Physics 218B
MHD

MHD Waves — { Information propagation in anisotropic medium

Linearizing the MHD equations:

$\hat{n} = \underline{B}_0/B_0$

$\alpha = \tilde{\rho}/\rho_0$

$\tilde{b} = \tilde{B}/B_0$

$\tilde{p} = \tilde{p}/p_0$

(continuity)

$$\frac{\partial}{\partial t}\tilde{\alpha} + \underline{\nabla}\cdot\tilde{\underline{v}} = 0$$

(momentum balance)

$$\rho_0 \frac{\partial \tilde{v}}{\partial t} = -P_0 \underline{\nabla}\tilde{p} + \frac{B_0^2}{4\pi}(\underline{\nabla}\times\underline{\tilde{b}})\times\hat{n}$$

(eqn. state)

$$\frac{\partial \tilde{p}}{\partial t} = \gamma\frac{\partial \tilde{\alpha}}{\partial t} \Rightarrow \frac{\tilde{p}}{P_0} = \gamma\,\tilde{\rho}/\rho_0$$

(magnetic field)

$$\frac{\partial \tilde{b}}{\partial t} + \underline{\nabla}\times(\hat{n}\times\tilde{\underline{v}}) = 0$$

Now $\tilde{A} = A e^{i(\underline{k}\cdot\underline{x} - \omega t)}$

$\Rightarrow$

1) $$-i\omega\,\underline{v}_1 = -\frac{P_0}{\rho_0} i\underline{k}\,p_1 + \frac{B_0^2}{4\pi\rho_0}(i\underline{k}\times\tilde{b})\times\hat{n}$$

2) $$-i\omega\,\tilde{b} + i\underline{k}\times(\hat{n}\times\tilde{v}) = 0$$

(consistent $\underline{\nabla}\cdot\underline{B} = \underline{k}\cdot\tilde{\underline{b}} = 0$)

3) ~~(iω)~~ $\tilde{p} = \gamma\tilde{\alpha}$ ~~(iω)~~

($\omega = 0 \Rightarrow$ entropy mode)
thermal diffn.

4) $$-i\omega\tilde{\alpha} + i\underline{k}\cdot\tilde{\underline{v}} = 0$$

Now, $c_s^2 = \gamma P_0/\rho_0$; $V_A^2 = B_0^2/4\pi\rho_0$

$\beta \equiv 4\pi P_0 / B_0^2 = \frac{1}{\gamma} c_s^2/V_A^2$ → [impt. ratio

plugging 2), 3), 4) into 1) ⇒

$$\omega^2 \underline{\tilde{V}} + V_A^2 \{ \underline{k} \times (\underline{k} \times (\hat{n} \times \underline{\tilde{V}}_1))\} \times \hat{n} - \underline{k}\, c_s^2\, \underline{k} \cdot \underline{\tilde{V}} = 0$$

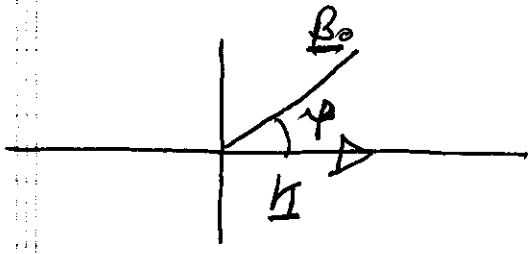
Cranking the cross-products gives:

$$(\omega^2 - k_\parallel^2 V_A^2)\underline{\tilde{V}} + \underline{k}\left[-(V_A^2 + c_s^2)\,\underline{k}\cdot\underline{\tilde{V}} + V_A^2 (\hat{n}\cdot\underline{k})\,\hat{n}\cdot\underline{\tilde{V}}\right]$$

$$+ \hat{n} V_A^2 (\hat{n}\cdot\underline{k})\,\underline{k}\cdot\underline{\tilde{V}} = 0$$

Consider: $\underline{k} = k\,\hat{x}$ — wave vector

$\hat{n} = \cos\psi\,\hat{x} + \sin\psi\,\hat{y}$ — field



$$\Rightarrow \left\{ \begin{aligned} &(\omega^2 - k^2 V_A^2 \cos^2\psi)\underline{\tilde{V}} + \hat{x}\left[-k^2(V_A^2 + c_s^2)\tilde{V}_x + k^2 V_A^2 \cos\psi\,(\tilde{V}_x \cos\psi \right. \\ &\left. + V_y \sin\psi) + k^2 V_A^2 \cos^2\psi\, \tilde{V}_x\right] + \hat{y}\left[k^2 V_A^2 \sin\psi\cos\psi\, \tilde{V}_x\right] \\ &= 0 \end{aligned} \right.$$

i.e. $\frac{\partial \vec{\psi}}{\partial t} = B_z \partial_z \vec{\phi}$ $\qquad$ $-i\omega \vec{\psi} = ik_z B_z \vec{\phi}$

$\rho_0 \frac{\partial}{\partial t} \nabla^2 \vec{\phi} = B_z \partial_z \nabla^2 \psi$ $\qquad$ $+i\omega \rho_0 k_\perp^2 \phi = -ik_z B_z k_\perp^2 \psi$

$\omega^2 = k_z^2 V_A^2$

then solutions to $\{ \quad \} = 0$ (above)
break into classes with

a.) $\tilde{V}_Z \neq 0 \Rightarrow$ contain motion $\perp$ to plane of $\underline{k}$ and $\underline{B_0}$ (i.e. transverse $\leftrightarrow$ shear Alfven)

b) $\tilde{V}_Z = 0 \Rightarrow$ contain only motion in plane of $\underline{k}$, B_0 (i.e. longitudinal)

a.) $\tilde{V}_z \neq 0 \Rightarrow$ solution if $\tilde{V}_x = \tilde{V}_y = 0$
$\tilde{V}_z \neq 0$

recovered in RMHD

and
$$\begin{cases} \omega^2 - k^2 V_A^2 \cos^2\psi = 0 \\ \omega^2 - k_\parallel^2 V_A^2 = 0 \end{cases}$$

shear Alfven

$$\begin{cases} V_A^2 = B_0^2/4\pi \rho_0 \\ T_{eff} = B_0^2/4\pi \rho_0 \end{cases}$$

b.) $\tilde{V}_z = 0, \quad V_x, V_y \neq 0$

$$\begin{pmatrix} \omega^2 + k^2 V_A^2 \cos^2\psi - k^2(V_A^2 + C_s^2), & k^2 V_A^2 \sin\psi \cos\psi \\ k^2 V_A^2 \sin\psi \cos\psi, & \omega^2 - k^2 V_A^2 \cos^2\psi \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \omega^4 - k^2(V_A^2 + C_s^2)\omega^2 + k^4 V_A^2 C_s^2 \cos^2\psi = 0$$

i.e.
$$V_{ph}^4 - (V_A^2 + C_s^2) V_{ph}^2 + C_s^2 V_A^2 \cos^2\psi = 0$$

$$V_{ph} = \omega/k$$

so
$$\boxed{V_p^2 = \frac{1}{2}\left\{ (V_A^2 + C_s^2) \pm \left[(V_A^2+C_s^2)^2 - 4V_A^2 C_s^2 \cos^2\psi\right]^{1/2} \right\}}$$

$+ \rightarrow$ fast magnetosonic / MHD modes
$- \rightarrow$ slow magnetosonic / MHD

Consider $k_{\parallel}$, $k_{\perp}$ limits

a.) $\cos\psi = 1$ $\quad k = k_{\parallel}$

$$V_p^2 = \frac{1}{2}\left\{ (V_A^2 + C_s^2) \pm \left[(V_A^2 - C_s^2)^2\right]^{1/2} \right\}$$

$$= V_A^2,\ C_s^2$$

fast = faster (V_A^2, C_s^2)
slow = slower

i.e. acoustic, shear Alfven (with diff'nt polarization previous)

b.) $\cos\psi = 0$ $\quad k = k_{\perp}$

$$V_p^2 = V_A^2 + C_s^2,\quad 0$$

i.e. slow vanished

fast $\rightarrow$ magnetosonic / compressional Alfven

Now, $V_{ph}^2 = \gamma \frac{P_0}{\rho_0} + 2\frac{P_{mag}}{\rho_0}$ $\qquad P_{mag} = \frac{B_0^2}{8\pi}$

↳ origin of factor 2?

Resolution: $\gamma_{eff} = 2$

→ 1D compression: $\gamma = \frac{d+2}{d}$, $d=1$ effectively

<u>QE</u>

→ alternatively, B/ρ frozen in

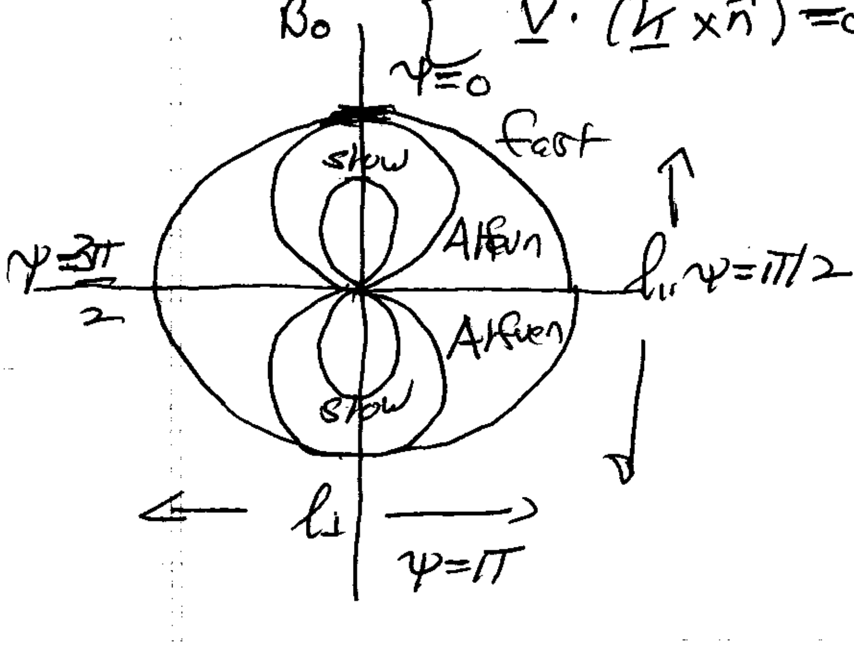
$\therefore B \sim \rho$

$$\Rightarrow \frac{B^2}{8\pi} \sim \rho^2 \sim P_{eff}$$

$$\frac{dP}{d\rho} = 2\frac{P}{\rho}.$$

<u>Summary Diagram:</u>

3 waves: $\begin{cases} \tilde{V} \approx \underline{k} \times \hat{n} \rightarrow \text{Alfven} \\ \underline{V} \cdot (\underline{k} \times \hat{n}) = 0 \Rightarrow \text{fast, slow MHD} \end{cases}$



$l_\perp > l_\parallel$

$\begin{cases} V_A^2 > C_s^2 \\ \text{emitter of all types at origin} \end{cases}$

<u>i.e.</u> on axis: $\psi = 0, \pi$

fast: $V_{ph} = V_A$
Alfven: $V_{ph} = V_A$
slow: $V_{ph} = C_s$

$\perp$ B_0: $\psi = \pi/2, 3\pi/2$

fast: $V_{ph} = \sqrt{V_A^2 + C_s^2}$ (ellipsoid)
slow: $V_{ph} = 0$
Alfven: $V_{ph} = 0$

<u>So</u> if emitter at origin:
emission $t=0$
$\Rightarrow$ after Δt, waves reach surfaces in diagram
$\Rightarrow$ $\perp$ $\underline{B_0}$ $\Rightarrow$ info propagates at magnetosonic speed (fastest)
$\parallel$ B_0 $\Rightarrow$ V_A (i.e. $V_{fast} = V_A$)

intermediate $\measuredangle$s $\Rightarrow$ fast mode carried info arrives first, slow last.

Read:
- L&L: Fluids → Shocks
- L+P: Phys. Kin → Collisionless shocks

→ Nonlinear Alfven Waves

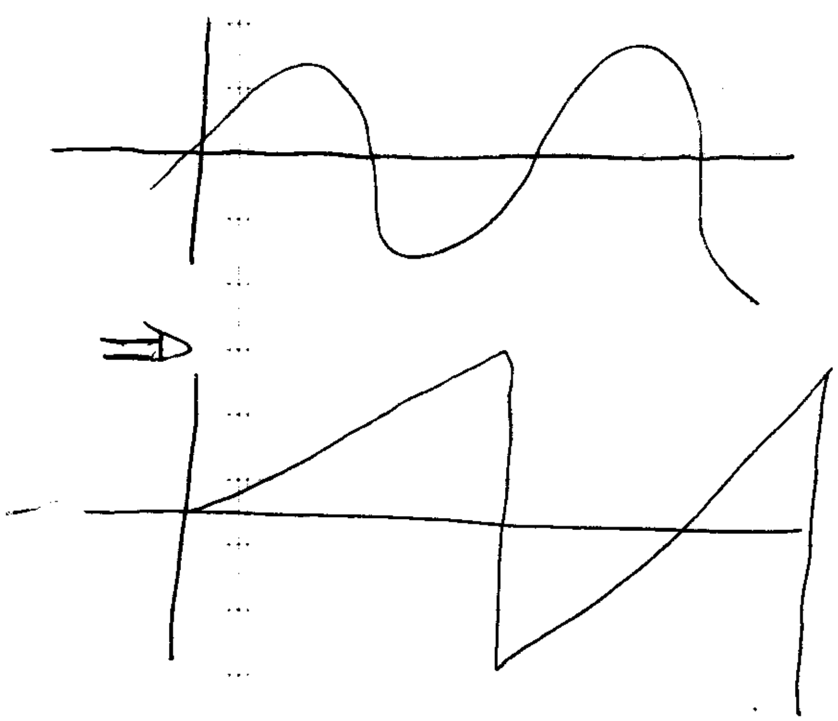
Interesting to ask re: nonlinear evolution of Alfven wave? → NL Wave dynamics

pure $\perp$ → pure compressional ⇒ steepening and shock, a/a' acoustic wave

pure $\parallel$ → pure shear Alfven ⇒ ?.

Central idea in shock → speed increases with density perturbation ⇒ steepening/over-turning and shock formation

i.e. in simple acoustics:



$$\begin{cases} P = P_0 (\rho/\rho_0)^\gamma \\ c_s^2 = \dfrac{dP}{d\rho} = \dfrac{P_0}{\rho_0} \gamma (\rho/\rho_0)^{\gamma-1} \\ c_s^2 \uparrow \text{ with } \rho \end{cases}$$

→ process of speed increasing with density (⇒ high density regions speeding up, low density slowing down) is one of steepening.

→ shock formation is process of making waveform set by:

steepening vs. dissipation → hydro shock

steepening vs. dispersion → collisionless shock

i.e. - hydro - shock:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} - \nu \frac{\partial^2}{\partial x^2} \rho = 0$$

$$\frac{\rho^2}{\Delta} \sim \frac{\nu}{\Delta^2} \rho \quad \Rightarrow \quad \Delta \sim \frac{\nu}{\rho}$$

⇒ shock thickness set by viscosity

- collisionless shock (i.e. ion acoustic) (from R. Z. Sagdeev)

$$\omega = \frac{k c_s}{(1 + k^2 \lambda_D^2)^{1/2}}$$

$$\cong k c_s \left(1 - \frac{1}{2} k^2 \lambda_D^2\right)$$

$$\therefore \quad \boxed{\frac{\partial a}{\partial t} + c_s \frac{\partial a}{\partial x} + \gamma a \frac{\partial a}{\partial x} + c_s \rho_s^2 \frac{\partial^3 a}{\partial x^3} = 0}$$

(KDV equation) → exactly solvable
→ soliton soln.

for scale:

$$\frac{\gamma a^2}{\Delta} \sim c_s^2 \rho_s^2 \frac{a}{\Delta^3} \Rightarrow \Delta^2 \sim \frac{c_s^2 \rho_s^2}{\gamma a}$$

→ set by dispersion, etc

→ Now how does (transverse) shear Alfven wave steepen?

Key point: need introduce parallel compressibility

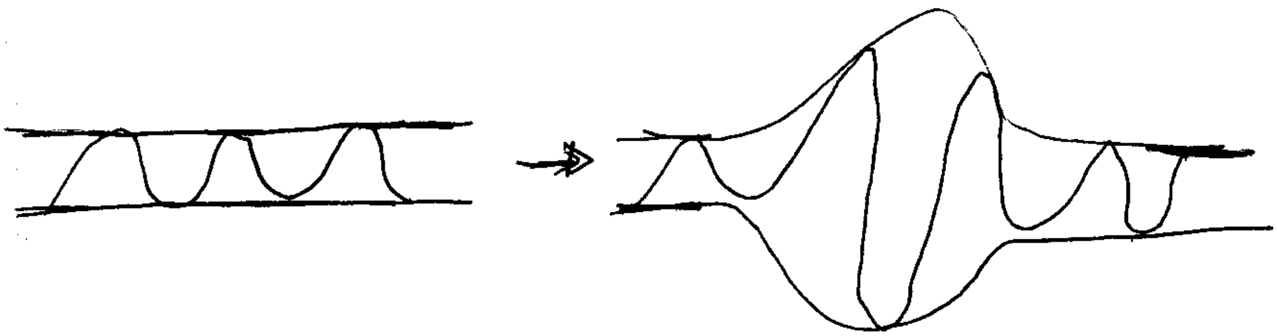
in pictures:

- Consider Alfven wave train:

$\rightarrow V_A$

→ now, modulate the wave packet:

i.e.



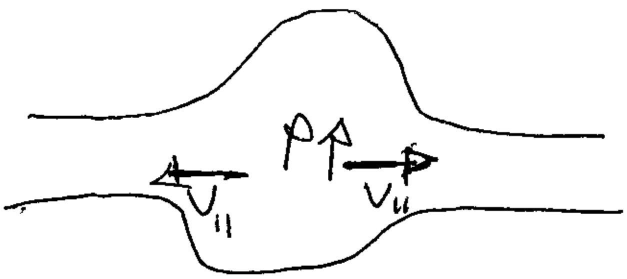
now, B^2 increase $\Rightarrow$ ρ increases

and $P = P_0 (\rho/\rho_0)^\gamma$ $\Rightarrow$ P increases

$\Rightarrow$ pressure increases at modulation

$\rightarrow$ pressure increase $\Rightarrow$ parallel flow
and parallel flow $\Rightarrow$ re-inforces δB by
$\underline{V} \times \underline{B}$

i.e.



How to Calculate:

- recall envelope approach to Zakharov equations

$$\omega^2 = \omega_p^2 + k^2 \lambda_D^2 \quad \Rightarrow$$

$$+ i\gamma\omega_p = \omega_p^2 \frac{\delta n}{n} + k^2 \lambda_D^2$$

$$\omega_{p0} \frac{\partial \mathcal{E}}{\partial t} = \omega_{p0}^2 \frac{\delta n}{n_0} \mathcal{E} - \lambda_0^2 \nabla^2 \mathcal{E}$$

and $$\frac{\partial \rho}{\partial t} = -\rho_0 \underline{\nabla} \cdot \underline{V}$$

$$\rho_0 \frac{\partial \underline{V}}{\partial t} = -\nabla \left(P_{Th} + P_{ES} \right)$$

etc.

– now,

$$\omega = k V_A$$

$$(\omega + i\gamma) = \left(k - i \frac{\partial}{\partial x} \right) V_A$$

$$V_A = B_0 / \sqrt{4\pi\rho}$$

$$= B_0 / \sqrt{4\pi\rho_0 (1 + \tilde{\rho}/\rho_0)} \cong V_A^{(0)} \left(1 - \frac{1}{2} \frac{\tilde{\rho}}{\rho_0} \right)$$

$$\cancel{\omega \mathcal{E}} + i \frac{\partial}{\partial t} \mathcal{E} = \cancel{k V_A \mathcal{E}} - i \frac{\partial}{\partial x} \left[\left(\frac{1}{2} \frac{\tilde{\rho}}{\rho_0} \right) \mathcal{E} \right] V_{A0}$$

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{V_{A0}}{2}\frac{\partial}{\partial x}\left[\frac{\tilde{\rho}}{\rho_0}\mathcal{E}\right]$$

Now, for $\tilde{\rho}/\rho_0$

$$\frac{\partial}{\partial t} V_{\parallel} = -\frac{\nabla_{\parallel}}{\rho_0}\left[p + \frac{B^2}{8\pi} + \frac{\rho V^2}{2}\right]$$

$$= -\frac{\nabla_{\parallel}}{\rho_0}\left[p + \frac{B^2}{4\pi}\right]$$

as, for Alfven waves:

$$\rho_0\frac{V^2}{2} = \frac{B^2}{8\pi}$$

$\rightarrow$ equipartition of energy

and continuity

(fluid accelerated along $\underline{B_0}$ by magnetic, thermal pressure)

$\therefore$

$$\frac{\partial}{\partial t} V_x = -\frac{\partial}{\partial x}\left[C_s^2\left(\frac{\tilde{\rho}}{\rho_0}\right) + V_A^2\left|\frac{\delta B}{B_0}\right|^2\right] \rightarrow \text{fluid}$$

$$\frac{\partial\tilde{\rho}}{\partial t} = -\rho_0\left(\frac{\partial}{\partial x}\tilde{V}\right)$$

$\rightarrow$

$$\frac{\partial^2}{\partial t^2}\frac{\tilde{\rho}}{\rho_0} = \frac{\partial^2}{\partial x^2}\left[C_s^2\frac{\tilde{\rho}}{\rho_0} + V_A^2\left|\frac{\delta B}{B_0}\right|^2\right]$$

So, like Zakharov system, have:

$$\frac{\partial \mathcal{E}}{\partial t} = -\frac{V_{A0}^2}{2}\frac{\partial}{\partial x}\left(\frac{\tilde{\rho}}{\rho_0}\mathcal{E}\right) \qquad \mathcal{E} \equiv \frac{\delta B}{B_0}$$

$$\frac{\partial^2}{\partial t^2}\tilde{\rho}/\rho_0 = \frac{\partial^2}{\partial x^2}\left[c_s^2\frac{\tilde{\rho}}{\rho_0} + V_{A0}^2 \quad |\mathcal{E}|^2\right]$$

Now, choose frame moving with Alfven wave packet, so $\frac{\partial}{\partial t} = -V_A\frac{\partial}{\partial x}$

$\Rightarrow$

$$V_A^2\frac{\partial^2}{\partial x^2}\left(\tilde{\rho}/\rho_0\right) = V_A^2\frac{\partial^2}{\partial x^2}\left[\beta\frac{\tilde{\rho}}{\rho_0} + |\mathcal{E}|^2\right]$$

$\therefore \quad \tilde{\rho}/\rho_0 \cong \frac{1}{(1-\beta)}|\mathcal{E}|^2 \rightarrow$ Alfven wave modulation-driven density perturbation

so have:

$$\boxed{\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial}{\partial x}\left[\frac{V_{A0}^2}{2(1-\beta)}|\mathcal{E}|^2\mathcal{E}\right] = 0}$$

— envelope equation for NL Alfven (shear) wave.

- can invoke:

resistive dissipation $\to -\eta\, \partial^2/\partial x^2$

dispersion (Hall effects) $\to -i\Omega_i \dfrac{c^2}{\omega_{pi}^2}\dfrac{\partial}{\partial x}$

to oppose steepening, form shock.

$\Rightarrow$ (net product) is "DNLS" – "Derivative NLS"

i.e. parallel compression as route to steepening

typically, DNLS written as:

$$\frac{\partial}{\partial t}\mathcal{E} + \frac{\partial}{\partial x}\left[\frac{V_{A0}^2}{(1-\beta)}|\mathcal{E}|^2\mathcal{E}\right] + i\delta\frac{\partial^2}{\partial x^2}\mathcal{E} = 0$$

- $\beta \to 1$ $\Rightarrow$ subsonic approximation fails

i.e. not sensible to choose frame co-moving with Alfven wave.

- a) DNLS is integrable, using inverse scattering theory methods.

b) if kinetics $\to$ KNLS
$\to$ drives n-part. Landau damped.

→ physical interpretation:

- DNLS reflects scattering/coupling of Alfven wave by ion-acoustic wavemode.

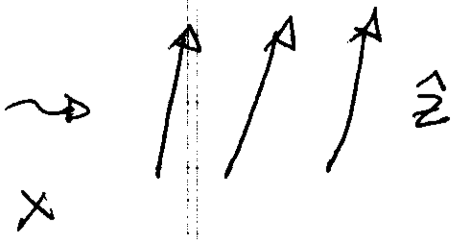
- obvious analogy:

Langmuir	MHD
Plasmon	Alfven
Phonon	Phonon
NLS	DNLS
Ponderomotive Force	‖ Ponderomotive Force

as in Langmuir, can treat via RPA methods, too.

→ Compressional / Magnetosonic Case

Consider simplified limit where:



1D

— $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0$$

$$\frac{\partial B_z}{\partial t} + v\frac{\partial}{\partial x} B_z = \cancel{B_z \frac{\partial}{\partial z} v_x} - \frac{\partial v_x}{\partial x} B_z$$

— $$\frac{\partial}{\partial t} B_z + \frac{\partial}{\partial x}(v B_z) = 0$$

— $$\frac{\partial v}{\partial t} + v\frac{\partial}{\partial x} v = \frac{-1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{B^2}{8\pi}\right)$$

Now, from ρ, B_z eqn. $B_z = b\rho$
i.e. B/ρ frozen in.

$\Rightarrow$ $B = b\rho$

$$\therefore \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \frac{-1}{\rho}\frac{\partial}{\partial x}\left(p(\rho) + \frac{b^2\rho^2}{8\pi}\right)$$
$$= -\frac{1}{\rho}\frac{\partial}{\partial x} p_{eff}.$$

we can map problem to solvable hydro problem with $P_{eff} = P(\rho) + \frac{b^2\rho^2}{8\pi}$, as eqn. state.

HW: $P \to 0 \Leftrightarrow$ Burgers.