Chapter 10:
Motion in Non-inertial Reference Frame

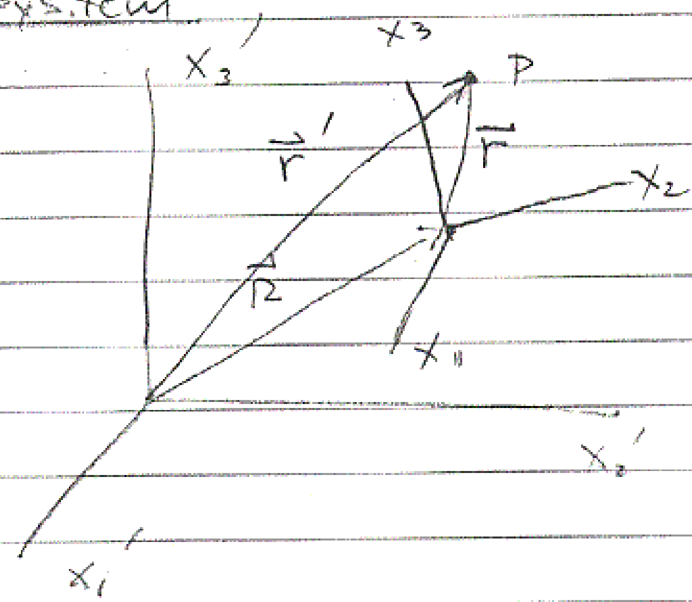
1. Intro

Although dynamics is simplest in IRF = inertial reference frames, it is not always convenient if, e.g., coordinates are attached to a NRF = non-inertial RF

examples
- rotating reference frame (e.g. Earth)
  - ballistic trayectories
  - geophysical fluid dynamics
- accelerating reference frame
- rigid body motion

2. Rotating Coord. System

Consider two sets coord. axis:



$\vec{X}'$ = IRF coord.
$\vec{X}$ = RRF coord.

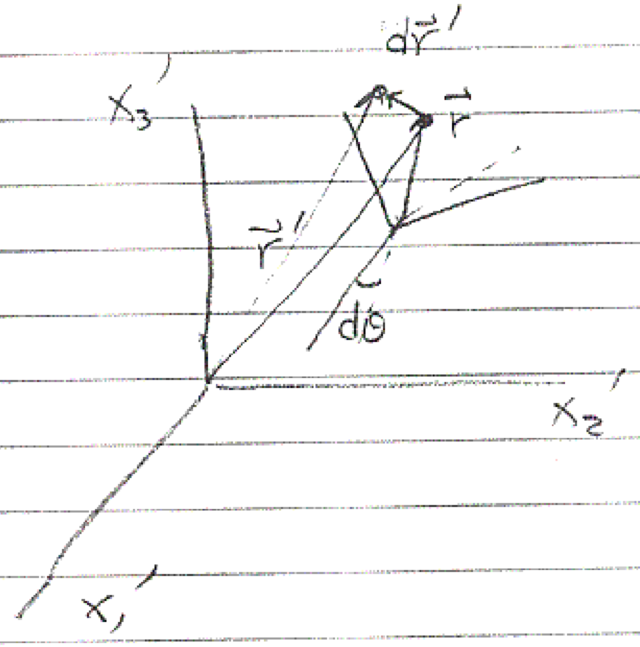
Let $\vec{R}$ be position of origin of RRF in IRF, and
$\vec{r}'$ be position of some point P in IRF.
$\vec{r}$ " " " P in RRF.

Then

$$\vec{r}' = \vec{R} + \vec{r}$$

Suppose P is at rest in RRF (e.g., a city on the surface of the Earth), Then, if RRF rotates through an infinitesimal angle $d\vec{\theta}$

$$d\vec{r}\,' = \vec{r} \times d\vec{\theta}$$



$d\vec{\theta}$ is a unit vector that points in the direction of the instantaneous rotation axis of the RRF

Taking time derivative:

$$\frac{d\vec{r}\,'}{dt} = \vec{r} \times \frac{d\vec{\theta}}{dt} \qquad \left(\vec{r} \text{ is fixed in RRF}\right)$$

$$= \vec{r} \times \vec{\omega} \qquad (\text{P fixed in RRF})$$

$\vec{\omega} \equiv \dfrac{d\vec{\theta}}{dt}$ is angular velocity of RRF

If P is moving in RRF (e.g. artillery shell)

$$\boxed{\frac{d\vec{r}\,'}{dt} = \frac{d\vec{r}}{dt} + \vec{r} \times \vec{\omega}}$$

$\frac{d\vec{r}'}{dt}$ is velocity in IRF

$\frac{d\vec{r}}{dt}$ is velocity in RRF.

The equation in $\boxed{}$ is true for any vector Q

$$\boxed{\left.\frac{d\vec{Q}}{dt}\right)_{IRF} = \left.\frac{d\vec{Q}}{dt}\right)_{RRF} + \vec{\omega} \times \vec{Q}}$$

Let's apply it to $\vec{\omega}$, the angular velocity vector:

$$\left.\frac{d\vec{\omega}}{dt}\right)_{IRF} = \left.\frac{d\vec{\omega}}{dt}\right)_{RRF} + \underbrace{\cancel{\vec{\omega} \times \vec{\omega}}}_{0}$$

$\dot{\vec{\omega}}$ = angular acceleration is the same in both fixed and rotating RFs.

Let's apply it to velocity of point P.

$$\left.\frac{d\vec{r}'}{dt}\right)_{IRF} = \left.\frac{d\vec{R}}{dt}\right)_{IRF} + \left.\frac{d\vec{r}}{dt}\right)_{IRF} \quad \text{by def.}$$

but $\left.\frac{d\vec{r}}{dt}\right)_{IRF} = \left.\frac{d\vec{r}}{dt}\right)_{RRF} + \vec{\omega} \times \vec{r}$

so

$$\left.\frac{d\vec{r}'}{dt}\right)_{IRF} = \left.\frac{d\vec{R}}{dt}\right)_{IRF} + \left.\frac{d\vec{r}}{dt}\right)_{RRF} + \vec{\omega} \times \vec{r}$$

If we define

$$\vec{v}_f \equiv \left.\frac{d\vec{r}\,'}{dt}\right)_{IRF}$$

$$\vec{V} \equiv \left.\frac{d\vec{R}}{dt}\right)_{IRF}$$

$$\vec{v}_r \equiv \left.\frac{d\vec{r}}{dt}\right)_{RRF}$$

we have

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega}\times\vec{r}$$

vel. due to rotation

vel. WRT rotating frame

vel. of origin of RRF
<u>in</u> IRF

vel. in IRF

Consider a vector $\vec{r}$ in the rotating reference frame.

$$\vec{r} = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3$$

unit vectors in RRF

and let its time derivative in RRF be

$$\left(\frac{d\vec{r}}{dt}\right)_{RRF}$$

We can show that its time derivative in the inertial frame is

$$\left(\frac{d\vec{r}}{dt}\right)_{IRF} = \left(\frac{d\vec{r}}{dt}\right)_{RRF} + \vec{\omega} \times \vec{r}$$

where $\vec{\omega}$ is angular velocity of rotating frame in fixed frame.

<u>Proof</u>

Taking time derivative of

$$\vec{r} = \sum_i x_i \hat{e}_i$$

$$\left(\frac{d\vec{r}}{dt}\right)_{IRF} = \sum_i \dot{x}_i \hat{e}_i + \sum_i x_i \dot{\hat{e}}_i$$

$$= \left(\frac{d\vec{r}}{dt}\right)_{RRF}$$

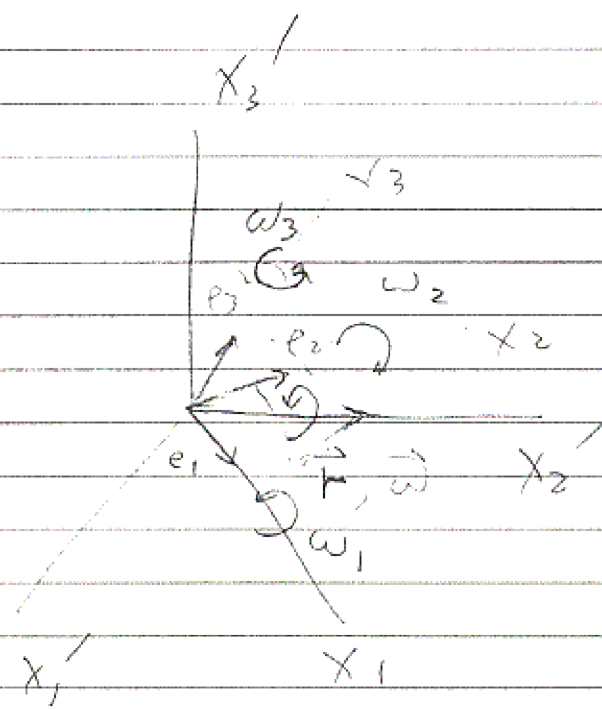
What are $\dot{\hat{e}}_i$? It is the rate at which unit vector is being rotated by virtue of $\vec{\omega}$ (measured in IRF)

10/6

$$\vec{F}_{eff} = \vec{F} - m\ddot{\vec{R}}_f - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$
$$- 2m\,\vec{\omega}\times\vec{v}_r$$

centrifugal force

coriolis force

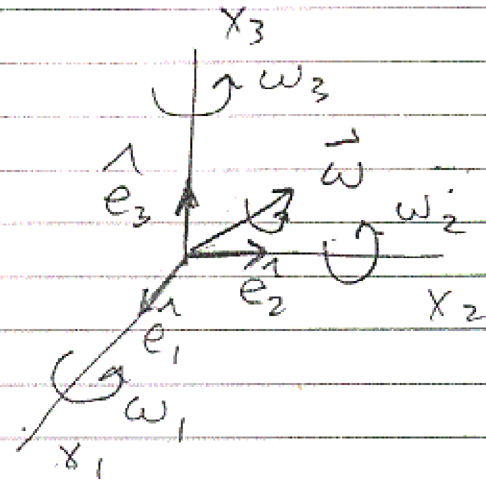


$$\frac{d\hat{e}_1}{dt} = \omega_3\,\hat{e}_2 - \omega_2\,\hat{e}_3$$ in IRF

Let $\omega_1 = \vec{\omega} \cdot \hat{e}_1$

$\omega_2 = \vec{\omega} \cdot \hat{e}_2$

$\omega_3 = \vec{\omega} \cdot \hat{e}_3$



How does $\hat{e}_1$ change?

- rotating $\hat{e}_1$ around x_1 does not change it
- rotating $\hat{e}_3$ around x_2 changes $\hat{e}_1$ at rate $\omega_2 \hat{e}_3$
- rotating $\hat{e}_2$ around x_3 changes $\hat{e}_1$ at rate $-\omega_3 \hat{e}_2$

$$\therefore \frac{d\hat{e}_1}{dt} = \omega_2 \hat{e}_3 - \omega_3 \hat{e}_2$$

likewise

$$\frac{d\hat{e}_2}{dt} = \omega_3 \hat{e}_1 - \omega_1 \hat{e}_3$$

$$\frac{d\hat{e}_3}{dt} = \omega_1 \hat{e}_2 - \omega_2 \hat{e}_1$$

or $\dot{e}_i = \vec{\omega} \times \hat{e}_i$

hence,

$$\left(\frac{d\vec{r}}{dt}\right)_{IRF} = \left(\frac{d\vec{r}}{dt}\right)_{RRF} + \sum_i x_i \, \vec{\omega} \times \hat{e}_i$$

$$\boxed{\left(\frac{d\vec{r}}{dt}\right)_{IRF} = \left(\frac{d\vec{r}}{dt}\right)_{RRF} + \vec{\omega} \times \vec{r}}$$

This is true for arbitrary $\vec{Q}$

3. Centrifugal and Coriolis Forces

N2 is valid only in IRF

$$\vec{F} = m\vec{a}_f = m\left(\frac{d\vec{v}_f}{dt}\right)_{IRF}$$

but

$$\vec{v}_f = \vec{V} + \vec{v}_r + \vec{\omega}\times\vec{r}$$

So

$$\left.\frac{d\vec{v}_f}{dt}\right)_{IRF} = \left.\frac{d\vec{V}}{dt}\right)_{IRF} + \left.\frac{d\vec{v}_r}{dt}\right)_{IRF} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{IRF}$$

$$= \ddot{\vec{R}}_f + \Big[\underbrace{\left.\frac{d\vec{v}_r}{dt}\right)_{RRF}}_{\vec{a}_r} + \vec{\omega}\times\vec{v}_r\Big]$$

$$+ \dot{\vec{\omega}}\times\vec{r}$$

$$+ \vec{\omega}\times\Big[\underbrace{\left.\frac{d\vec{r}}{dt}\right)_{RRF}}_{\vec{v}_r} + \vec{\omega}\times\vec{r}\Big]$$

$$\therefore \vec{F} = m\vec{a}_f = m\ddot{\vec{R}}_f + m\vec{a}_r + \boxed{m\vec{\omega}\times\vec{v}_r}$$

$$+ m\dot{\vec{\omega}}\times\vec{r} + \boxed{m\vec{\omega}\times\vec{v}_r}$$

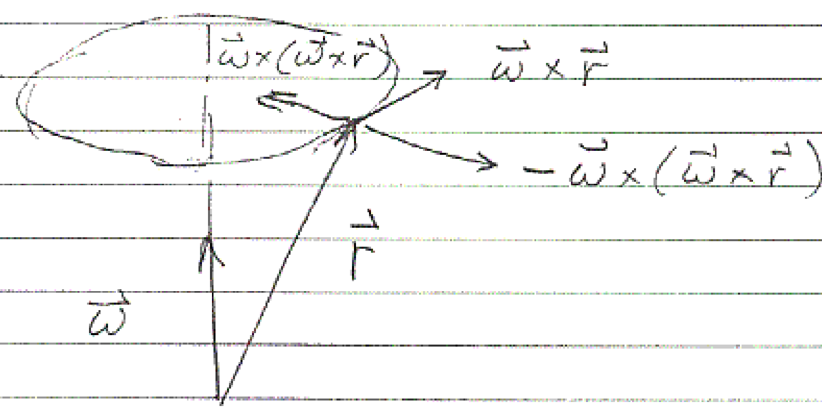
$$+ m\vec{\omega}\times\vec{\omega}\times\vec{r}$$

To observer in rotating frame, if we define effective forces F_{eff} such that

$$\vec{F}_{eff} \equiv m\vec{a}_r\,,\quad \text{then}$$

$$
\begin{aligned}
\vec{F}_{eff} = \; & \vec{F} && \text{force in IRF} \\
& -m\ddot{\vec{R}}_f && \text{pseudo-force linear accel.} \\
& -m\dot{\vec{\omega}}\times\vec{r} && \text{'' angular accel.} \\
& -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) && \text{centrifugal force} \\
& -2m\vec{\omega}\times\vec{v}_r && \text{Coriolis force}
\end{aligned}
$$

The centrifugal pseudo-force is directed away from rotation axis:



Coriolis pseudo-force is proportional to $\vec{v}_r$; i.e., vanishes if P is stationary in RRF.

→ is responsible for cyclonic motion in Earth's atmosphere

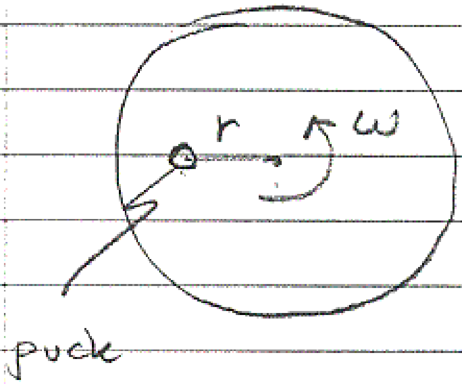
Rotating, frictionless disk example

Imagine a rotating, frictionless disk (merry-go-round) rotating with constant angular velocity ω.

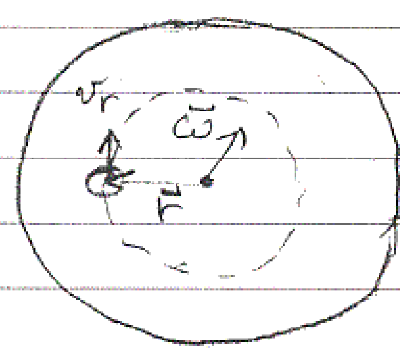
$$\vec{F}_{eff} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_r$$

Imagine placing a hockey puck on it at some radius r <u>at rest</u> <u>in the IRF</u>. What is $\vec{F}_{eff}$ and what is trajectory in RRF?

IRF



RRF



$$\vec{v}_r = -\vec{\omega}\times\vec{r}$$

$$\vec{F}_{eff} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

$$= m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

Since $\vec{\omega}\times\vec{r}$ is tangential by construction, $\vec{\omega}\times\vec{\omega}\times\vec{r}$ is radially inward; keeps puck <u>in orbit</u>

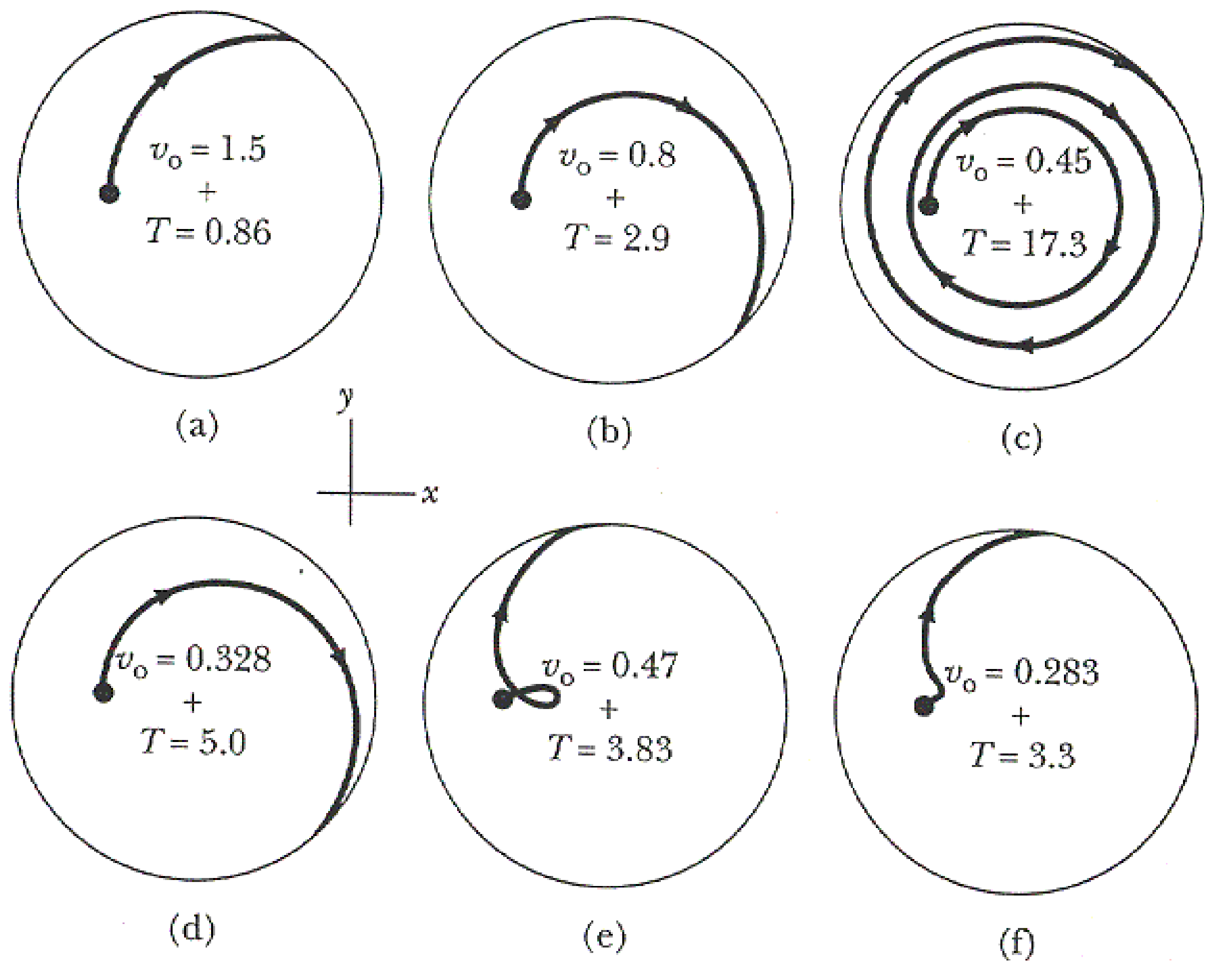


FIGURE 10-4 The motion of the hockey puck of Example 10.2 as observed in the rotating system for various initial directions and velocities v_0 at the times T noted. The angular velocity ω(1 rad/s) is out of the page.