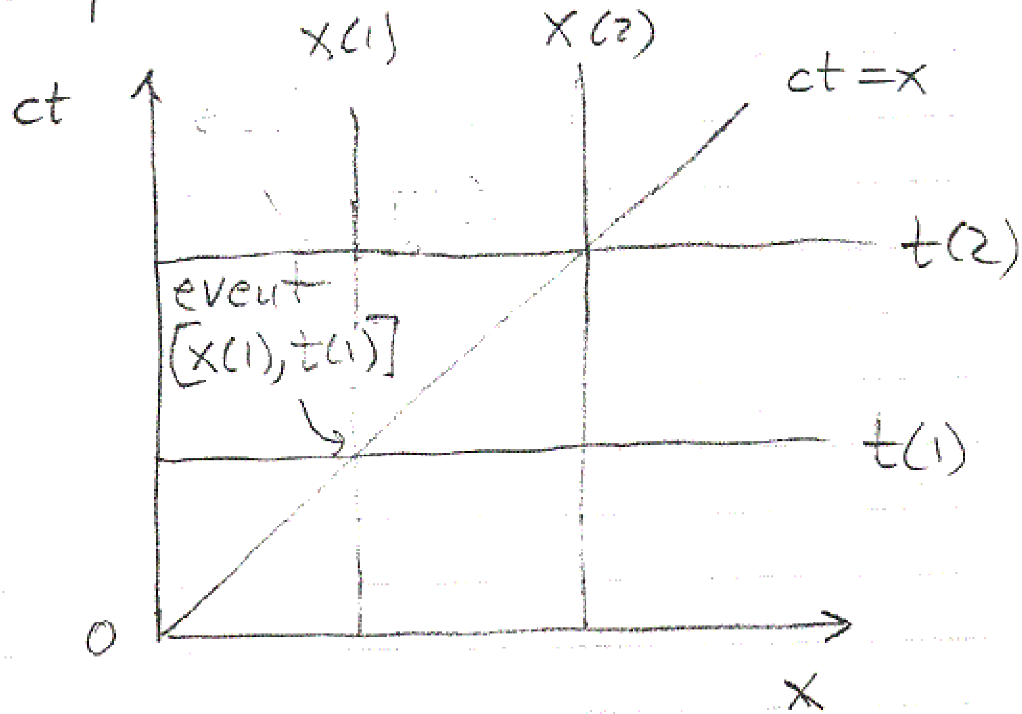
9. Geometric illustration of Lorentz contraction and time dilation

To see what is going on, we introduce a graphical device called a spacetime diagram. x is plotted horizontally, t vertically. So that the axes have the same units we plot ct vs x



In this diagram, the x-location of a light ray emitted at O at $t=0$ is a 45° line. An event in the K frame is a point in this diagram.
In K frame, lines of constant t are horizontal lines; lines of constant x are vertical lines.
The K′ frame is moving with speed v to the right. What does its spacetime coordinate system

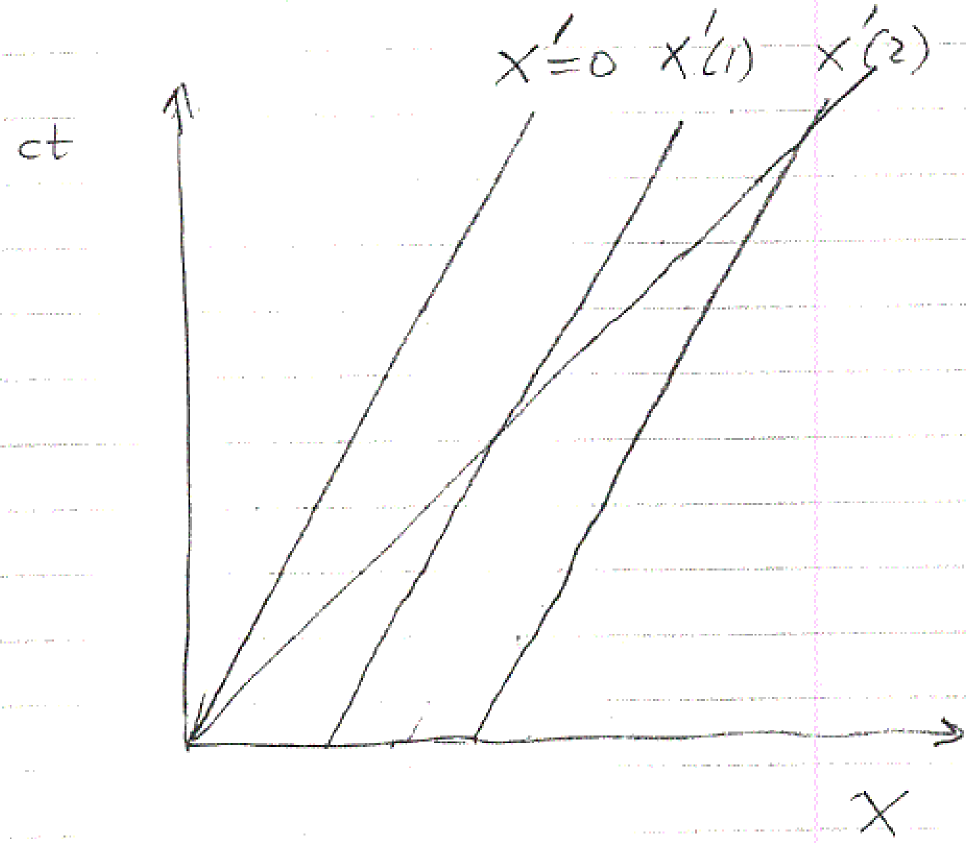
look like as seen in the K frame?

Lines of constant x'

The origin of K' is defined as $x'=0$
It moves with speed v

$$\frac{dx}{dt} = v = \beta c$$

$$\therefore \quad \frac{d(ct)}{dx} = \frac{1}{\beta}$$



For general $x' \neq 0$, we have

$$x' = \gamma(x - vt)$$

$$\frac{x'}{\gamma} = x - vt \Rightarrow \boxed{ct = \frac{1}{\beta}x - \frac{x'}{\beta\gamma}}$$

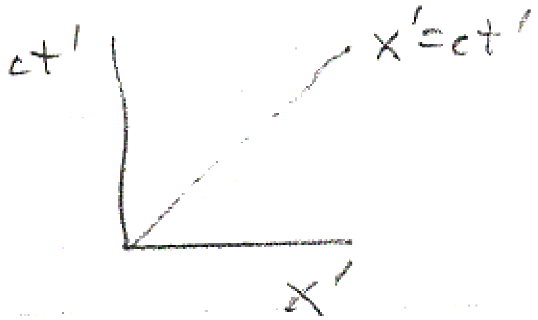
Lines of constant x' also have slope $1/\beta$

Lines of constant t'

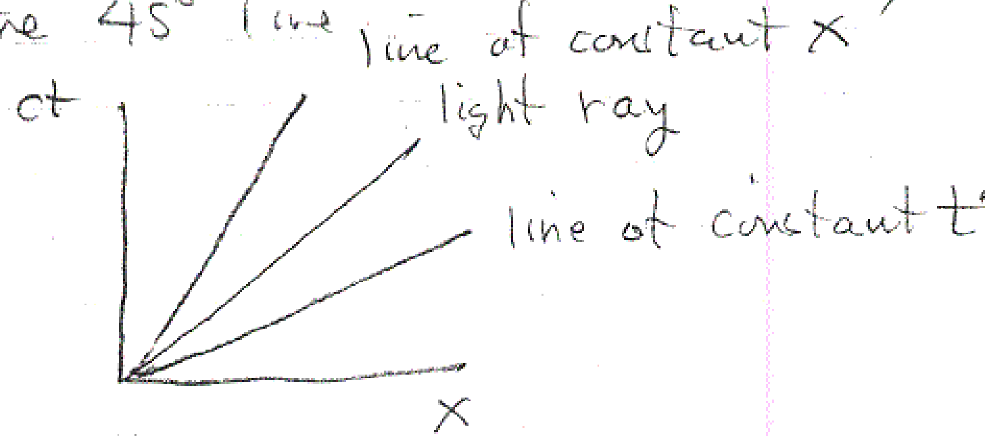
By the constancy of the speed of light a flash emitted when O and O' coincide obeys

$$x' = ct'$$

Relative to K' spacetime diagram, this is also a diagonal line



Therefore, lines of constant t' in K frame make an equal and opposite angle to the 45° line



For general $t' \neq 0$, we have

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

We want x vs. t for constant t'. Substituting $x' = \gamma(x - vt)$ we have

$$t' = \gamma t + \frac{x}{\gamma v}(1-\gamma^2)$$

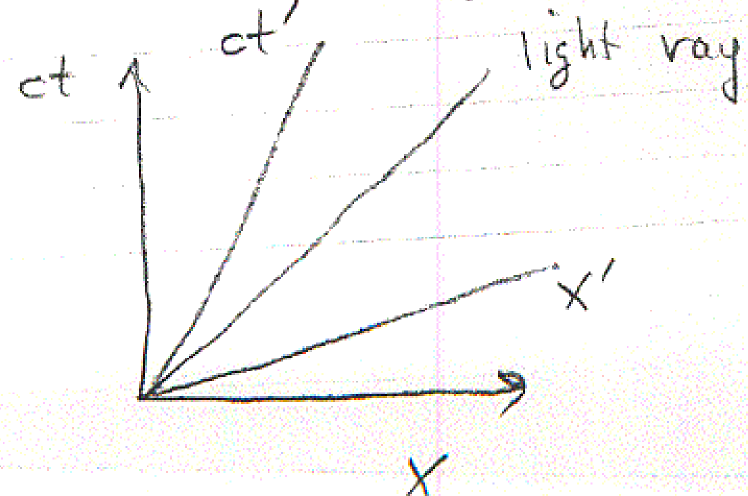
$$1-\gamma^2 = 1 - \frac{1}{1-\beta^2} = \frac{-\beta^2}{1-\beta^2} = -\beta^2\gamma^2$$

$$ct' = \gamma ct - \frac{\beta^2\gamma^2 cx}{\gamma v}$$

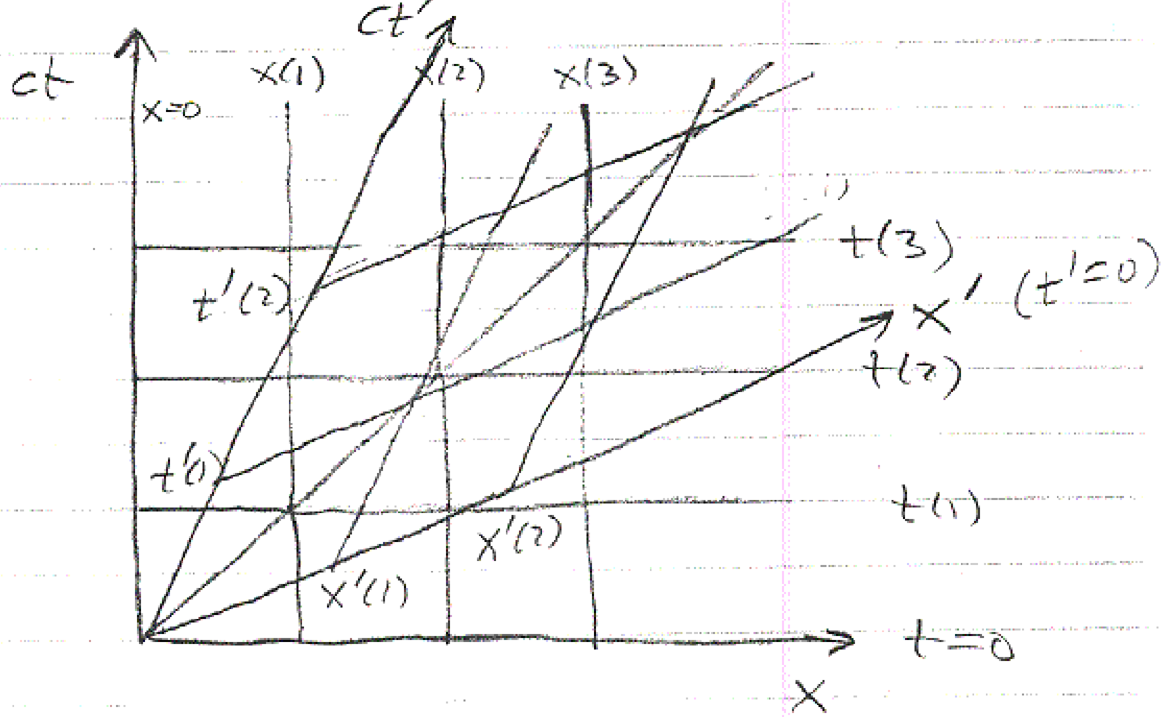
$$= \gamma ct - \beta\gamma x$$

$$\therefore \boxed{ct = \frac{ct'}{\gamma} + \beta x}$$

So lines of constant t' have slope β in K spacetime diagram. Just as lines of constant x are parallel to t axis, and vice versa, lines of constant x' are parallel to t' axis, and vice versa. Hence we can now draw K's ST diagram relative to K's.



K′ ST coordinates relative to K



How to calibrate the axes?

10. <u>Invariance of spacetime interval</u>

Consider two events E(1) and E(2)
Relative to K, let them have coords

$$E(1) : [x(1), t(1)]$$
$$E(2) : [x(2), t(2)]$$

Define the <u>spacetime interval</u>

$$\Delta s^2 \equiv (x(2)-x(1))^2 - c^2(t(2)-t(1))^2$$
$$= \Delta x^2 - c^2 \Delta t^2 .$$

Relative to K′, events have coords

$$E'(1) : [x'(1), t'(1)]$$
$$E'(2) : [x'(2), t'(2)]$$

If we define

$$\Delta s'^2 = \Delta x'^2 - c\Delta t'^2$$

then one can show, using LT, that

$$\boxed{\Delta s'^2 = \Delta s^2}$$

Proof is left as an exercise for students.

<u>Calibrating axes</u>

Consider clock at rest in K frame at $x=0$. Let its period be $c\Delta t = 1$. The ST interval between ticks is

$$\begin{aligned} \Delta s^2 &= x^2 - c^2t^2 \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

Draw the hypebola $\Delta s'^2 = x'^2 - (c\Delta t')^2 = -1$