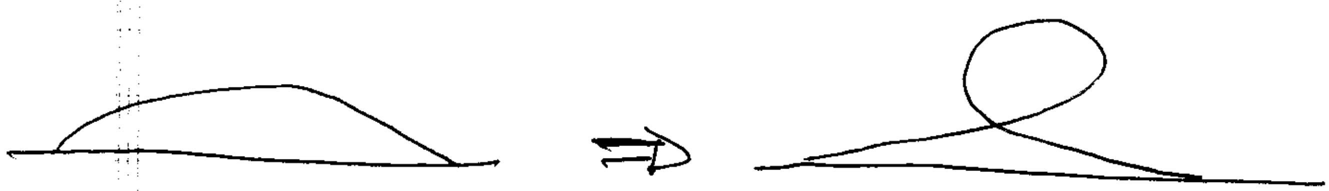
B field anchored in high-density valleys. B/c freezing ⇒ high ρ ⇒ high B. Thus,

- Parker mechanism will strengthen magnetic field and raise ("overload") density in sunspots

⇒

- furthure cooling/darkening due to convection inhibition and mass increase (⇒ radiation)

- i.e. twist ⇒ (kink process)



⇒ reconnection, prominences, etc.