4.31

We have a cube at the origin of side a with polarization $\mathbf{P} = k\mathbf{r}$. The bound surface and bulk charge densities are given by

$$\sigma_b = \hat{n} \cdot \mathbf{P}, \qquad \rho_b = -\nabla \cdot \mathbf{P}. \tag{1}$$

By symmetry, each of the six sides of the cube carries the same bound surface charge, so we'll focus on just one side, compute the total charge on that side, and multiply the result by six. Consider the side with surface normal $\hat{n} = \hat{x}$. From (1), we find $\sigma_b = kx|_{x=a/2} = ak/2$. Integrating this over the surface of the cube yields $6 \times a^3k/2 = 3a^3k$.

Next, we'll consider the bound bulk charges. Here, we find from (1) that the charge density is $\rho_b = -3k$. This is constant over the cube, so we just multiply by the volume to get the total bulk charge: $-3a^3k$. Adding this to the total surface charge found above yields a total charge of zero for the cube.

4.32

Since we have spherical symmetry in this problem, we can use Gauss' Law to find $\mathbf{D}$ everywhere:

$$\mathbf{D} = \frac{q}{4\pi}\frac{\hat{r}}{r^2}. \tag{2}$$

From this, we can find $\mathbf{E}$ and $\mathbf{P}$ inside the sphere:

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon}, \qquad \mathbf{P} = \epsilon_0\chi_e\mathbf{E}. \tag{3}$$

Here, $\epsilon = \epsilon_0(1 + \chi_e)$. Outside the sphere, $\mathbf{E} = \mathbf{D}/\epsilon_0$ and $\mathbf{P} = 0$.

From the expression for $\mathbf{P}$, it is straight-forward to find the bound surface charge density at the outer surface of the sphere:

$$\sigma_b = \hat{n} \cdot \mathbf{P}|_{r=R} = |\mathbf{P}|(r = R) = \frac{\chi_e q}{4\pi(1 + \chi_e)R^2}. \tag{4}$$

The bound surface charge is thus

$$Q_b^{surface} = \frac{q\chi_e}{1 + \chi_e}. \tag{5}$$

The bulk bound charge density is given by

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2|\mathbf{P}|). \tag{6}$$

Since $r^2|\mathbf{P}|$ is a constant, you might be inclined to say that ρ_b is zero, but that would lead to a contradiction since we know that we must have $Q_b^{bulk} = -Q_b^{surface} \neq 0$. The problem is that the expression we have used for the divergence operator in spherical coordinates is not well defined at $r = 0$. So we have at least shown that $\rho_b = 0$ everywhere but at $r = 0$, where $\rho_b \neq 0$. This means that ρ_b must be proportional to a Dirac delta function:

$$\rho_b = -\frac{\chi_e q}{1+\chi_e}\delta(\mathbf{r}). \tag{7}$$

We have fixed the coefficient by requiring that $Q_b^{bulk} = -Q_b^{surface}$.

Note that you can also arrive at this result by considering example 4.5 in the book. Here, Griffiths considers a conducting charged sphere inside the spherical dielectric, instead of a point charge. In this case, $\rho_b = 0$ everywhere, and the conductor and dielectric surfaces have equal and opposite total surface charge, giving $Q_b^{total} = 0$. If we make the conducting sphere smaller and smaller (ie take $a \to 0$), we find that we still get a nonvanishing charge on the inner surface even though the surface area vanishes in this limit (Q_b^{inner} doesn't depend on r). In this case, we might think of the extra bound charge as being on the "surface" of the point charge. Whichever way we do the problem, we see that the charge on the surface of the dielectric is canceled by a bound charge at $r = 0$.

4.33

To do this problem, we need to consider the boundary conditions for the electric field (see eqns 4.29 and 4.40 in the book):

$$E_1^{\|} = E_2^{\|}, \qquad \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}. \tag{8}$$

From Fig 4.34 in the book, it is evident that

$$E^{\|} = E\sin\theta, \qquad E^{\perp} = E\cos\theta. \tag{9}$$

We thus have

$$E_1\sin\theta_1 = E_2\sin\theta_2, \qquad \epsilon_1 E_1\cos\theta_1 = \epsilon_2 E_2\cos\theta_2. \tag{10}$$

Solving each of these equations for E_1/E_2 and equating the results gives

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_2}{\epsilon_1}. \tag{11}$$

4.36

(a)

We start by assuming the potential is the same as we would get without the dielectric, namely

$$V = \frac{V_0 R}{r}. \tag{12}$$

We know this must be the potential since, in the absence of the dielectric, spherical symmetry tells us that the charges on the surface of the conductor must be evenly distributed. In this case, the potential outside the sphere is the same as that of a point charge at the origin, so it must be proportional to $1/r$. The boundary condition, $V(r = R) = V_0$ fixes the coefficient. This potential gives

$$\mathbf{E} = -\nabla V = \frac{V_0 R}{r^2}\hat{r}, \qquad \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}. \tag{13}$$

We then find for the bound charges that

$$\begin{aligned} \rho_b &= -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r) = 0, \\ \sigma_b &= \hat{n} \cdot \mathbf{P} = -\hat{r} \cdot \mathbf{P} = -|\mathbf{P}|(r = R) = \left\{ \begin{array}{cc} 0 & \text{empty space hemisphere} \\ -\epsilon_0 \chi_e V_0/R & \text{dielectric hemisphere} \end{array} \right\}. \end{aligned} \tag{14}$$

For the free charges, we have

$$\begin{aligned} \rho_f &= \epsilon_0 \nabla \cdot \mathbf{E} = 0, \\ \sigma_f &= \epsilon_{out} E_{out}^{\perp} - \epsilon_{in} E_{in}^{\perp} = \left\{ \begin{array}{cc} \epsilon_0 V_0/R & \text{empty space hemisphere} \\ \epsilon_0(1 + \chi_e)V_0/R & \text{dielectric hemisphere} \end{array} \right\}. \end{aligned} \tag{15}$$

In the last expression, E_{out} corresponds to the electric field in either empty space or in the dielectric, while $E_{in} = 0$ is the field in the conductor.

(b)

If we combine the above charge distributions, we find that $\rho = 0$ everywhere, and $\sigma = \epsilon_0 V_0/R$ everywhere on the surface of the conductor. Since this charge distribution is uniformly distributed over the surface of the conductor, it generates the same potential as for a single point source of charge $4\pi R^2 \times \epsilon_0 V_0/R = 4\pi\epsilon_0 V_0 R$. The potential for a point source of charge q is just $q/4\pi\epsilon_0 r$, so we get back the potential we assumed in part (a).

(c)

We have found a self-consistent solution to the problem, so by the theorem quoted in problem 4.35 of the book, we have found *the* solution. All the conditions of the theorem are satisfied in the statement of the problem, namely the susceptibilities of each dielectric and the potential on every surface are specified. You may have noticed that our solution appears to have an inconsistency though: the bound charges don't add to zero. This is due to the fact that our system is not finite—our dielectric extends to infinity. A finite dielectric would have an additional surface, and this surface would carry just the right surface charge density to cancel that which we computed in part (a).

(d)

We would be able to solve configuration (b) with the same potential, but not configuration (a). This is because in (a), the boundary between the dielectric and empty space (ie outside the conductor) is not parallel to the electric field. This means that there will be bound charges on this surface as well, destroying the spherical symmetry we needed to have the potential of part (a), and yielding an inconsistency. In the original problem and in configuration (b), the component of the electric field perpendicular to this boundary vanishes, which implies that the perpendicular component of the polarization $\vec{P}$ vanishes, meaning that there is no surface charge. So all is well in these cases.