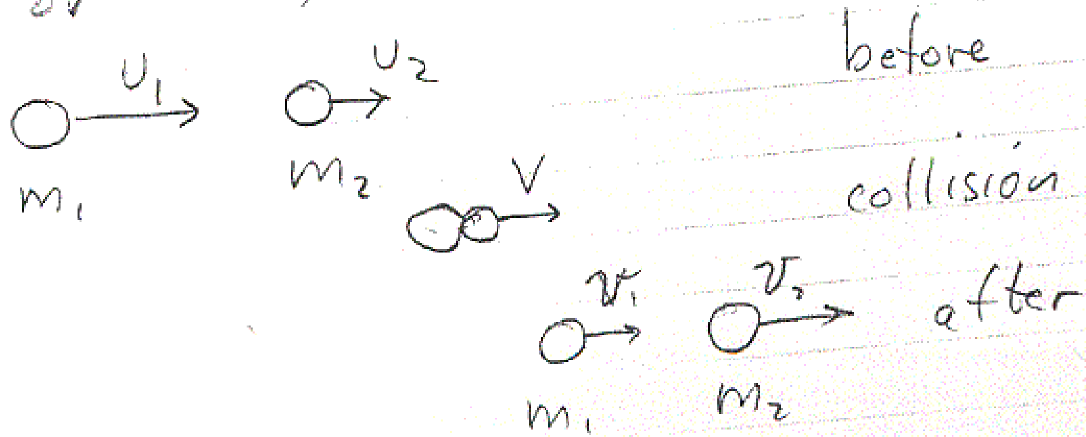
8. Inelastic Collisions

when 2 bodies interact, energy can be gained or lost because of processes internal to the bodies. If Q is the energy gained or lost, from cons. en. we have



$$Q + \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$Q = 0$: elastic collision
$Q > 0$: exothermic collision (e.g. fission)
$Q < 0$: endothermic collision (heat)

The coefficient of restitution ε is defined as

$$\varepsilon = \frac{|v_2 - v_1|}{|u_2 - u_1|}$$

Experimentally, ε depends only on the composition of the bodies, and not on velocities (Newton's rule).

perfectly $\longrightarrow 0 < \varepsilon < 1 \longleftarrow$ elastic

Example 9.9 shows that $\varepsilon = 1$ for an elastic, head-on collision.

Impulse

During a collision (elastic or inelastic), forces involved operate over a brief interval of time; and are called impulsive forces. The impulse is defined as the change of a bodies momentum:

$$P \equiv (mv)_2 - (mv)_1$$

but, by N2

$$F = \frac{d}{dt}(mv)$$

Integrating

$$\int_1^2 F\,dt = \int_1^2 \frac{d}{dt}(mv) = mv\Big|_1^2 \equiv P$$

$$\boxed{P \equiv \int_1^2 F\,dt}$$

in general, P is a vector.

9. Scattering cross sections

We would now like to use the preceding results to derive formulae that predict how a particle will scatter off another given the impact parameter and force law.

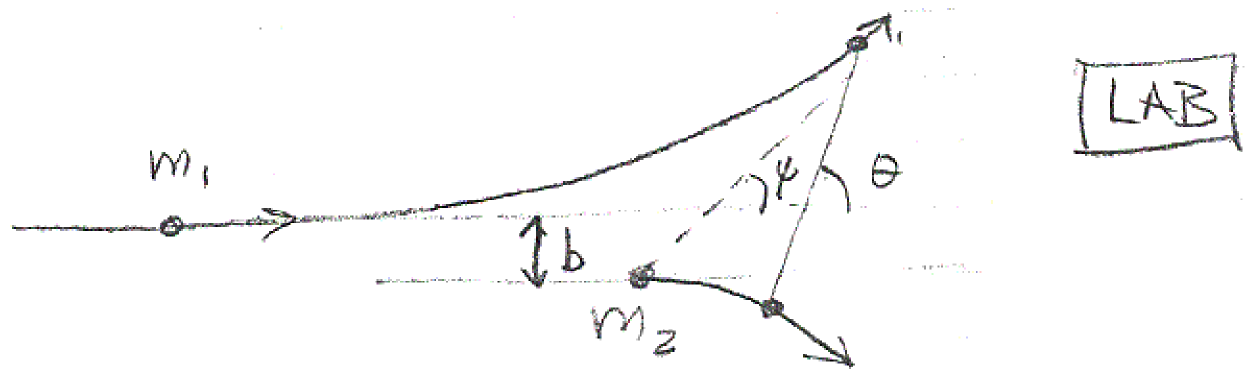


Diagram above shows case of a <u>repulsive force</u> (e.g., 2 like-charged particles).
The distance b is called <u>impact parameter</u>
It is the distance of closest approach in the absence of scattering.

If we know force, we can compute trajectories since this is 2-body problem. Often, we are not interested in detailed trajectories, but only asymptotic <u>scattering angle</u> ψ or θ. When firing a beam of particles at a target, one may not be able to control b, rather scattering will occur for a range of b. In that cas, we would like to calculate the probability that the incident particle is scattered through and angle θ.

<u>differential scattering cross section</u>

$\sigma(\theta)$

In CM frame, define

$$\sigma(\theta) = \frac{\left(\begin{array}{c}\#\text{ interactions per target particle}\\ \text{that scatter into } d\Omega' \text{ at angle } \theta\end{array}\right)}{\#\text{ of incident particles / unit area}}$$

then, if dN is number of particles scattered into $d\Omega'$ per unit time, and I is intensity of incident beam, we have

$$\sigma(\theta)\, d\Omega' = \frac{dN}{I}$$

or

$$\frac{dN}{d\Omega'} = \sigma(\theta)\, I$$

$$\left[\frac{dN}{d\Omega'}\right] = \frac{\#}{\text{steradian}}$$

$$\left[I\right] = \frac{\#}{\text{area}}$$

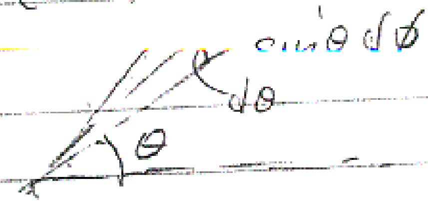
therefore

$$\left[\sigma\right] = \frac{\text{area}}{\text{steradian}}$$

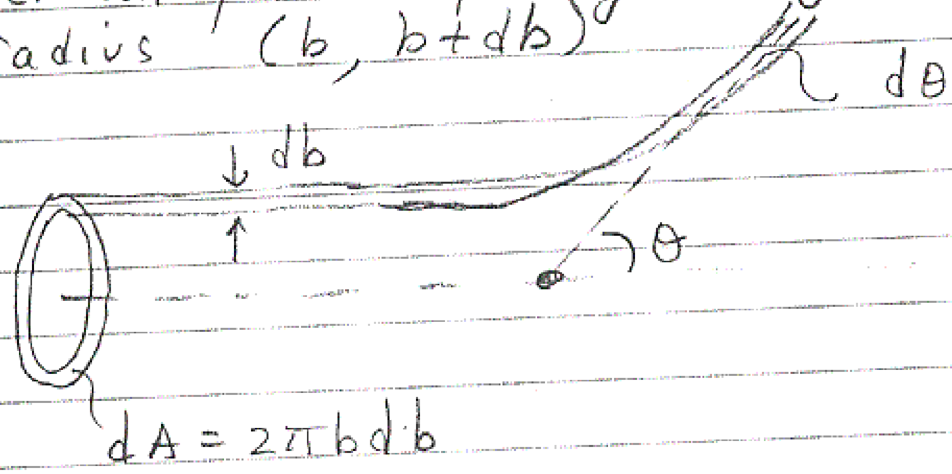
which is why it is called a <u>cross section</u>.

If scattering is axially symmetric (as for central

$$d\Omega' = 2\pi \sin\theta \, d\theta$$



Consider all particles passing through annulus of radius $(b, b+db)$



Since particle number is conserved,

$$I \cdot 2\pi b \, db = -I \, \sigma(\theta) \cdot 2\pi \sin\theta \, d\theta$$

$\frac{db}{d\theta}$ is negative

From this, we derive

$$\boxed{\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

Given the force law, we can calculate σ.

Recall from ch. 8 the equation for the 2-body problem with a central force

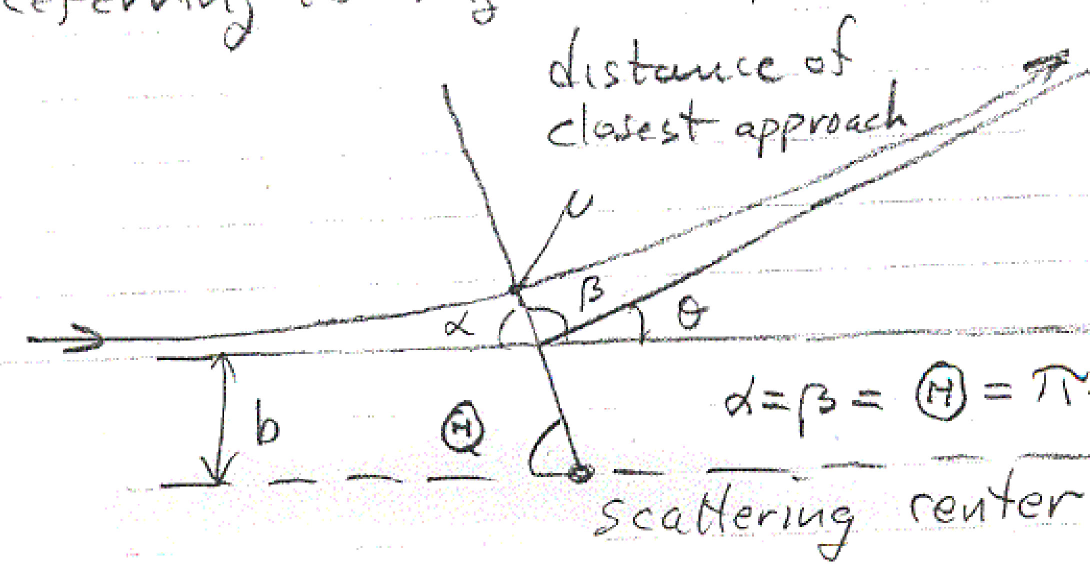
$$d\theta = \frac{\ell/\mu r^2 \quad dr}{\pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}$$

see aside

where

- $\ell = \mu b u_1 = b\sqrt{2\mu T_0'}$ angular momentum
- $T_0' = \frac{1}{2}\mu u_1^2$ KE in CM frame
- $\mu = \frac{m_1 m_2}{m_1+m_2}$ reduced mass
- $E = T_0'$ is total energy in CM frame
- $U(r)$ is potential energy

Referring to Fig. 9-22, we have:



$\alpha = \beta = \Theta = \pi - 2\theta$

aside

$$T + U = E = \text{constant}$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r)$$

$$\ell = \mu r^2 \dot{\theta}$$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{\ell^2}{\mu r^2} + U(r)}$$

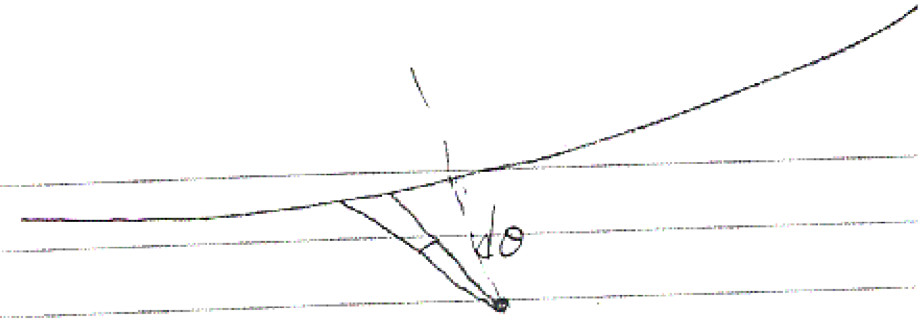
$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}$$

we don't care about t; would like trajectory $r(\theta)$ or $\theta(r)$

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$$

but $\dot{\theta} = \ell/\mu r^2$

$$\therefore \boxed{d\theta = \frac{\ell/\mu r^2}{\pm\sqrt{\frac{2}{\mu}(E-U) - \frac{\ell^2}{\mu^2 r^2}}}\,dr}$$



$$\theta = -\int_{-\infty}^{\infty} d\theta = \int \frac{\ell/\mu r^2}{\sqrt{\frac{2}{\mu}(T_0' - U) - \frac{\ell^2}{\mu^2 r}}}\, dr$$

$$= -2\int_{r_{min}}^{\infty} \frac{\ell/r^2 \; dr}{\sqrt{2\mu(T_0' - U) - \frac{\ell^2}{r}}}$$

$$\ell = b\sqrt{2\mu T_0'}$$

denom: $\sqrt{2\mu(T_0' - U) - b^2 2\mu T_0'/r}$

$$= \sqrt{(2\mu T_0')\left(1 - \frac{U}{T_0'}\right) - b^2/r^2}$$

hence

$$\theta = -2\int_{r_{min}}^{\infty} \frac{(b/r^2)\, dr}{\sqrt{1 - U/T_0 - b^2/r^2}}$$

$$\Theta = \pi - 2\theta \implies d\Theta = -2\, d\theta$$

So
$$\boxed{\Theta = \int_{r_{min}}^{\infty} \frac{(b/r^2)\,dr}{\sqrt{1 - b^2/r^2 - (U/T_0)}}} \qquad 9.123$$

Recap:
Given m_1, m_2, we know μ
μ, v_1, we know $T_0' = E$
b, we know ℓ

$U(r)$ we can calculate r_{min}

Using 9.123, we can then calculate
$$\theta = \pi - 2\Theta$$
by integration.
Once we know $\theta(b)$
$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{b}{\sin\theta}\left|\frac{d\theta}{db}\right|^{-1}.$$

10. Rutherford scattering

An important application of the theory developed above is scattering of charged particles by electrostatic potential
$$U(r) = \frac{k}{r}, \quad k = \frac{q_1 q_2}{4\pi\epsilon_0}$$

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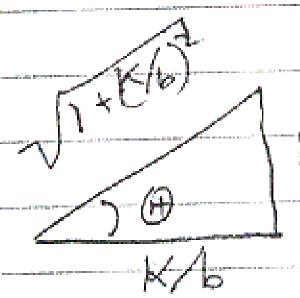
Direct substitution into 9.123 gives

$$\Theta = \int_{r_{min}}^{\infty} \frac{(b/r)\,dr}{\sqrt{r^2 - (k/T_0')r - b^2}}$$

This can be integrated to obtain

$$\cos\Theta = \frac{(K/b)}{\sqrt{1+(K/b)^2}}$$



where $K \equiv k/2T_0'$.

Using simple trig

$$\tan\Theta = b/K$$

but $\theta = \pi - 2\Theta$, $\Theta = \pi/2 - \theta/2$

so

$$\boxed{b = K\cot(\theta/2)}$$

thus

$$\frac{db}{d\theta} = -\frac{K}{2}\,\frac{1}{\sin^2(\theta/2)}$$

Plugging into formula for $\sigma(\theta)$

$$\sigma(\theta) = \frac{K^2}{2}\cdot\frac{\cot(\theta/2)}{\sin\theta\,\sin^2(\theta/2)}$$

Using $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$

we derive an explicit formula for Rutherford scattering:

$$\sigma(\theta) = \frac{K^2}{4} \cdot \frac{1}{\sin^4(\theta/2)}$$

$$\boxed{\sigma(\theta) = \frac{k^2}{(4T_0')^2} \cdot \frac{1}{\sin^4(\theta/2)}}$$

For case $m_1 = m_2$, $T_0' = \frac{1}{2} T_0$

$$\sigma(\theta) = \frac{k^2}{4T_0^2} \cdot \frac{1}{\sin^4(\theta/2)}, \quad m_1 = m_2$$

Total scattering cross section

$$\sigma_t = \int_{4\pi} \sigma(\theta)\, d\Omega' = 2\pi \int_0^{\pi} \sigma(\theta) \sin\theta \, d\theta$$

Interestingly, this integral is ∞, because of long range $1/r$ potential. In a real plasma, charge is screened for $r > R_{debeye}$, thus integral becomes finite.

$\Rightarrow$ scattering is dominated by many small angle collisions.