11. Dynamics of Rigid Bodies

A rigid body can be thought of as a SOP where the $\vec{r}_\alpha$ (coords. relative to CM) are fixed. If we ignore atomic vibrations, a solid is rigid body so long as there are no forces large enough to cause dislocations. Clearly, a fluid is not a rigid body.

The dynamics of rigid bodies is surprisingly complex. That's because a RB has 6 degrees of freedom

- 3 translational (x, y, z)
- 3 rotational (e.g. pitch, yaw, roll)

The purpose of this chapter is to derive equations of motion (EOM) for RB, and apply to common examples, (e.g. tops, pendula).

3. The inertia tensor

The resistance of a body to spatial acceleration is its inertial mass

$$\vec{F} = m\vec{a}$$

The resistance of a body to angular

acceleration is its rotational inertia.
In simple, planar problems we had

torque $\rightarrow N = \dot{L} \leftarrow$ t.r.c. ang. mom.
ang. mom. $\rightarrow L = I\omega \leftarrow$ ang. velocity

rotational inertia

Combining $N = I\dot{\omega}$

angular acceleration

We want to generalize this equation to 3D, where $\vec{\omega}$, $\underline{L}$, $\underline{N}$ are vectors.
Q: what does I become?
A: a tensor, called the inertia tensor

$$\overleftrightarrow{I} = \begin{Bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{Bmatrix}$$

$$= \{I_{ij}\} \quad i=1,2,3; \; j=1,2,3$$

As we will show, in a fixed frame (IRF)

$$L_i = \sum_j I_{ij}\,\omega_j$$

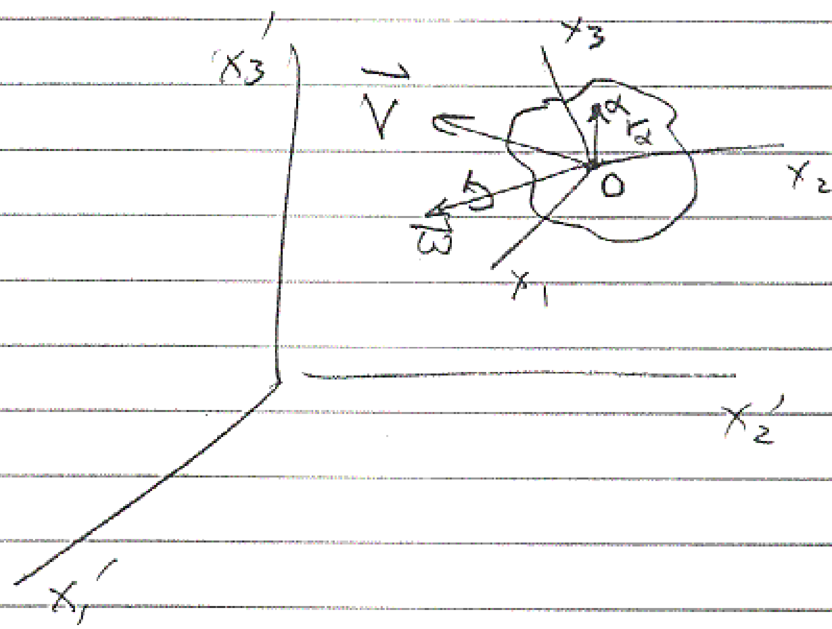
$$N_i = \dot{L}_i = \sum_j I_{ij}\,\dot{\omega}_j$$

deriving $\overleftrightarrow{I}$

Recall from Ch. 10, the general expression for the velocity of a point P in a NIRF

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega} \times \vec{r} \qquad (10.17)$$

Let point P be particle α in a RB



Then, by definition of a RB, $\left.\frac{d\vec{r}_\alpha}{dt}\right)_r = \vec{v}_r = 0$

Hence $\boxed{\vec{v}_\alpha = \vec{V} + \vec{\omega} \times \vec{r}_\alpha}$

Henceforth, we will call the rotating reference frame the body frame (BF) and the coord. sys. attached to the body the body coordinate system (BCS).

The angular momentum of particle α relative to origin O of BCS is

$$\vec{L}_\alpha = \vec{r}_\alpha \times \vec{p}_\alpha = \vec{r}_\alpha \times (m \cdot \vec{\omega} \times \vec{r}_\alpha)$$

The AM of the RB is sum over α

$$\vec{L} = \sum_\alpha m \cdot \vec{r}_\alpha \times (\vec{\omega} \times \vec{r}_\alpha)$$

Using vector identity $\vec{A} \times (\vec{B} \times \vec{A}) = A^2 \vec{B} - \vec{A}(\vec{A} \cdot \vec{B})$
we get

$$\vec{L} = \sum_\alpha m_\alpha \left[r_\alpha^2 \vec{\omega} - \vec{r}_\alpha (\vec{r}_\alpha \cdot \vec{\omega}) \right]$$

Component-wise, we can write

$$L_i = \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j} \omega_j \right)$$

which we can manipulate using following <u>trick</u>

$$\omega_i = \sum_j \omega_j \delta_{ij}$$

Kronecker delta

Inserting & rearranging

$$L_i = \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - \omega_j x_{\alpha,i} x_{\alpha,j} \right)$$

$$= \sum_j \omega_j \underbrace{\sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)}_{I_{ij}}$$

$$= \sum_j \omega_j I_{ij}$$

$$\boxed{I_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)}$$

Let's develop a little intuition about $\overleftrightarrow{I}$. If we revert to xyz notation, and define $r_\alpha^2 = \sum_k x_{\alpha,k}^2$, then, working out the components of $\overleftrightarrow{I}$, we have

$$\overleftrightarrow{I} = \begin{Bmatrix} \sum_\alpha m_\alpha (r_\alpha^2 - x_\alpha^2) & -\sum_\alpha m_\alpha x_\alpha y_\alpha & -\sum_\alpha m_\alpha x_\alpha z_\alpha \\ -\sum_\alpha m_\alpha y_\alpha x_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - y_\alpha^2) & -\sum_\alpha m_\alpha y_\alpha z_\alpha \\ -\sum_\alpha m_\alpha z_\alpha x_\alpha & -\sum_\alpha m_\alpha z_\alpha y_\alpha & \sum_\alpha m_\alpha (r_\alpha^2 - z_\alpha^2) \end{Bmatrix}$$

$$\equiv \begin{Bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{Bmatrix}$$

We see that $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$; i.e., $\overleftrightarrow{I}$ is <u>symmetric</u>. That means there are only 6 unique components of the inertia tensor.

$\Rightarrow$ diagonal components called <u>moments of inertia</u>

$\Rightarrow$ off-diagonal components called <u>products of inertia</u>

For a continuous medium, with mass density $\rho(\vec{r})$,

$$I_{ij} = \int_V \rho(\vec{r})\left(\delta_{ij} r^2 - x_i x_j\right) d^3\vec{r}$$

Example)

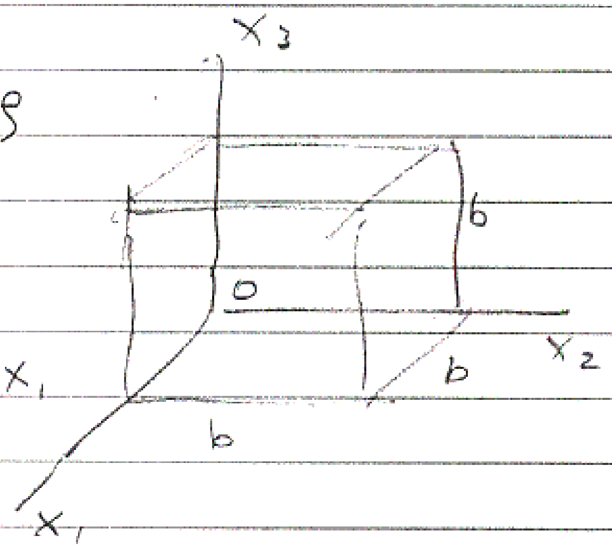
Here we see that I_{ij} depend on where the origin O is placed w.r.t. the body.

Example 11.3 in text

uniform cube, density ρ

side length b

$M = \rho b^3$



$$I_{11} = \rho \int_0^b dx_3 \int_0^b dx_2 \left(x_2^2 + x_3^2\right) \int_0^b dx_1$$

$$= \rho b \int_0^b dx_3 \left(\frac{b^3}{3} + x_3^2 b\right)$$

$$= \rho b \left(\frac{b^4}{3} + \frac{b^4}{3}\right) = \frac{2}{3}\rho b^5 = \frac{2}{3} M b^2$$

$$I_{12} = -\rho \int_0^b x_1\, dx_1 \int_0^b x_2\, dx_2 \int_0^b dx_3$$

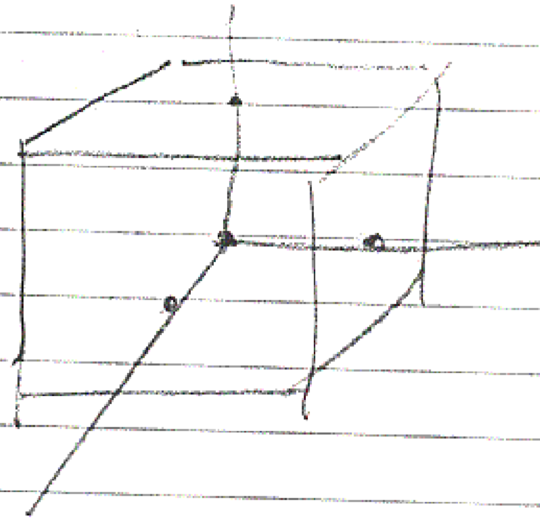
$$= -\rho\left(\frac{b^2}{2}\right)\left(\frac{b^2}{2}\right)(b) = -\frac{\rho b^5}{4} = -\frac{1}{4} M b^2$$

$\beta = Mb^2$,

$$I = \begin{Bmatrix} \frac{2}{3}\beta & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & \frac{2}{3}\beta & -\frac{1}{4}\beta \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta \end{Bmatrix}$$

Repeat, placing O at center of cube

$$I_{11} = \rho \int_{-\frac{b}{2}}^{\frac{b}{2}} dx_3 \int_{-\frac{b}{2}}^{\frac{b}{2}} dx_2 \,(x_2^2 + x_3^2) \int_{-\frac{b}{2}}^{b/2} dx_1$$



$$= \rho b \int_{-\frac{b}{2}}^{b/2} dx_3 \left(\frac{x_2^3}{3}\Bigg|_{-\frac{b}{2}}^{\frac{b}{2}} + x_3^2 b \right)$$

$$= \rho b \int_{-\frac{b}{2}}^{b/2} dx_3 \left(\frac{b^3}{12} + x_3^2 b \right)$$

$$= \rho b \left(\frac{b^4}{12} + \frac{b^4}{12} \right) = \frac{\rho b^5}{6} = \frac{1}{6} M b^2$$

$$I_{12} = -\rho \int_{-\frac{b}{2}}^{\frac{b}{2}} x_1 \, dx_1 \int_{-\frac{b}{2}}^{\frac{b}{2}} x_2 \, dx_2 \int_{-\frac{b}{2}}^{\frac{b}{2}} dx_3$$

$$= -\rho (0)(0) b = 0$$

$$\hat{I} = \begin{Bmatrix} \frac{1}{6}\beta & 0 & 0 \\ 0 & \frac{1}{6}\beta & 0 \\ 0 & 0 & \frac{1}{6}\beta \end{Bmatrix}$$