Chapter 4

Continuum Mechanics

Continuous systems possess an infinite number of degrees of freedom. They are described by a set of fields $\phi_a(\boldsymbol{x},t)$ which depend on space and time. These fields may represent local displacement, pressure, velocity, *etc.* The equations of motion of the fields are again determined by extremizing the action, which, in turn, is an integral of the *Lagrangian density* over all space and time. Extremization yields a set of (generally coupled) *partial* differential equations.

4.1 Euler-Lagrange Equations for Classical Field Theories

Suppose $\phi_a(\boldsymbol{x})$ depends on n independent variables, $\{x^1, x^2, \ldots, x^n\}$. Consider the functional

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_{\Omega} d\boldsymbol{x}\, \mathcal{L}(\phi_a\, \partial_\mu \phi_a, \boldsymbol{x}) \ , \tag{4.1}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in R^n. Then the first variation of S is

$$\begin{aligned} \delta S &= \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial\, \delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \delta\phi_a + \int\limits_{\Omega} d\boldsymbol{x} \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a \ , \end{aligned} \tag{4.2}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ of $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right]_{\boldsymbol{x}} \ , \tag{4.3}$$

where the subscript means we are to evaluate the term in brackets at $\boldsymbol{x}$. In a mechanical system, one of the n independent variables (usually x^0), is the time t. However, we may

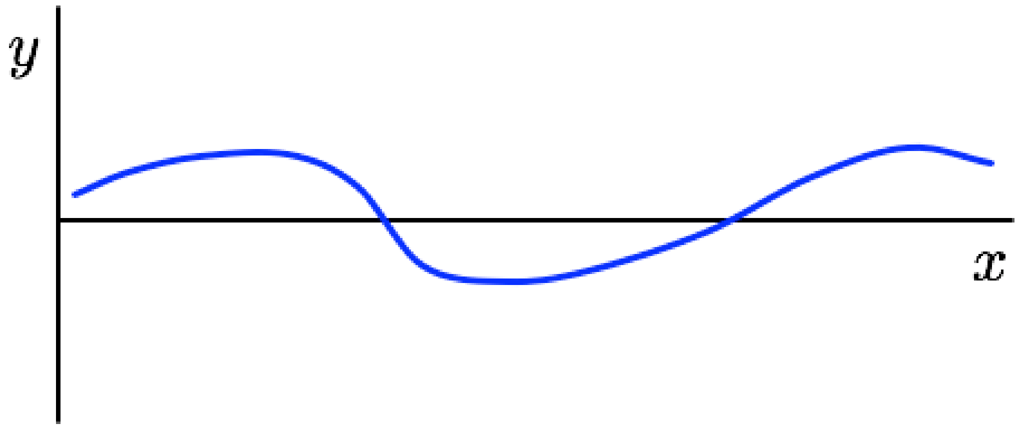

Figure 4.1: A string is described by the vertical displacement field $y(x,t)$.

be interested in a time-independent context in which we wish to extremize the energy functional, for example. In any case, setting the first variation of S to zero yields the Euler-Lagrange equations,

$$\delta S = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\right) = 0 \tag{4.4}$$

As an example, consider a string of linear mass density $\mu(x)$ under tension $\tau(x)$.[1] Let the string move in a plane, such that its shape is described by a smooth function $y(x)$, the vertical displacement of the string at horizontal position x. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. Consider a differential element of the string extending from x to $x + dx$. The change in length relative to the unstretched ($y = 0$) configuration is

$$d\ell = \sqrt{dx^2 + dy^2} - dx \simeq \tfrac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx \ . \tag{4.5}$$

The differential potential energy is then

$$dU = \tau(x)\, d\ell = \tfrac{1}{2}\, \tau(x)\, {y'}^2(x)\, dx \ . \tag{4.6}$$

The differential kinetic energy is simply

$$dT = \tfrac{1}{2}\, \mu(x)\, \dot{y}^2(x)\, dx \ . \tag{4.7}$$

Thus, the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\tau(x)\left(\frac{\partial y}{\partial x}\right)^2 , \tag{4.8}$$

[1] As an example of a string with a position-dependent tension, consider a string of length ℓ freely suspended from one end at $z = 0$ in a gravitational field. The tension is then $\tau(z) = \mu g(\ell - z)$.

whence the Euler-Lagrange equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) \\ &= \frac{\partial}{\partial x}\left[\tau(x)\,\frac{\partial y}{\partial x}\right] - \mu(x)\,\frac{\partial^2 y}{\partial t^2} \ , \end{aligned} \tag{4.9}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. When τ and μ are constant, we obtain $\mu\ddot{y} = \tau y''$, which is the Helmholtz equation. We've assumed boundary conditions where $\delta y(x_a, t) = \delta y(x_b, t) = \delta y(x, t_a) = \delta y(x, t_b) = 0$.

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - J_\mu\,A^\mu \ . \tag{4.10}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu\,F^{\mu\nu} = 4\pi J^\nu \ , \tag{4.11}$$

which are Maxwell's equations.

4.1.1 Conserved Currents in Field Theory

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{\eta}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{4.12}$$

where $\tilde{\eta}_\sigma = \tilde{\eta}_\sigma(q, \zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x}, t) \ , \tag{4.13}$$

where $\{\phi_a(\boldsymbol{x}, t)\}$ are a set of fields, which are functions of the independent variables $\{x, y, z, t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct, x, y, z)$. The generalization of $dQ/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0 \ , \tag{4.14}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\,J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} \ . \tag{4.15}$$

We call $Q = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu J^\mu$ over the spatial region Ω gives

$$\frac{dQ}{dt} = \int_\Omega d^3x\, \partial_0\, J^0 = -\int_\Omega d^3x\, \boldsymbol{\nabla} \cdot \boldsymbol{J} = -\oint_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}} \cdot \boldsymbol{J} = 0 \ , \tag{4.16}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a complex scalar field, with Lagrangian density[2]

$$\mathcal{L}(\psi, , \psi^*, \partial_\mu\psi, \partial_\mu\psi^*) = \tfrac{1}{2} K\, (\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \ . \tag{4.17}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\, e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\, e^{-i\zeta}\,\psi^* \ , \tag{4.18}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\,(\partial_\mu\psi)} \cdot (i\psi) + \frac{\partial\mathcal{L}}{\partial\,(\partial_\mu\psi^*)} \cdot (-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \ . \end{aligned} \tag{4.19}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

4.1.2 Gross-Pitaevskii Model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\,\big(|\psi|^2 - n_0\big)^2 \ . \tag{4.20}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x}, t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned} \delta S[\psi^*, \psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\ &= \int\! dt \!\int\! d^dx\, \bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^* \cdot \boldsymbol{\nabla}\psi \\ &\qquad\qquad - 2g\,\big(|\psi|^2 - n_0\big)\,(\psi^*\delta\psi + \psi\delta\psi^*)\bigg\} \\ &= \int\! dt \!\int\! d^dx\, \bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \bigg]\,\delta\psi \\ &\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi \bigg]\,\delta\psi^* \bigg\} \ , \end{aligned} \tag{4.21}$$

[2] We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \text{diag}\,(+,-,-,-)$.

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*, \psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + 2g \left(|\psi|^2 - n_0 \right) \psi \tag{4.22}$$

as well as its complex conjugate,

$$-i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi^* + 2g \left(|\psi|^2 - n_0 \right) \psi^* \ . \tag{4.23}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\frac{\delta S}{\delta \psi} = \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi} \right) \tag{4.24}$$

$$\frac{\delta S}{\delta \psi^*} = \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*} \right) , \tag{4.25}$$

with $x^\mu = (t, \boldsymbol{x})$[3] Plugging in

$$\frac{\partial \mathcal{L}}{\partial \psi} = -2g \left(|\psi|^2 - n_0 \right) \psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi} = i\hbar\, \psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla} \psi} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla} \psi^* \tag{4.26}$$

and

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\hbar\, \psi - 2g \left(|\psi|^2 - n_0 \right) \psi \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi^*} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla} \psi^*} = -\frac{\hbar^2}{2m} \boldsymbol{\nabla} \psi \ , \tag{4.27}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a U(1) invariance, under

$$\psi(\boldsymbol{x}, t) \to \tilde{\psi}(\boldsymbol{x}, t) = e^{i\zeta}\, \psi(\boldsymbol{x}, t) \quad , \quad \psi^*(\boldsymbol{x}, t) \to \tilde{\psi}^*(\boldsymbol{x}, t) = e^{-i\zeta}\, \psi^*(\boldsymbol{x}, t) \ . \tag{4.28}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi} \frac{\partial \tilde{\psi}}{\partial \zeta} \bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*} \frac{\partial \tilde{\psi}^*}{\partial \zeta} \bigg|_{\zeta=0}$$

$$J^0 = -\hbar\, |\psi|^2 \tag{4.29}$$

$$\boldsymbol{J} = -\frac{\hbar^2}{2im} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) \ . \tag{4.30}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar \boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \ , \tag{4.31}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im} \left(\psi^* \boldsymbol{\nabla} \psi - \psi \boldsymbol{\nabla} \psi^* \right) \ . \tag{4.32}$$

are the particle density and the particle current, respectively.

[3]In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

4.2 Strings

The action for the string is

$$S = \int_{x_a}^{x_b} dx \int_{t_a}^{t_b} dt\, \mathcal{L}(y, y', \dot{y}) \ , \tag{4.33}$$

where $y(x,t)$ is the vertical displacement field. Typically, we have $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2$. The first variation of S is

$$\delta S = \int_{x_a}^{x_b} dx \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) \right] \tag{4.34}$$

$$+ \int_{t_a}^{t_b} dt \left[\frac{\partial \mathcal{L}}{\partial y'}\, \delta y \right]_{x=x_b}^{x=x_a} + \int_{x_a}^{x_b} dx \left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\, \delta y \right]_{t=t_a}^{t=t_b} \ , \tag{4.35}$$

which simply recapitulates the general result from eqn. 4.2. There are two boundary terms, one of which is an integral over time and the other an integral over space. The first boundary term vanishes provided $\delta y(x,t_a) = \delta y(x,t_b) = 0$. The second boundary term vanishes provided $\tau(x)\, y'(x)\, \delta y(x) = 0$ at $x = x_a$ and $x = x_b$, for all t. Assuming $\tau(x)$ does not vanish, this can happen in one of two ways: at each endpoint either $y(x)$ is fixed or $y'(x)$ vanishes.

We've already found the Euler-Lagrange equation,

$$\frac{\partial}{\partial x}\left[\tau(x)\, \frac{\partial y}{\partial x}\right] - \mu(x)\, \frac{\partial^2 y}{\partial t^2} = 0 \ . \tag{4.36}$$

When $\tau(x) = \tau$ and $\mu(x) = \mu$ are both constants, we obtain the Helmholtz equation,

$$\frac{1}{c^2}\, \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \ , \tag{4.37}$$

which is the wave equation for the string, where $c = \sqrt{\tau/\mu}$ has dimensions of velocity. We will now see that c is the speed of wave propagation on the string.

4.2.1 D'Alambert's Solution to the Wave Equation

Let us define two new variables,

$$u \equiv x - ct \qquad , \qquad v \equiv x + ct \ . \tag{4.38}$$

We then have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\, \frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\, \frac{\partial}{\partial v} \qquad = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{4.39}$$

$$\frac{1}{c}\, \frac{\partial}{\partial t} = \frac{1}{c}\, \frac{\partial u}{\partial t}\, \frac{\partial}{\partial u} + \frac{1}{c}\, \frac{\partial v}{\partial t}\, \frac{\partial}{\partial v} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \ . \tag{4.40}$$

Thus,

$$\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -4\,\frac{\partial^2}{\partial u\,\partial v}\ . \tag{4.41}$$

Thus, the wave equation may be solved:

$$\frac{\partial^2 y}{\partial u\,\partial v} = 0 \qquad \Longrightarrow \qquad y(u,v) = f(u) + g(v)\ , \tag{4.42}$$

where $f(u)$ and $g(v)$ are arbitrary functions. For the moment, we work with an infinite string, so we have no spatial boundary conditions to satisfy. Note that $f(u)$ describes a right-moving disturbance, and $g(v)$ describes a left-moving disturbance:

$$y(x,t) = f(x-ct) + g(x+ct)\ . \tag{4.43}$$

We do, however, have boundary conditions in time. At $t=0$, the configuration of the string is given by $y(x,0)$, and its instantaneous vertical velocity is $\dot{y}(x,0)$. We then have

$$y(x,0) = f(x) + g(x) \tag{4.44}$$

$$\dot{y}(x,0) = -c\,f'(x) + c\,g'(x)\ , \tag{4.45}$$

hence

$$f'(x) = \tfrac{1}{2}\,y'(x,0) - \tfrac{1}{2c}\,\dot{y}(x,0) \tag{4.46}$$

$$g'(x) = \tfrac{1}{2}\,y'(x,0) + \tfrac{1}{2c}\,\dot{y}(x,0)\ , \tag{4.47}$$

and integrating we obtain the right and left moving components

$$f(\xi) = \tfrac{1}{2}\,y(\xi,0) - \tfrac{1}{2c}\int\limits_0^{\xi}\!d\xi'\;\dot{y}(\xi',0) - \mathcal{C} \tag{4.48}$$

$$g(\xi) = \tfrac{1}{2}\,y(\xi,0) + \tfrac{1}{2c}\int\limits_0^{\xi}\!d\xi'\;\dot{y}(\xi',0) + \mathcal{C}\ , \tag{4.49}$$

where $\mathcal{C}$ is an arbitrary constant. Adding these together, we obtain the full solution

$$y(x,t) = \tfrac{1}{2}\Big[y(x-ct,0) + y(x+ct,0)\Big] + \tfrac{1}{2c}\int\limits_{x-ct}^{x+ct}\!d\xi\;\dot{y}(\xi,0)\ , \tag{4.50}$$

valid for all times.

4.2.2 Reflection at an Interface

Consider a semi-infinite string on the interval $\big[0,\infty\big]$, with $y(0,t)=0$. We can still invoke D'Alambert's solution, $y(x,t) = f(x-ct) + g(x+ct)$, but we must demand

$$y(0,t) = f(-ct) + g(ct) = 0 \quad \Rightarrow \quad f(\xi) = -g(-\xi)\ . \tag{4.51}$$

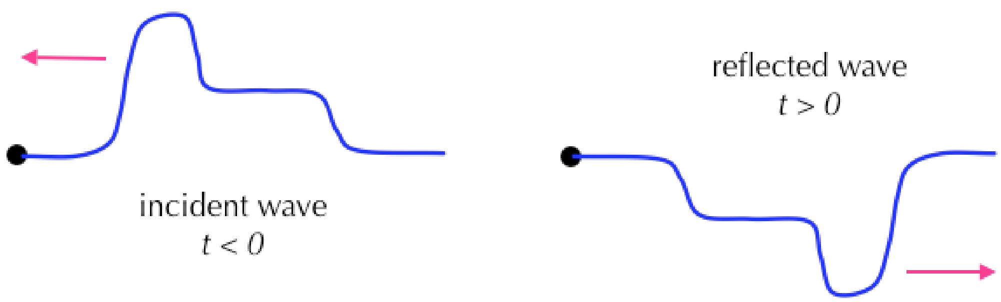

Figure 4.2: Reflection of a pulse at an interface at $x = 0$, with $y(0,t) = 0$.

Thus,

$$y(x,t) = g(ct + x) - g(ct - x) \ . \tag{4.52}$$

Now suppose $g(\xi)$ describes a pulse, and is nonzero only within a neighborhood of $\xi = 0$. For large negative values of t, the right-moving part, $-g(ct - x)$, is negligible everywhere, since $x > 0$ means that the argument $ct - x$ is always large and negative. On the other hand, the left moving part $g(ct + x)$ is nonzero for $x \approx -ct > 0$. Thus, for $t < 0$ we have a left-moving pulse incident from the right. For $t > 0$, the situation is reversed, and the left-moving component is negligible, and we have a right moving reflected wave. However, the minus sign in eqn. 4.51 means that the reflected wave is *inverted.*

If instead of fixing the endpoint at $x = 0$ we attach this end of the string to a massless ring which frictionlessly slides up and down a vertical post, then we must have $y'(0,t) = 0$, else there is a finite vertical force on the massless ring, resulting in infinite acceleration. We again write $y(x,t) = f(x - ct) + g(x + ct)$, and we invoke

$$y'(0,t) = f'(-ct) + g'(ct) \quad \Rightarrow \quad f'(\xi) = -g'(-\xi) \ , \tag{4.53}$$

which, upon integration, yields $f(\xi) = g(-\xi)$, and therefore

$$y(x,t) = g(ct + x) + g(ct - x) \ . \tag{4.54}$$

The reflected pulse is now 'right-side up', in contrast to the situation with a fixed endpoint.

Next, consider the case depicted in Fig. 4.4, where a point mass m is affixed to an infinite string at $x = 0$. Let us suppose that at large negative values of t, a right moving

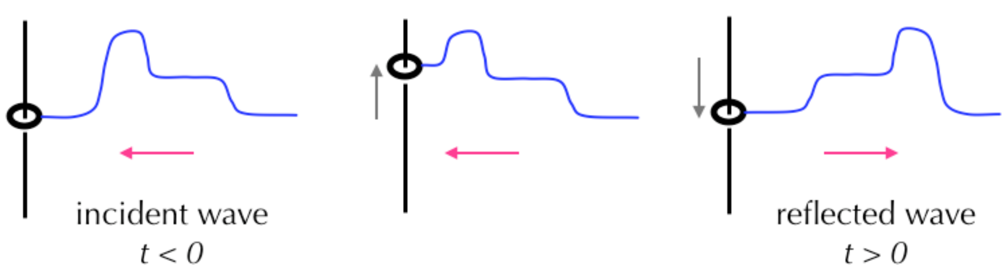

Figure 4.3: Reflection of a pulse at an interface at $x = 0$, with $y'(0,t) = 0$.

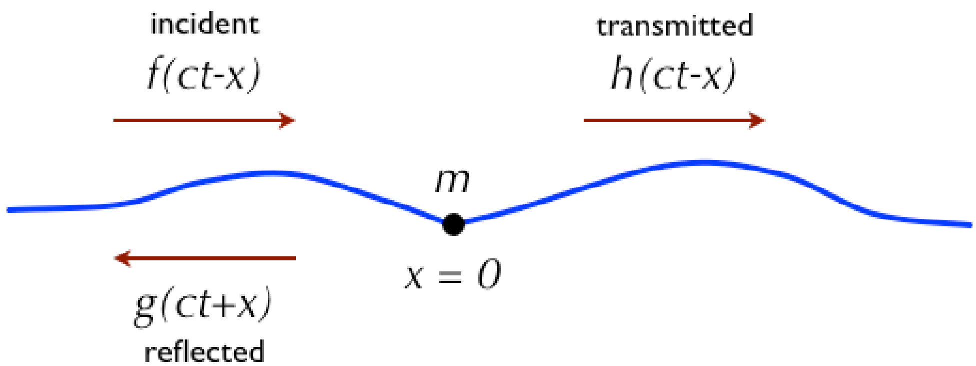

Figure 4.4: Reflection and transmission at an impurity. A point mass m is affixed to an infinite string at $x = 0$.

wave $f(ct - x)$ is incident from the left. The full solution may then be written as a sum of incident, reflected, and transmitted waves:

$$x < 0 \quad : \quad y(x,t) = f(ct - x) + g(ct + x) \tag{4.55}$$

$$x > 0 \quad : \quad y(x,t) = h(ct - x) \ . \tag{4.56}$$

At $x = 0$, we invoke Newton's second Law, $F = ma$:

$$m\,\ddot{y}(0,t) = \tau\, y'(0^+,t) - \tau\, y'(0^-,t) \ . \tag{4.57}$$

Any discontinuity in the derivative $y'(x,t)$ at $x = 0$ results in an acceleration of the point mass. Note that

$$y'(0^-,t) = -f'(ct) + g'(ct) \qquad , \qquad y'(0^+,t) = -h'(ct) \ . \tag{4.58}$$

Further invoking continuity at $x = 0$, *i.e.* $y(0^-,t) = y^(0^+,t)$, we have

$$h(\xi) = f(\xi) + g(\xi) \ , \tag{4.59}$$

and eqn. 4.57 becomes

$$g''(\xi) + \frac{2\tau}{mc^2}\, g'(\xi) = -f''(\xi) \ . \tag{4.60}$$

We solve this equation by Fourier analysis:

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{ik\xi} \qquad , \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\, f(\xi)\, e^{-ik\xi} \ . \tag{4.61}$$

Defining $\kappa \equiv 2\tau/mc^2 = 2\mu/m$, we have

$$\big[-k^2 + i\kappa k \big]\, \hat{g}(k) = k^2\, \hat{f}(k) \ . \tag{4.62}$$

We then have

$$\hat{g}(k) = -\frac{k}{k - i\kappa}\, \hat{f}(k) \equiv r(k)\, \hat{f}(k) \tag{4.63}$$

$$\hat{h}(k) = \frac{-i\kappa}{k - i\kappa}\, \hat{f}(k) \equiv t(k)\, \hat{f}(k) \ , \tag{4.64}$$

where $r(k)$ and $t(k)$ are the reflection and transmission amplitudes, respectively. Note that

$$t(k) = 1 + r(k) \ . \tag{4.65}$$

In real space, we have

$$\begin{aligned} h(\xi) &= \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, \hat{f}(k)\, e^{ik\xi} && (4.66) \\ &= \int\limits_{-\infty}^{\infty} d\xi' \left[\int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')} \right] f(\xi') && (4.67) \\ &\equiv \int\limits_{-\infty}^{\infty} d\xi'\, \mathcal{T}(\xi - \xi')\, f(\xi') \ , && (4.68) \end{aligned}$$

where

$$\mathcal{T}(\xi - \xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, t(k)\, e^{ik(\xi-\xi')} \ , \tag{4.69}$$

is the transmission kernel in real space. For our example with $r(k) = -i\kappa/(k - i\kappa)$, the integral is done easily using the method of contour integration:

$$\mathcal{T}(\xi - \xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{-i\kappa}{k - i\kappa}\, e^{ik(\xi-\xi')} = \kappa\, e^{-\kappa(\xi-\xi')}\, \Theta(\xi - \xi') \ . \tag{4.70}$$

Therefore,

$$h(\xi) = \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, f(\xi') \ , \tag{4.71}$$

and of course $g(\xi) = h(\xi) - f(\xi)$. Note that $m = \infty$ means $\kappa = 0$, in which case $r(k) = -1$ and $t(k) = 0$. Thus we recover the inversion of the pulse shape under reflection found earlier.

For example, let the incident pulse shape be $f(\xi) = b\, \Theta\big(a - |\xi|\big)$. Then

$$\begin{aligned} h(\xi) &= \kappa \int\limits_{-\infty}^{\xi} d\xi'\, e^{-\kappa(\xi-\xi')}\, b\, \Theta(a - \xi')\, \Theta(a + \xi') \\ &= b\, e^{-\kappa\xi} \Big[e^{\kappa \min(a,\xi)} - e^{-\kappa a} \Big]\, \Theta(\xi + a) \ . && (4.72) \end{aligned}$$

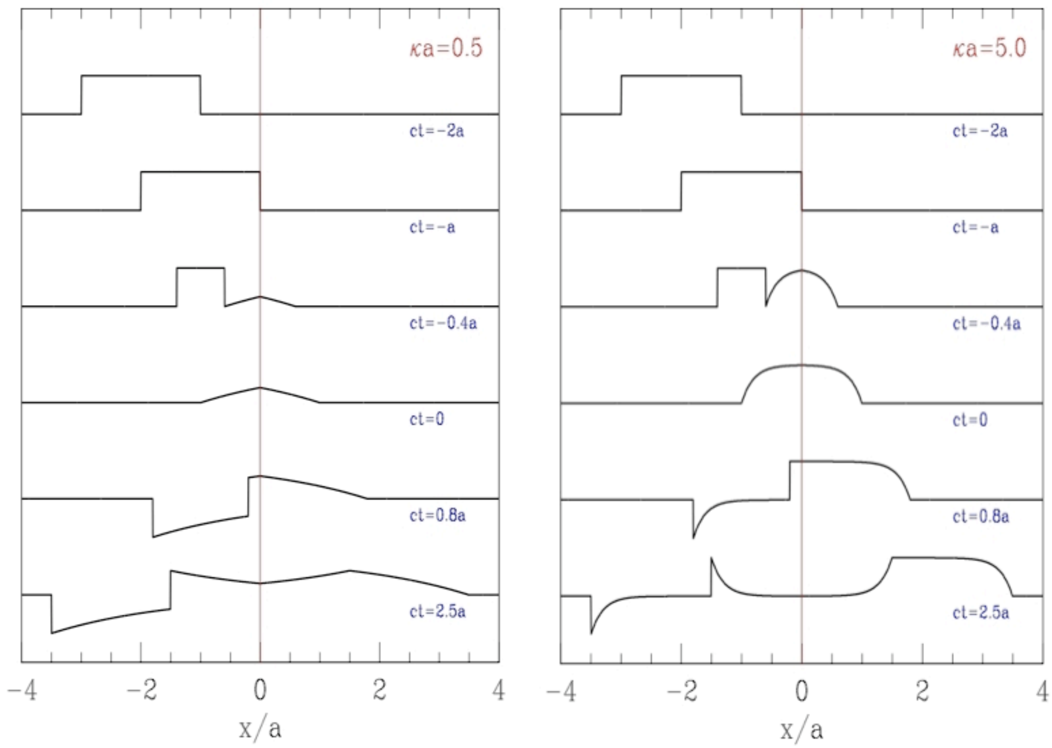

Figure 4.5: Reflection and transmission of a square wave pulse by a point mass at $x = 0$ The configuration of the string is shown for six different times, for $\kappa a = 0.5$ (left panel) and $\kappa a = 5.0$ (right panel). Note that the $\kappa a = 0.5$ case, which corresponds to a large mass $m = 2\mu/\kappa$, results in strong reflection with inversion, and weak transmission. For large κ, corresponding to small mass m, the reflection is weak and the transmission is strong.

Taking cases,

$$h(\xi) = \begin{cases} 0 & \text{if } \xi < -a \\ b\left(1 - e^{-\kappa(a+\xi)}\right) & \text{if } -a < \xi < a \\ 2b\, e^{-\kappa\xi} \sinh(\kappa a) & \text{if } \xi > a \ . \end{cases} \tag{4.73}$$

In Fig. 4.5 we show the reflection and transmission of this square pulse for two different values of κa.

4.2.3 Finite Strings

Suppose $x_a = 0$ and $x_b = L$ are the boundaries of the string, where $y(0,t) = y(L,t) = 0$. Again we write

$$y(x,t) = f(x - ct) + g(x + ct) \ . \tag{4.74}$$

Applying the boundary condition at $x_a = 0$ gives, as earlier,

$$y(x,t) = g(ct + x) - g(ct - x) \ . \tag{4.75}$$

Next, we apply the boundary condition at $x_b = L$, which results in

$$g(ct+L) - g(ct-L) = 0 \quad \Longrightarrow \quad g(\xi) = g(\xi + 2L) \ . \tag{4.76}$$

Thus, $g(\xi)$ is periodic, with period $2L$. Any such function may be written as a Fourier sum,

$$g(\xi) = \sum_{n=1}^{\infty} \left\{ \mathcal{A}_n \cos\left(\frac{n\pi\xi}{L}\right) + \mathcal{B}_n \sin\left(\frac{n\pi\xi}{L}\right) \right\} . \tag{4.77}$$

The full solution for $y(x,t)$ is then

$$\begin{aligned} y(x,t) &= g(ct+x) - g(ct-x) \\ &= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right\} , \end{aligned} \tag{4.78}$$

where $A_n = \sqrt{2\mu L}\,\mathcal{A}_n$ and $B_n = \sqrt{2\mu L}\,\mathcal{B}_n$. This is known as Bernoulli's solution.

We define the functions

$$\psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) . \tag{4.79}$$

We also write

$$k_n \equiv \frac{n\pi x}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad n = 1, 2, 3, \ldots, \infty \ . \tag{4.80}$$

Thus, $\psi_n(x) = \sqrt{2/\mu L}\ \sin(k_n x)$ has $(n+1)$ nodes at $x = jL/n$, for $j \in \{0, \ldots, n\}$. Note that

$$\langle \psi_m | \psi_n \rangle \equiv \int_0^L dx\, \mu\, \psi_m(x)\, \psi_n(x) = \delta_{mn} \ . \tag{4.81}$$

Furthermore, this basis is complete:

$$\mu \sum_{n=1}^{\infty} \psi_n(x)\, \psi_n(x') = \delta(x - x') \ . \tag{4.82}$$

Our general solution is thus equivalent to

$$y(x,0) = \sum_{n=1}^{\infty} A_n\, \psi_n(x) \tag{4.83}$$

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\, B_n\, \psi_n(x) \ . \tag{4.84}$$

The Fourier coefficients $\{A_n, B_n\}$ may be extracted from the initial data using the orthonormality of the basis functions and their associated resolution of unity:

$$A_n = \int_0^L dx\, \mu\, \psi_n(x)\, y(x,0) \tag{4.85}$$

$$B_n = \frac{L}{n\pi c} \int_0^L dx\, \mu\, \psi_n(x)\, \dot{y}(x,0) \ . \tag{4.86}$$

As an example, suppose our initial configuration is a triangle, with

$$y(x,0) = \begin{cases} \frac{2b}{L}\, x & \text{if } 0 \le x \le \frac{1}{2}L \\ \\ \frac{2b}{L}\,(L-x) & \text{if} \frac{1}{2}L \le x \le L \ , \end{cases} \tag{4.87}$$

and $\dot{y}(x,0) = 0$. Then $B_n = 0$ for all n, while

$$\begin{aligned} A_n &= \left(\frac{2\mu}{L}\right)^{1/2} \cdot \frac{2b}{L} \left\{ \int\limits_0^{L/2} \! dx\, x \, \sin\left(\frac{n\pi x}{L}\right) + \int\limits_{L/2}^{L} \! dx\,(L-x)\, \sin\left(\frac{n\pi x}{L}\right) \right\} \\ &= (2\mu L)^{1/2} \cdot \frac{4b}{n^2\pi^2} \, \sin\left(\tfrac{1}{2}n\pi\right) \delta_{n,\text{odd}} \ . \end{aligned} \tag{4.88}$$

Another way to write this is to separately give the results for even and odd coefficients:

$$A_{2k} = 0 \qquad , \qquad A_{2k+1} = \frac{4b}{\pi^2}\,(2\mu L)^{1/2} \cdot \frac{1}{(2k+1)^2} \ . \tag{4.89}$$

Note that each $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is antisymmetric about the midpoint $x = \frac{1}{2}L$, for all k. Since our initial conditions are that $y(x,0)$ is symmetric about $x = \frac{1}{2}L$, none of the even order eigenfunctions can enter into the expansion, precisely as we have found. The D'Alambert solution to this problem is particularly simple and is shown in Fig. 4.6. Note that $g(x) = \frac{1}{2}y(x,0)$ must be extended to the entire real line. We know that $g(x) = g(x+2L)$ is periodic with spatial period $2L$, but how to we extend $g(x)$ from the interval $\big[0,L\big]$ to the interval $\big[-L,0\big]$? To do this, we use $y(x,0) = g(x) - g(-x)$, which says that $g(x)$ must be *antisymmetric*, *i.e.* $g(x) = -g(-x)$. Equivalently, $\dot{y}(x,0) = cg'(x) - cg'(-x) = 0$, which integrates to $g(x) = -g(-x)$.

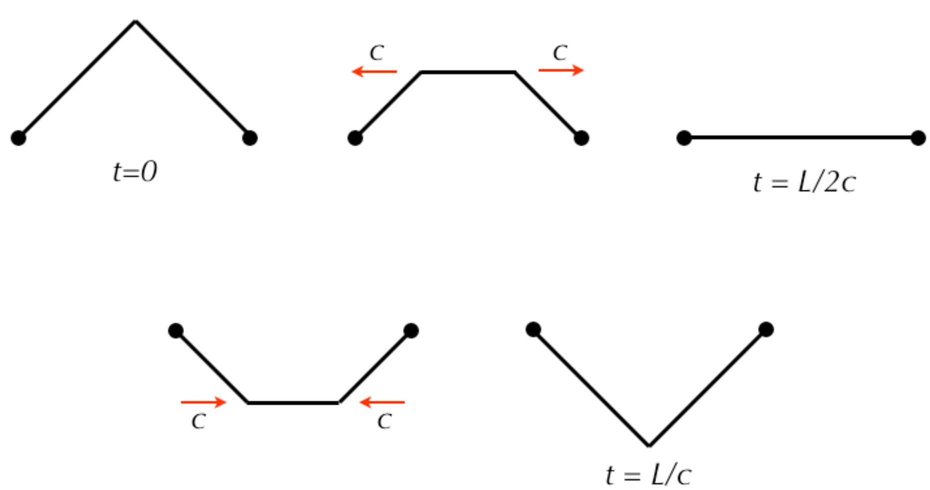

Figure 4.6: Evolution of a string with fixed ends starting from an isosceles triangle shape.

4.2.4 Sturm-Liouville Theory

Consider the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\,\mu(x)\,\dot{y}^2 - \tfrac{1}{2}\,\tau(x)\,y'^2 - \tfrac{1}{2}\,v(x)\,y^2\ . \tag{4.90}$$

The last term is new and has the physical interpretation of a harmonic potential which attracts the string to the line $y = 0$. The Euler-Lagrange equations are then

$$-\frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = -\mu(x)\,\frac{\partial^2 y}{\partial t^2}\ . \tag{4.91}$$

This equation is invariant under time translation. Thus, if $y(x,t)$ is a solution, then so is $y(x, t+t_0)$, for any t_0. This means that the solutions can be chosen to be eigenstates of the operator ∂_t, which is to say $y(x,t) = \psi(x)\,e^{-i\omega t}$. Because the coefficients are real, both y and y^* are solutions, and taking linear combinations we have

$$y(x,t) = \psi(x)\,\cos(\omega t + \phi)\ . \tag{4.92}$$

Plugging this into eqn. 4.91, we obtain

$$-\frac{d}{dx}\Big[\tau(x)\,\psi'(x)\Big] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x)\ . \tag{4.93}$$

This is the Sturm-Liouville equation. There are four types of boundary conditions that we shall consider:

1. Fixed endpoint: $\psi(x) = 0$, where $x = x_{a,b}$.
2. Natural: $\tau(x)\,\psi'(x) = 0$, where $x = x_{a,b}$.
3. Periodic: $\psi(x) = \psi(x+L)$, where $L = x_b - x_a$.
4. Mixed homogeneous: $\alpha\,\psi(x) + \beta\,\psi'(x) = 0$, where $x = x_{a,b}$.

The Sturm-Liouville equation is an eigenvalue equation. The eigenfunctions $\{\psi_n(x)\}$ satisfy

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_n'(x)\Big] + v(x)\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)\ . \tag{4.94}$$

Now suppose we a second solution $\psi_m(x)$, satisfying

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_m'(x)\Big] + v(x)\,\psi_m(x) = \omega_m^2\,\mu(x)\,\psi_m(x)\ . \tag{4.95}$$

Now multiply $(4.94)^*$ by $\psi_m(x)$ and (4.95) by $\psi_n^*(x)$ and subtract, yielding

$$\psi_n^*\,\frac{d}{dx}\Big[\tau\,\psi_m'\Big] - \psi_m\,\frac{d}{dx}\Big[\tau\,{\psi'}_n^*\Big] = \big({\omega_n^*}^2 - \omega_m^2\big)\,\mu\,\psi_m\,\psi_n^* \tag{4.96}$$

$$= \frac{d}{dx}\Big[\tau\,\psi_n^*\,\psi_m' - \tau\,\psi_m\,{\psi'}_n^*\Big]\ . \tag{4.97}$$

We integrate this equation over the length of the string, to get

$$\big(\omega_n^{*2}-\omega_m^2\big)\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi_n^*(x)\,\psi_m(x)=\Big[\tau(x)\,\psi_n^*(x)\,\psi'_m(x)-\tau(x)\,\psi_m(x)\,\psi'^*_n(x)\Big]_{x=x_b}^{x=x_a}$$
$$=0\ . \tag{4.98}$$

The RHS vanishes for any of the four types of boundary conditions articulated above.

Thus, we have

$$\big(\omega_n^{*2}-\omega_m^2\big)\,\big\langle\psi_n\big|\psi_m\big\rangle=0\ , \tag{4.99}$$

where the inner product is defined as

$$\big\langle\psi\big|\phi\big\rangle\equiv\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi^*(x)\,\phi(x)\ . \tag{4.100}$$

Note that the distribution $\mu(x)$ is non-negative definite. Setting $m=n$, we have $\big\langle\psi_n\big|\psi_n\big\rangle\geq 0$, and hence $\omega_n^{*2}=\omega_n^2$, which says that $\omega_n^2\in\mathbf{R}$. When $\omega_m^2\neq\omega_n^2$, the eigenfunctions are orthogonal with respect to the above inner product. In the case of degeneracies, we may invoke the Gram-Schmidt procedure, which orthogonalizes the eigenfunctions within a given degenerate subspace. Since the Sturm-Liouville equation is linear, we may normalize the eigenfunctions, taking

$$\big\langle\psi_m\big|\psi_n\big\rangle=\delta_{mn}. \tag{4.101}$$

Finally, since the coefficients in the Sturm-Liouville equation are all real, we can and henceforth do choose the eigenfunctions themselves to be real.

4.2.5 Green's Functions

Suppose we add a forcing term,

$$\mu(x)\,\frac{\partial^2 y}{\partial t^2}-\frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg]+v(x)\,y=\mathrm{Re}\Big[\mu(x)\,f(x)\,e^{-i\omega t}\Big]\ . \tag{4.102}$$

We write the solution as

$$y(x,t)=\mathrm{Re}\Big[y(x)\,e^{-i\omega t}\Big]\ , \tag{4.103}$$

where

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dy(x)}{dx}\bigg]+v(x)\,y(x)-\omega^2\,\mu(x)\,y(x)=\mu(x)\,f(x)\ , \tag{4.104}$$

or

$$\Big[K-\omega^2\mu(x)\Big]\,y(x)=\mu(x)\,f(x)\ , \tag{4.105}$$

where K is a differential operator,

$$K\equiv-\frac{d}{dx}\,\tau(x)\,\frac{d}{dx}+v(x)\ . \tag{4.106}$$

Note that the eigenfunctions of K are the $\{\psi_n(x)\}$:

$$K\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x)\ . \tag{4.107}$$

The formal solution to equation 4.105 is then

$$y(x) = \Big[K - \omega^2\mu\Big]^{-1}_{x,x'}\,\mu(x')\,f(x') \tag{4.108}$$

$$= \int\limits_{x_a}^{x_b} dx'\,\mu(x')\,G_\omega(x,x')\,f(x'). \tag{4.109}$$

What do we mean by the term in brackets? If we define the *Green's function*

$$G_\omega(x,x') \equiv \Big[K - \omega^2\mu\Big]^{-1}_{x,x'}\ , \tag{4.110}$$

what this means is

$$\Big[K - \omega^2\mu(x)\Big]\,G_\omega(x,x') = \delta(x-x')\ . \tag{4.111}$$

Note that the Green's function may be expanded in terms of the (real) eigenfunctions, as

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2}\ , \tag{4.112}$$

which follows from completeness of the eigenfunctions:

$$\mu(x)\sum_{n=1}^{\infty}\psi_n(x)\,\psi_n(x') = \delta(x-x')\ . \tag{4.113}$$

The expansion in eqn. 4.112 is formally exact, but difficult to implement, since it requires summing over an infinite set of eigenfunctions. It is more practical to construct the Green's function from solutions to the homogeneous Sturm Liouville equation, as follows. When $x \neq x'$, we have that $(K - \omega^2\mu)\,G_\omega(x,x') = 0$, which is a homogeneous ODE of degree two. Consider first the interval $x \in [x_a, x']$. A second order homogeneous ODE has two solutions, and further invoking the boundary condition at $x = x_a$, there is a unique solution, up to a multiplicative constant. Call this solution $y_1(x)$. Next, consider the interval $x \in [x', x_b]$. Once again, there is a unique solution to the homogeneous Sturm-Liouville equation, up to a multiplicative constant, which satisfies the boundary condition at $x = x_b$. Call this solution $y_2(x)$. We then can write

$$G_\omega(x,x') = \begin{cases} A(x')\,y_1(x) & \text{if } x_a \le x < x' \\ \\ B(x')\,y_2(x) & \text{if } x' < x \le x_b\ . \end{cases} \tag{4.114}$$

Here, $A(x')$ and $B(x')$ are undetermined functions. We now invoke the inhomogeneous Sturm-Liouville equation,

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dG_\omega(x,x')}{dx}\bigg] + v(x)\,G_\omega(x,x') - \omega^2\mu(x)\,,G_\omega(x,x') = \delta(x-x')\ . \tag{4.115}$$

We integrate this from $x = x' - \epsilon$ to $x = x' + \epsilon$, where ϵ is a positive infinitesimal. This yields

$$\tau(x')\Big[A(x')\, y_1'(x') - B(x')\, y_2'(x')\Big] = 1 \ . \tag{4.116}$$

Continuity of $G_\omega(x, x')$ itself demands

$$A(x')\, y_1(x') = B(x')\, y_2(x') \ . \tag{4.117}$$

Solving these two equations for $A(x')$ and $B(x')$, we obtain

$$A(x') = -\frac{y_2(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \qquad , \qquad B(x') = -\frac{y_1(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \ , \tag{4.118}$$

where $\mathcal{W}_{y_1,y_2}(x)$ is the *Wronskian*

$$\begin{aligned} \mathcal{W}_{y_1,y_2}(x) &= \det \begin{pmatrix} y_1(x) & y_2(x) \\ \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) \ . \end{aligned} \tag{4.119}$$

Now it is easy to show that $\mathcal{W}_{y_1,y_2}(x)\, \tau(x) = \mathcal{W}\, \tau$ is a constant. This follows from the fact that

$$\begin{aligned} 0 &= y_2\, K\, y_1 - y_2\, K\, y_1 \\ &= \frac{d}{dx}\bigg\{\tau(x)\Big[y_1\, y_2' - y_2\, y_1'\Big]\bigg\} \ . \end{aligned} \tag{4.120}$$

Thus, we have

$$G_\omega(x, x') = \begin{cases} -y_1(x)\, y_2(x')/\mathcal{W} & \text{if } x_a \le x < x' \\ \\ -y_1(x')\, y_2(x)/\mathcal{W} & \text{if } x' < x \le x_b \ , \end{cases} \tag{4.121}$$

or, in compact form,

$$G_\omega(x, x') = -\frac{y_1(x_<)\, y_2(x_>)}{\mathcal{W}\, \tau} \ , \tag{4.122}$$

where $x_< = \min(x, x')$ and $x_> = \max(x, x')$.

As an example, consider a uniform string (*i.e.* μ and τ constant, $v = 0$) with fixed endpoints at $x_a = 0$ and $x_b = L$. The normalized eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{\mu L}}\, \sin\!\left(\frac{n\pi x}{L}\right) \ , \tag{4.123}$$

and the eigenvalues are $\omega_n = n\pi c/L$. The Green's function is

$$G_\omega(x, x') = \frac{2}{\mu L} \sum_{n=1}^\infty \frac{\sin(n\pi x/L)\, \sin(n\pi x'/L)}{(n\pi c/L)^2 - \omega^2} \ . \tag{4.124}$$

Now construct the homogeneous solutions:

$$(K-\omega^2\mu)\,y_1=0 \quad , \quad y_1(0)=0 \qquad \Longrightarrow \qquad y_1(x)=\sin\bigg(\frac{\omega x}{c}\bigg) \tag{4.125}$$

$$(K-\omega^2\mu)\,y_2=0 \quad , \quad y_2(L)=0 \qquad \Longrightarrow \qquad y_2(x)=\sin\bigg(\frac{\omega(L-x)}{c}\bigg)\ . \tag{4.126}$$

The Wronskian is

$$\mathcal{W}=y_1\,y_2'-y_2\,y_1'=-\frac{\omega}{c}\,\sin\bigg(\frac{\omega L}{c}\bigg)\ . \tag{4.127}$$

Therefore, the Green's function is

$$G_\omega(x,x')=\frac{\sin\big(\omega x_</c\big)\,\sin\big(\omega(L-x_>)/c\big)}{(\omega\tau/c)\,\sin(\omega L/c)}\ . \tag{4.128}$$

4.2.6 Perturbation Theory for the Green's Function

Suppose we have solved for the Green's function for the linear operator K_0 and mass density $\mu_0(x)$. *I.e.* we have

$$\big(K_0-\omega^2\mu_0(x)\big)\,G^0_\omega(x,x')=\delta(x-x')\ . \tag{4.129}$$

We now imagine perturbing $\tau_0\to\tau_0+\lambda\tau_1$, $v_0\to v_0+\lambda v_2$, $\mu_0\to\mu_0+\lambda\mu_1$. What is the new Green's function $G_\omega(x,x')$? We must solve

$$\big(L_0+\lambda L_1\big)\,G_\omega(x,x')=\delta(x-x')\ , \tag{4.130}$$

where

$$L^0_\omega\equiv K_0-\omega^2\,\mu_0 \tag{4.131}$$

$$L^1_\omega\equiv K_1-\omega^2\,\mu_1\ . \tag{4.132}$$

Dropping the ω subscript for simplicity, the full Green's function is then given by

$$\begin{aligned}
G_\omega&=\Big[L^0_\omega+\lambda L^1_\omega\Big]^{-1}\\
&=\Big[\,\big(G^0_\omega\big)^{-1}+\lambda L^1_\omega\,\Big]^{-1}\\
&=\Big[\,1+\lambda\,G^0_\omega\,L^1_\omega\Big]^{-1}\,G^0_\omega\\
&=G^0_\omega-\lambda\,G^0_\omega\,L^1_\omega\,G^0_\omega+\lambda^2\,G^0_\omega\,L^1_\omega\,G^0_\omega\,L^1_\omega\,G^0_\omega+\ldots\ .
\end{aligned} \tag{4.133}$$

The 'matrix multiplication' is of course a convolution, *i.e.*

$$G_\omega(x,x')=G^0_\omega(x,x')-\lambda\int\limits_{x_a}^{x_b}\!dx_1\,G^0_\omega(x,x_1)\,L^1_\omega\big(x_1,\tfrac{d}{dx_1}\big)\,G^0_\omega(x_1,x')+\ldots\ . \tag{4.134}$$

Each term in the perturbation expansion of eqn. 4.133 may be represented by a diagram, as depicted in Fig. 4.7.

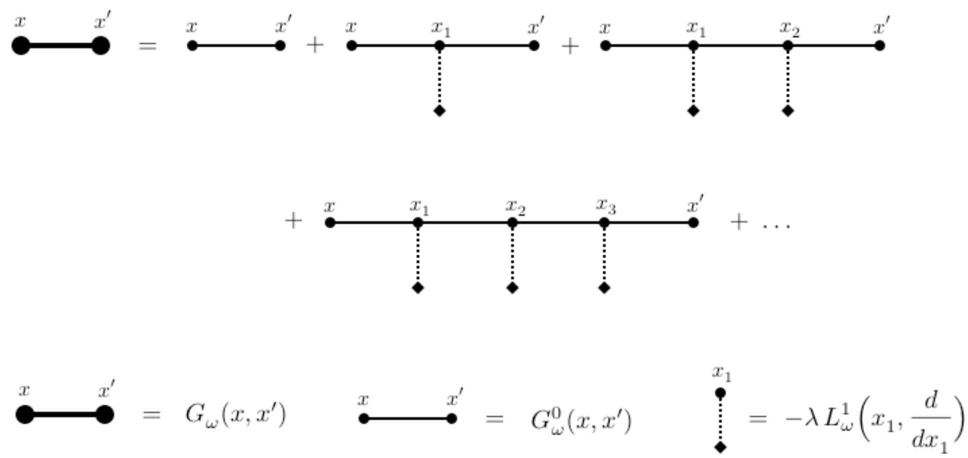

Figure 4.7: Diagrammatic representation of the perturbation expansion in eqn. 4.133..

As an example, consider a string with $x_a = 0$ and $x_b = L$ with a mass point m affixed at the point $x = d$. Thus, $\mu_1(x) = m\,\delta(x-d)$, and $L^1_\omega = -m\omega^2\,\delta(x-d)$, with $\lambda = 1$. The perturbation expansion gives

$$\begin{aligned} G_\omega(x,x') &= G^0_\omega(x,x') + m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x') + m^2\omega^4\, G^0_\omega(x,d)\, G^0_\omega(d,d)\, G^0_\omega(d,x') + \ldots \\ &= G^0_\omega(x,x') + \frac{m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x')}{1 - m\omega^2\, G^0_\omega(d,d)} \ . \end{aligned} \tag{4.135}$$

Note that the eigenfunction expansion,

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \ , \tag{4.136}$$

says that the exact eigenfrequencies are poles of $G_\omega(x,x')$, and furthermore the residue at each pole is

$$\operatorname*{Res}_{\omega=\omega_n} G_\omega(x,x') = -\frac{1}{2\omega_n}\,\psi_n(x)\,\psi_n(x') \ . \tag{4.137}$$

According to eqn. 4.135, the poles of $G_\omega(x,x')$ are located at solutions to[4]

$$m\omega^2\, G^0_\omega(d,d) = 1 \ . \tag{4.138}$$

For simplicity let us set $d = \frac{1}{2}L$, so the mass point is in the middle of the string. Then according to eqn. 4.128,

$$\begin{aligned} G^0_\omega\big(\tfrac{1}{2}L, \tfrac{1}{2}L\big) &= \frac{\sin^2(\omega L/2c)}{(\omega\tau/c)\,\sin(\omega L/c)} \\ &= \frac{c}{2\omega\tau}\,\tan\left(\frac{\omega L}{2c}\right) \ . \end{aligned} \tag{4.139}$$

[4]Note in particular that there is no longer any divergence at the location of the original poles of $G^0_\omega(x,x')$. These poles are cancelled.

The eigenvalue equation is therefore

$$\tan\left(\frac{\omega L}{2c}\right) = \frac{2\tau}{m\omega c} \ , \tag{4.140}$$

which can be manipulated to yield

$$\frac{m}{M}\,\lambda = \operatorname{ctn}\lambda \ , \tag{4.141}$$

where $\lambda = \omega L/2c$ and $M = \mu L$ is the total mass of the string. When $m = 0$, the LHS vanishes, and the roots lie at $\lambda = (n + \frac{1}{2})\pi$, which gives $\omega = \omega_{2n+1}$. Why don't we see the poles at the even mode eigenfrequencies ω_{2n}? The answer is that these poles are present in the Green's function. They do not cancel for $d = \frac{1}{2}L$ because the perturbation does not couple to the even modes, which all have $\psi_{2n}(\frac{1}{2}L) = 0$. The case of general d may be instructive in this regard. One finds the eigenvalue equation

$$\frac{\sin(2\lambda)}{2\lambda\,\sin\big(2\epsilon\lambda\big)\sin\big(2(1-\epsilon)\lambda\big)} = \frac{m}{M} \ , \tag{4.142}$$

where $\epsilon = d/L$. Now setting $m = 0$ we recover $2\lambda = n\pi$, which says $\omega = \omega_n$, and all the modes are recovered.

4.2.7 Perturbation Theory for Eigenvalues and Eigenfunctions

We wish to solve

$$\big(K_0 + \lambda K_1\big)\,\psi = \omega^2\big(\mu_0 + \lambda\mu_1\big)\,\psi \ , \tag{4.143}$$

which is equivalent to

$$L^0_\omega\,\psi = -\lambda L^1_\omega\,\psi \ . \tag{4.144}$$

Multiplying by $\big(L^0_\omega\big)^{-1} = G^0_\omega$ on the left, we have

$$\psi(x) = -\lambda\int\limits_{x_a}^{x_b}\!dx'\,G_\omega(x,x')\,L^1_\omega\,\psi(x') \tag{4.145}$$

$$= \lambda\sum_{m=1}^{\infty}\frac{\psi_m(x)}{\omega^2-\omega_m^2}\int\limits_{x_a}^{x_b}\!dx'\,\psi_m(x')\,L^1_\omega\,\psi(x') \ . \tag{4.146}$$

We are free to choose any normalization we like for $\psi(x)$. We choose

$$\big\langle\psi\big|\psi_n\big\rangle = \int\limits_{x_a}^{x_b}\!dx\,\mu_0(x)\,\psi_n(x)\,\psi(x) = 1 \ , \tag{4.147}$$

which entails

$$\omega^2-\omega_n^2 = \lambda\int\limits_{x_a}^{x_b}\!dx\,\psi_n(x)\,L^1_\omega\,\psi(x) \tag{4.148}$$

as well as

$$\psi(x) = \psi_n(x) + \lambda \sum_{\substack{k \\ (k \neq n)}} \frac{\psi_k(x)}{\omega^2 - \omega_k^2} \int\limits_{x_a}^{x_b} dx' \, \psi_k(x') \, L^1_\omega \, \psi(x') \ . \tag{4.149}$$

By expanding ψ and ω^2 in powers of λ, we can develop an order by order perturbation series.

To lowest order, we have

$$\omega^2 = \omega_n^2 + \lambda \int\limits_{x_a}^{x_b} dx \, \psi_n(x) \, L^1_{\omega_n} \psi_n(x) \ . \tag{4.150}$$

For the case $L^1_\omega = -m\,\omega^2\,\delta(x-d)$, we have

$$\begin{aligned} \frac{\delta\omega_n}{\omega_n} &= -\tfrac{1}{2} m \big[\psi_n(d)\big]^2 \\ &= -\frac{m}{M} \, \sin^2\!\Big(\frac{n\pi d}{L}\Big) \ . \end{aligned} \tag{4.151}$$

For $d = \frac{1}{2}L$, only the odd n modes are affected, as the even n modes have a node at $x = \frac{1}{2}L$.

Carried out to second order, one obtains for the eigenvalues,

$$\begin{aligned} \omega^2 = \omega_n^2 &+ \lambda \int\limits_{x_a}^{x_b} dx \, \psi_n(x) \, L^1_{\omega_n} \, \psi_n(x) \\ &+ \lambda^2 \sum_{\substack{k \\ (k \neq n)}} \frac{\Big| \int_{x_a}^{x_b} dx \, \psi_k(x) \, L^1_{\omega_n} \, \psi_n(x) \Big|^2}{\omega_n^2 - \omega_k^2} + \mathcal{O}(\lambda^3) \\ &- \lambda^2 \int\limits_{x_a}^{x_b} dx \, \psi_n(x) \, L^1_{\omega_n} \, \psi_n(x) \cdot \int\limits_{x_a}^{x_b} dx' \, \mu_1(x') \, \big[\psi_n(x)\big]^2 + \mathcal{O}(\lambda^3) \ . \end{aligned} \tag{4.152}$$

4.2.8 Variational Method

Consider the functional

$$\omega^2\big[\psi(x)\big] = \frac{\frac{1}{2} \int\limits_{x_a}^{x_b} dx \, \Big\{ \tau(x) \, \psi'^2(x) + v(x) \, \psi^2(x) \Big\}}{\frac{1}{2} \int\limits_{x_a}^{x_b} dx \, \mu(x) \, \psi^2(x)} \equiv \frac{\mathcal{N}}{\mathcal{D}} \ . \tag{4.153}$$

The variation is

$$\begin{aligned} \delta\omega^2 &= \frac{\delta\mathcal{N}}{\mathcal{D}} - \frac{\mathcal{N}\,\delta\mathcal{D}}{\mathcal{D}^2} \\ &= \frac{\delta\mathcal{N} - \omega^2\,\delta\mathcal{D}}{\mathcal{D}} \ . \end{aligned} \tag{4.154}$$

Thus,

$$\delta\omega^2 = 0 \qquad \Longrightarrow \qquad \delta\mathcal{N} = \omega^2\,\delta\mathcal{D} \ , \tag{4.155}$$

which says

$$-\frac{d}{dx}\bigg[\,\tau(x)\,\frac{d\psi(x)}{dx}\,\bigg] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \ , \tag{4.156}$$

which is the Sturm-Lioiuville equation. In obtaining this equation, we have dropped a boundary term, which is correct provided

$$\Big[\,\tau(x)\,\psi'(x)\,\psi(x)\,\Big]_{x=x_a}^{x=x_b} = 0 \ . \tag{4.157}$$

This condition is satisfied for any of the first three classes of boundary conditions: $\psi = 0$ (fixed endpoint), $\tau\,\psi' = 0$ (natural), or $\psi(x_a) = \psi(x_b)$, $\psi'(x_a) = \psi'(x_b)$ (periodic). For the fourth class of boundary conditions, $\alpha\psi + \beta\psi' = 0$ (mixed homogeneous), the Sturm-Liouville equation may still be derived, provided one uses a slightly different functional,

$$\omega^2\big[\psi(x)\big] = \frac{\widetilde{\mathcal{N}}}{\mathcal{D}} \qquad \text{with} \qquad \widetilde{\mathcal{N}} = \mathcal{N} + \frac{\alpha}{2\beta}\Big[\tau\big(x_b\big)\,\psi^2\big(x_b\big) - \tau\big(x_a\big)\,\psi^2\big(x_a\big)\Big] \ , \tag{4.158}$$

since then

$$\begin{aligned} \delta\widetilde{\mathcal{N}} - \widetilde{\mathcal{N}}\,\delta D = \int\limits_{x_a}^{x_b}\!dx\,\bigg\{ -\frac{d}{dx}\bigg[\,\tau(x)\,\frac{d\psi(x)}{dx}\,\bigg] + v(x)\,\psi(x) - \omega^2\mu(x)\,\psi(x)\bigg\}\,\delta\psi(x) \\ + \Bigg[\tau(x)\bigg(\psi'(x) + \frac{\alpha}{\beta}\,\psi(x)\bigg)\,\delta\psi(x)\Bigg]_{x=x_a}^{x=x_b} \ , \end{aligned} \tag{4.159}$$

and the last term vanishes as a result of the boundary conditions.

For all four classes of boundary conditions we may write

$$\omega^2\big[\psi(x)\big] = \frac{\int\limits_{x_a}^{x_b}\!dx\,\psi(x)\,\overbrace{\Big[-\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x)\Big]}^{K}\,\psi(x)}{\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi^2(x)} \tag{4.160}$$

If we expand $\psi(x)$ in the basis of eigenfunctions of the Sturm-Liouville operator K,

$$\psi(x) = \sum_{n=1}^{\infty} \mathcal{C}_n\,\psi_n(x) \ , \tag{4.161}$$

we obtain

$$\omega^2\big[\psi(x)\big] = \omega^2(\mathcal{C}_1,\ldots,\mathcal{C}_\infty) = \frac{\sum_{j=1}^{\infty}|\mathcal{C}_j|^2\,\omega_j^2}{\sum_{k=1}^{\infty}|\mathcal{C}_k|^2} \ . \tag{4.162}$$

If $\omega_1^2 \le \omega_2^2 \le \ldots$, then we see that $\omega^2 \ge \omega_1^2$, so an arbitrary function $\psi(x)$ will always yield an upper bound to the lowest eigenvalue.

As an example, consider a violin string ($v = 0$) with a mass m affixed in the center. We write $\mu(x) = \mu + m\,\delta(x - \frac{1}{2}L)$, hence

$$\omega^2\big[\psi(x)\big] = \frac{\tau \int\limits_0^L dx\, \psi'^2(x)}{m\,\psi^2(\frac{1}{2}L) + \mu \int\limits_0^L dx\, \psi^2(x)} \tag{4.163}$$

Now consider a trial function

$$\psi(x) = \begin{cases} A\,x^\alpha & \text{if} \quad 0 \le x \le \frac{1}{2}L \\ \\ A\,(L-x)^\alpha & \text{if} \quad \frac{1}{2}L \le x \le L\ . \end{cases} \tag{4.164}$$

The functional $\omega^2\big[\psi(x)\big]$ now becomes an ordinary function of the trial parameter α, with

$$\omega^2(\alpha) = \frac{2\tau \int_0^{L/2} dx\, \alpha^2\, x^{2\alpha-2}}{m\,\big(\frac{1}{2}L\big)^{2\alpha} + 2\mu \int\limits_0^{L/2} dx\, s^{2\alpha}} = \left(\frac{2c}{L}\right)^2 \cdot \frac{\alpha^2(2\alpha+1)}{(2\alpha-1)\big[1+(2\alpha+1)\frac{m}{M}\big]}\ , \tag{4.165}$$

where $M = \mu L$ is the mass of the string alone. We minimize $\omega^2(\alpha)$ to obtain the optimal solution of this form:

$$\frac{d\omega^2}{d\alpha} = 0 \qquad \Longrightarrow \qquad 4\alpha^2 - 2\alpha - 1 + (2\alpha+1)^2\,(\alpha-1)\,\frac{m}{M} = 0\ . \tag{4.166}$$

For $m/M \to 0$, we obtain $\alpha = \frac{1}{4}\big(1+\sqrt{5}\big) \approx 0.809$. The variational estimate for the eigenvalue is then 6.00% larger than the exact answer $\omega_1^0 = \pi c/L$. In the opposite limit, $m/M \to \infty$, the inertia of the string may be neglected. The normal mode is then piecewise linear, in the shape of an isosceles triangle with base L and height y. The equation of motion is then $m\ddot{y} = -2\tau \cdot (y/\frac{1}{2}L)$, assuming $|y/L| \ll 1$. Thus, $\omega_1 = (2c/L)\sqrt{M/m}$. This is reproduced exactly by the variational solution, for which $\alpha \to 1$ as $m/M \to \infty$.

4.2.9 Energy Density and Energy Current

The Hamiltonian density for a string is

$$\mathcal{H} = \varPi\,\dot{y} - \mathcal{L}\ , \tag{4.167}$$

where

$$\varPi = \frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu\,\dot{y} \tag{4.168}$$

is the momentum density. Thus,

$$\mathcal{H} = \frac{\varPi^2}{2\mu} + \tfrac{1}{2}\tau\,y'^2\ . \tag{4.169}$$

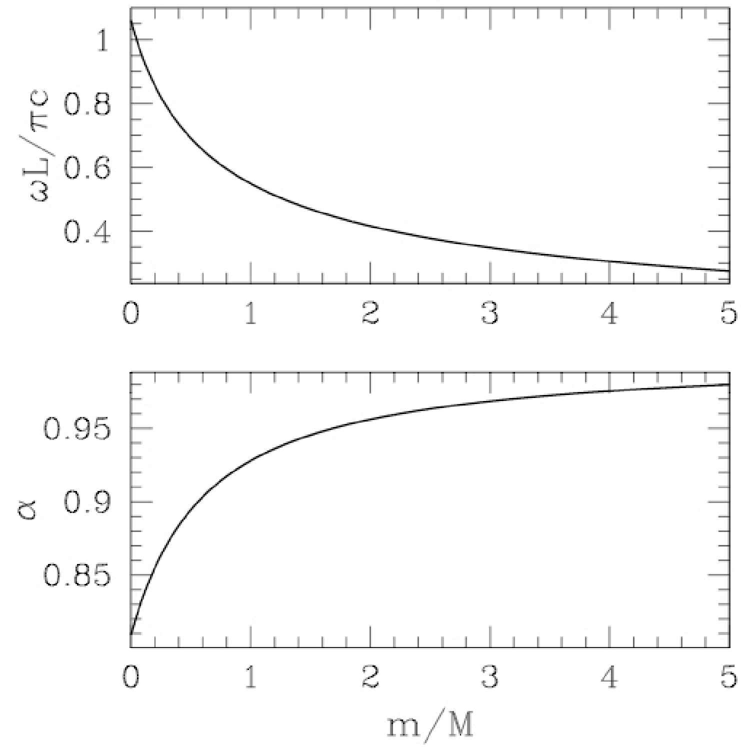

Figure 4.8: One-parameter variational solution for a string with a mass m affixed at $x = \frac{1}{2}L$.

Expressed in terms of $\dot{y}$ rather than Π, this is the energy density $\mathcal{E}$,

$$\mathcal{E} = \tfrac{1}{2}\mu\,\dot{y}^2 + \tfrac{1}{2}\tau\,y'^2 \ . \tag{4.170}$$

We now evaluate $\dot{\mathcal{E}}$ for a solution to the equations of motion:

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial t} &= \mu\,\frac{\partial y}{\partial t}\,\frac{\partial^2 y}{\partial t^2} + \tau\,\frac{\partial y}{\partial x}\,\frac{\partial^2 y}{\partial t\,\partial x} \\ &= \tau\,\frac{\partial y}{\partial t}\,\frac{\partial}{\partial x}\bigg(\tau\,\frac{\partial y}{\partial x}\bigg) + \tau\,\frac{\partial y}{\partial x}\,\frac{\partial^2 y}{\partial t\,\partial x} \\ &= \frac{\partial}{\partial x}\bigg[\tau\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t}\bigg] \equiv -\frac{\partial j_{\mathcal{E}}}{\partial x} \ ,\end{aligned} \tag{4.171}$$

where the *energy current density* (or energy flux) is

$$j_{\mathcal{E}} = -\tau\,\frac{\partial y}{\partial x}\,\frac{\partial y}{\partial t} \ . \tag{4.172}$$

We therefore have that solutions of the equation of motion also obey the energy *continuity equation*

$$\frac{\partial\mathcal{E}}{\partial t} + \frac{\partial j_{\mathcal{E}}}{\partial x} = 0 \ . \tag{4.173}$$

Let us integrate the above equation between points x_1 and x_2. We obtain

$$\frac{\partial}{\partial t} \int\limits_{x_1}^{x_2} dx\, \mathcal{E}(x,t) = -\int\limits_{x_1}^{x_2} dx\, \frac{\partial j_{\mathcal{E}}(x,t)}{\partial x} = j_{\mathcal{E}}(x_1,t) - j_{\mathcal{E}}(x_2,t) \ , \tag{4.174}$$

which says that the time rate of change of the energy contained in the interval $\big[x_1, x_2\big]$ is equal to the difference between the entering and exiting energy flux.

When $\tau(x) = \tau$ and $\mu(x) = \mu$, we have

$$y(x,t) = f(x - ct) + g(x + ct) \tag{4.175}$$

and we find

$$j_{\mathcal{E}}(x,t) = c\tau\big[f'(x - ct)\big]^2 - c\tau\big[g'(x + ct)\big]^2 \ , \tag{4.176}$$

which is a sum of rightward and leftward directed energy fluxes.

4.3 Continua in higher dimensions

In higher dimensions, we generalize the operator K as follows:

$$K = -\frac{\partial}{\partial x^\alpha}\, \tau_{\alpha\beta}(\boldsymbol{x})\, \frac{\partial}{\partial x^\beta} + v(\boldsymbol{x}) \ . \tag{4.177}$$

The eigenvalue equation is again

$$K\psi(\boldsymbol{x}) = \omega^2\, \mu(\boldsymbol{x})\, \psi(\boldsymbol{x}) \ , \tag{4.178}$$

and the Green's function again satisfies

$$\Big[K - \omega^2\, \mu(\boldsymbol{x})\Big] G_\omega(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \ , \tag{4.179}$$

and has the eigenfunction expansion,

$$G_\omega(\boldsymbol{x}, \boldsymbol{x}') = \sum_{n=1}^{\infty} \frac{\psi_n(\boldsymbol{x})\, \psi_n(\boldsymbol{x}')}{\omega_n^2 - \omega^2} \ . \tag{4.180}$$

The eigenfunctions form a complete and orthonormal basis:

$$\mu(\boldsymbol{x}) \sum_{n=1}^{\infty} \psi_n(\boldsymbol{x})\, \psi_n(\boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{4.181}$$

$$\int\limits_{\Omega} d\boldsymbol{x}\, \mu(\boldsymbol{x})\, \psi_m(\boldsymbol{x})\, \psi_n(\boldsymbol{x}) = \delta_{mn} \ , \tag{4.182}$$

where Ω is the region of space in which the continuous medium exists. For purposes of simplicity, we consider here fixed boundary conditions $u(\boldsymbol{x},t)\big|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of Ω. The general solution to the wave equation

$$\bigg[\mu(\boldsymbol{x})\, \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^\alpha}\, \tau_{\alpha\beta}(\boldsymbol{x})\, \frac{\partial}{\partial x^\beta} + v(\boldsymbol{x})\bigg]\, u(\boldsymbol{x},t) = 0 \tag{4.183}$$

is

$$u(\boldsymbol{x},t)=\sum_{n=1}^{\infty}\mathcal{C}_n\,\psi_n(\boldsymbol{x})\,\cos(\omega_n\,t+\delta_n)\ . \tag{4.184}$$

The variational approach generalizes as well. We define

$$\mathcal{N}\big[\psi(\boldsymbol{x})\big]=\int\limits_{\Omega}\!d\boldsymbol{x}\left[\tau_{\alpha\beta}\,\frac{\partial\psi}{\partial x^\alpha}\,\frac{\partial\psi}{\partial x^\beta}+v\,\psi^2\right] \tag{4.185}$$

$$\mathcal{D}\big[\psi(\boldsymbol{x})\big]=\int\limits_{\Omega}\!d\boldsymbol{x}\;\mu\,\psi^2\ , \tag{4.186}$$

and

$$\omega^2\big[\psi(\boldsymbol{x})\big]=\frac{\mathcal{N}\big[\psi(\boldsymbol{x})\big]}{\mathcal{D}\big[\psi(\boldsymbol{x})\big]}\ . \tag{4.187}$$

Setting the variation $\delta\omega^2=0$ recovers the eigenvalue equation $K\psi=\omega^2\mu\,\psi$.

4.4 Membranes

Consider a surface where the height z is a function of the lateral coordinates x and y:

$$z=u(x,y)\ . \tag{4.188}$$

The equation of the surface is then

$$F(x,y,z)=z-u(x,y)=0\ . \tag{4.189}$$

Let the differential element of surface area be dS. The projection of this element onto the (x,y) plane is

$$\begin{aligned}dA&=dx\,dy\\&=\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{z}}\,dS\ .\end{aligned} \tag{4.190}$$

The unit normal $\hat{\boldsymbol{n}}$ is given by

$$\hat{\boldsymbol{n}}=\frac{\boldsymbol{\nabla}F}{\big|\boldsymbol{\nabla}F\big|}=\frac{\hat{\boldsymbol{z}}-\boldsymbol{\nabla}u}{\sqrt{1+(\boldsymbol{\nabla}u)^2}}\ . \tag{4.191}$$

Thus,

$$dS=\frac{dx\,dy}{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{z}}}=\sqrt{1+(\boldsymbol{\nabla}u)^2}\,dx\,dy\ . \tag{4.192}$$

The potential energy for a deformed surface can take many forms. In the case we shall consider here, we consider only the effect of surface tension σ, and we write the potential energy functional as

$$\begin{aligned}U\big[u(x,y,t)\big]&=\sigma\!\int\!dS\\&=U_0+\tfrac{1}{2}\!\int\!dA\,(\boldsymbol{\nabla}u)^2+\ldots\ .\end{aligned} \tag{4.193}$$

The kinetic energy functional is

$$T\big[u(x,y,t)\big] = \tfrac{1}{2}\!\int\! dA\, \mu(\boldsymbol{x})\, (\partial_t u)^2\ . \tag{4.194}$$

Thus, the action is

$$S\big[u(\boldsymbol{x},t)\big] = \int\! d^2\!x\, \mathcal{L}(u, \boldsymbol{\nabla} u, \partial_t u, \boldsymbol{x})\ , \tag{4.195}$$

where the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu(\boldsymbol{x})\, (\partial_t u)^2 - \tfrac{1}{2}\sigma(\boldsymbol{x})\, (\boldsymbol{\nabla} u)^2\ , \tag{4.196}$$

where here we have allowed both $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ to depend on the spatial coordinates. The equations of motion are

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial\, \partial_t u} + \boldsymbol{\nabla}\cdot \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla} u} - \frac{\partial \mathcal{L}}{\partial u} & (4.197)\\ &= \mu(\boldsymbol{x})\, \frac{\partial^2 u}{\partial t^2} - \boldsymbol{\nabla}\cdot \Big\{ \sigma(\boldsymbol{x})\, \boldsymbol{\nabla} u \Big\}\ . & (4.198) \end{aligned}$$

4.4.1 Helmholtz Equation

When μ and σ are each constant, we obtain the Helmholtz equation:

$$\bigg(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \bigg)\, u(\boldsymbol{x},t) = 0\ , \tag{4.199}$$

with $c = \sqrt{\sigma/\mu}$. The D'Alambert solution still works – waves of arbitrary shape can propagate *in a fixed direction* $\hat{\boldsymbol{k}}$:

$$u(\boldsymbol{x},t) = f(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct)\ . \tag{4.200}$$

This is called a *plane wave* because the three dimensional generalization of this wave has wavefronts which are planes. In our case, it might better be called a *line wave*, but people will look at you funny if you say that, so we'll stick with *plane wave*. Note that the locus of points of constant f satisfies

$$\phi(\boldsymbol{x},t) = \hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct = \text{constant}\ , \tag{4.201}$$

and setting $d\phi = 0$ gives

$$\hat{\boldsymbol{k}}\cdot\frac{d\boldsymbol{x}}{dt} = c\ , \tag{4.202}$$

which means that the velocity along $\hat{\boldsymbol{k}}$ is c. The component of $\boldsymbol{x}$ perpendicular to $\hat{\boldsymbol{k}}$ is arbitrary, hence the regions of constant ϕ correspond to lines which are orthogonal to $\hat{\boldsymbol{k}}$.

Owing to the linearity of the wave equation, we can construct arbitrary superpositions of plane waves. The most general solution is written

$$u(\boldsymbol{x},t) = \int\!\frac{d^2\!k}{(2\pi)^2}\, \Big[A(\boldsymbol{k})\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - ckt)} + B(\boldsymbol{k})\, e^{i(\boldsymbol{k}\cdot\boldsymbol{x} + ckt)} \Big]\ . \tag{4.203}$$

The first term in the bracket on the RHS corresponds to a plane wave moving in the $+\hat{\boldsymbol{k}}$ direction, and the second term to a plane wave moving in the $-\hat{\boldsymbol{k}}$ direction.

4.4.2 Rectangles

Consider a rectangular membrane where $x \in [0, a]$ and $y \in [0, b]$, and subject to the boundary conditions $u(0, y) = u(a, y) = u(x, 0) = u(x, b) = 0$. We try a solution of the form

$$u(x, y, t) = X(x)\, Y(y)\, T(t) \ . \tag{4.204}$$

This technique is known as *separation of variables*. Dividing the Helmholtz equation by u then gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \ . \tag{4.205}$$

The first term on the LHS depends only on x. The second term on the LHS depends only on y. The RHS depends only on t. Therefore, each of these terms must individually be constant. We write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \ , \tag{4.206}$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2} \ . \tag{4.207}$$

Thus, $\omega = \pm c|\boldsymbol{k}|$. The most general solution is then

$$X(x) = A\, \cos(k_x x) + B \sin(k_x x) \tag{4.208}$$

$$Y(y) = C\, \cos(k_y y) + D \sin(k_y y) \tag{4.209}$$

$$T(t) = E\, \cos(\omega t) + B \sin(\omega t) \ . \tag{4.210}$$

The boundary conditions now demand

$$A = 0 \quad , \quad C = 0 \quad , \quad \sin(k_x a) = 0 \quad , \quad \sin(k_y b) = 0 \ . \tag{4.211}$$

Thus, the most general solution subject to the boundary conditions is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathcal{A}_{mn} \sin\!\left(\frac{m\pi x}{a}\right) \sin\!\left(\frac{n\pi y}{b}\right) \cos\!\big(\omega_{mn} t + \delta_{mn}\big) \ , \tag{4.212}$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2} \ . \tag{4.213}$$

4.4.3 Circles

For a circular membrane, such as a drumhead, it is convenient to work in two-dimensional polar coordinates (r, φ). The Laplacian is then

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \ . \tag{4.214}$$

We seek a solution to the Helmholtz equation which satisfies the boundary conditions $u(r = a, \boldsymbol{\varphi}, t) = 0$. Once again, we invoke the separation of variables method, writing

$$u(r, \boldsymbol{\varphi}, t) = R(r)\, \Phi(\boldsymbol{\varphi})\, T(t) \ , \tag{4.215}$$

resulting in

$$\frac{1}{R}\,\frac{1}{r}\,\frac{\partial}{\partial r}\bigg(r\,\frac{\partial R}{\partial r}\bigg) + \frac{1}{r^2}\,\frac{1}{\Phi}\,\frac{\partial^2 \Phi}{\partial \boldsymbol{\varphi}^2} = \frac{1}{c^2}\,\frac{1}{T}\,\frac{\partial^2 T}{\partial t^2} \ . \tag{4.216}$$

The azimuthal and temporal functions are

$$\Phi(\boldsymbol{\varphi}) = e^{im\varphi} \quad , \quad T(t) = \cos(\omega t + \delta) \ , \tag{4.217}$$

where m is an integer in order that the function $u(r, \boldsymbol{\varphi}, t)$ be single-valued. The radial equation is then

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \bigg(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\bigg) R = 0 \ . \tag{4.218}$$

This is Bessel's equation, with solution

$$R(r) = A\, J_m\Big(\frac{\omega r}{c}\Big) + B\, N_m\Big(\frac{\omega r}{c}\Big) \ , \tag{4.219}$$

where $J_m(z)$ and $N_m(z)$ are the Bessel and Neumann functions of order m, respectively. Since the Neumann functions diverge at $r = 0$, we must exclude them, setting $B = 0$ for each m.

We now invoke the boundary condition $u(r = a, \boldsymbol{\varphi}, t) = 0$. This requires

$$J_m\Big(\frac{\omega a}{c}\Big) = 0 \qquad \Longrightarrow \qquad \omega = \omega_{m\ell} = x_{m\ell}\,\frac{c}{a} \ , \tag{4.220}$$

where $J_m(x_{m\ell}) = 0$, *i.e.* $x_{m\ell}$ is the $\ell^{\rm th}$ zero of $J_m(x)$. The mose general solution is therefore

$$u(r, \boldsymbol{\varphi}, t) = \sum_{m=0}^{\infty} \sum_{\ell=1}^{\infty} \mathcal{A}_{m\ell}\, J_m\big(x_{m\ell}\, r/a\big)\, \cos\big(m\boldsymbol{\varphi} + \beta_{m\ell}\big)\, \cos(\omega_{m\ell}\, t + \delta_{m\ell}) \ . \tag{4.221}$$